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On Finite Forms of Certain Bilateral Basic Hypergeometric Series and Their Applications

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Abstract In this paper we derive finite forms of the summation formulas for bilateral basic hypergeometric series ${}_3\psi_3$, ${}_4\psi_4$ and ${}_5\psi_5$. We therefrom obtain the summation formulae obtained recently by Wenchang CHU and Xiaoxia WANG. As applications of these summation formulae, we deduce the well-known Jacobi's two and four square theorems, a formula for the number of representations of an integer n as sum of four triangular numbers and some theta function identities.

Keywords bilateral basic hypergeometric series; finite forms; theta functions; sums of squares and sums of triangular numbers

MR(2010) Subject Classification 33D15

1. Introduction

In [1], Dougall derived the summation formulae for the bilateral ${}_2H_2$ and very well-posed ${}_5H_5$ hypergeometric series, which are perhaps the first summation formulae appearing in the literature. Later Ramanujan [2] gave a summation formula for ${}_1\psi_1$ series, where ${}_r\psi_r$ is defined by

$$ {}_r\psi_r\left(\begin{matrix} a_1, & a_2, \ldots, & a_r \\ b_1, & b_2, \ldots, & b_r \end{matrix}; q; z\right) := \sum_{n=-\infty}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_r)_n}{(b_1)_n(b_2)_n\cdots(b_r)_n} z^n. \tag{1.1} $$

Here $|q|<1$, $|\frac{b_1\cdots b_r}{a_1\cdots a_r}|<|z|<1$,

$$ (a)_\infty := (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \quad (a)_n := (a;q)_n := \frac{(a)_\infty}{(aq^n)_\infty} $$

and

$$ (a_1,a_2,a_3,\ldots,a_m)_n = (a_1)_n(a_2)_n(a_3)_n\cdots(a_m)_n, \quad n \text{ is an integer or } \infty. $$

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Further contributions to summations and transformations for bilateral basic hypergeometric series were made by Bailey [3,4], Slater [5], Jackson [6] and Jackson [7]. For more details one may refer to the book [8] by Gasper and Rahman.

In the recent past many authors including Schlosser [9], Zhang and Hu [10,11], Somashekara and Narasimha Murthy [12], Somashekara, Narasimha Murthy and Shalini [13], Chen and Fu [14] have contributed to the proofs of some summation and transformation formulae for bilateral basic hypergeometric series. In [15], Chu and Wang derived the following summation formulae:

$$ {}_3\psi_3\left(\begin{matrix} b, & c, & d \\ q/b, & q/c, & q/d, \end{matrix} ;q;\frac{q}{bcd}\right) = \frac{(q,\ q/bc,\ q/bd,\ q/cd;q)_\infty}{(q/b,\ q/c,\ q/d,\ q/bcd;q)_\infty}, \tag{1.2} $$

$$ {}_3\psi_3\left(\begin{matrix} b, & c, & d \\ q^2/b, & q^2/c, & q^2/d, \end{matrix} ;q;\frac{q^2}{bcd}\right) = \frac{(q,\ q^2/bc,\ q^2/bd,\ q^2/cd;q)_\infty}{(q^2/b,\ q^2/c,\ q^2/d,\ q^2/bcd;q)_\infty}, \tag{1.3} $$

$$ {}_4\psi_4\left(\begin{matrix} qw, & b, & c, & d \\ w, & q/b, & q/c, & q/d, \end{matrix} ;q;\frac{q}{bcd}\right) = \frac{(q,\ q/bc,\ q/bd,\ q/cd;q)_\infty}{(q/b,\ q/c,\ q/d,\ q/bcd;q)_\infty}, \tag{1.4} $$

$$ \begin{aligned} &{}_5\psi_5\left(\begin{matrix} qu, & qv, & b, & c, & d \\ u, & v, & q/b, & q/c, & q/d, \end{matrix} ;q;\frac{q^{-1}}{bcd}\right) \\ &= \frac{(q,\ 1/bc,\ 1/bd,\ 1/cd;q)_\infty}{(q/b,\ q/c,\ q/d,\ q^{-1}/bcd;q)_\infty}\frac{(1-1/quv)}{(1-1/u)(1-1/v)} \end{aligned} \tag{1.5} $$

by using very well-posed ${}_6\phi_5$-series summation formula [8, equation(II.20), p.356].

One way of deriving such formulae is by finding their finite forms. Way back in 1915, MacMahon [16] had given the finite version

$$ (qz;q^2)_\infty(q/z;q^2)_\infty = \sum_{j=-m}^{n}\binom{m+n}{j+m}(-z^2)^j q^{j^2} \tag{1.6} $$

of the well-known Jacobi's triple product identity namely

$$ (qz;q^2)_\infty(q/z;q^2)_\infty(q^2;q^2)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} z^n. \tag{1.7} $$

In 2005, Chen, Chu and Gu [17] gave the following finite form

$$ \sum_{k=0}^{n}(1+zq^k)\binom{m}{k}\frac{(z;q)_{m+1}}{(z^2q^k;q)_{m+1}}z^k q^{k^2} \equiv 1 \tag{1.8} $$

of the famous Watson's quintuple product identity

$$ (qz;q)_\infty(1/z;q)_\infty(qz;q^2)_\infty(q/z;q^2)_\infty(q;q)_\infty = \sum_{n=-\infty}^{\infty}(1-zq^n)q^{3n(n-1)/2}(qz^3)^n. \tag{1.9} $$

For different finite forms of the Jacobi's triple product identity and the Watson's quintuple product identity one may refer to the works of Ma [18], Chu and Jia [19] and Guo and Zeng [20]. Schlosser [21] derived the following finite form

$$ \sum_{k=-n}^{n}\frac{(q^{-n},a,bq^n)_k}{(q^{n+1},c,abq^{1-n}/c)_k}q^k = \frac{(c/a)_{2n}(c/b,\ bq/c,\ q,\ q)_n}{(q)_{2n}(c,q/a,b,c/ab)_n} \tag{1.10} $$

of the well-known Ramanujan's ${}_1\psi_1$ summation formula [2,22]

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}. \tag{1.11}$$

Recently Bayad, Somashekara and Narasimha Murthy [23] have obtained the finite forms of Bailey's ${}_2\psi_2$ transformation formulae. Also, in [24], Somashekara and Narasimha Murthy have derived finite forms of Ramanujan's reciprocity theorem and of its three and four variable generalizations.

In this sequel, we derive the finite forms of all the four summation formulae (1.2)–(1.5) and thereby give the proofs of them.

According to Dickson [25, p.6], Fermat made the following famous comment about 355 years ago: "I was the first to discover the very beautiful and entirely general theorem that every number is either triangular or the sum of 2 or 3 triangular numbers; every number is either a square or the sum of 2, 3 or 4 squares; either pentagonal or the sum of 2, 3, 4 or 5 pentagonal numbers; and so on ad infinitum...". Here "number" means "positive integer" and the triangular, square and pentagonal numbers are respectively described by: $n(n+1)/2$, n^2 and $n(3n-1)/2$, $n = 1, 2, \ldots$.

Only that part of the Fermat's statement regarding the squares and triangular numbers have become the most celebrated problems in Number Theory. If $r_k(n)$ and $t_k(n)$ denote the number of representations of an integer n as sum of k squares and k triangular numbers respectively, the problem is to find formulae for determining $r_k(n)$ and $t_k(n)$, in terms of simple arithmetical functions such as divisor functions. The well-known two and four square theorems of Jacobi are:

$$r_2(n) = 4[d_1(n) - d_3(n)] \quad \text{and} \quad r_4(n) = 8 \sum_{d|n,\ 4\nmid d} d$$

where $d_i(n)$ denotes the number of divisors d of n with $d \equiv i \pmod 4$. Ramanujan [26], Mordell [27], Andrews [28], Hirschhorn [29,30], Bhargava and Adiga [31], Bhargava, Adiga and Somashekara [32], Cooper and Hirschhorn [33] are among many others who contributed to the problem on sums of squares.

Adiga [34], Ono, Robins and Wahl [35] and Liu [36] have derived the following formula for sum of four tringular numbers

$$t_4(n) = \sum_{d|2n+1} d.$$

As an application of ${}_3\psi_3$ summation formula we deduce the Jacobi's two and four square theorems and the formula for the sum of four triangular numbers.

In Section 2, we present some standard identities which we employ to prove our main results. In Section 3, we present the finite forms of ${}_3\psi_3$, ${}_4\psi_4$ and ${}_5\psi_5$ summations. In Section 4, we use ${}_3\psi_3$-sum to deduce Jacobi's two and four Square theorems and the formula for the sum of 4-triangular numbers. We further deduce some theta function identities from the ${}_3\psi_3$ -sum.

2. Some standard identities for basic hypergeometric series

In this section, we list some standard identities for basic hypergeometric series which will be used to prove our main results. A q-shifted factorial identity [8, equation (I.2), p.351] is

$$(a)_{-n} = \frac{1}{(aq^{-n})_n} = \frac{(-q/a)^n}{(q/a)_n} q^{\binom{n}{2}}, \quad n, \text{ a non-negative integer.} \tag{2.1}$$

Ramanujan's general theta function $f(a,b)$ is given by

$$f(a,b) := 1 + \sum_{n=1}^{\infty} (ab)^{n(n-1)/2}(a^n + b^n) = \sum_{-\infty}^{\infty} (ab)^{n(n+1)/2}(b)^{n(n-1)/2}, \quad |ab| < 1.$$

See [2,22] for more details. The special cases of $f(a,b)$ are given by

$$\varphi(q) := f(q,q) = 1 + 2\sum_{k=1}^{\infty} q^{k^2} = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, \tag{2.2}$$

$$\psi(q) := f(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.3}$$

$$f(-q) := f(-q,-q^2) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2} = (q;q)_\infty, \tag{2.4}$$

$$\chi(q) := (-q;q^2)_\infty. \tag{2.5}$$

Jackson's q-analogue of Dougall's ${}_7F_6$ sum [8, (II.22), p.356] is given by

$$\sum_{k=0}^{n} \frac{(a, qa^{1/2}, -qa^{1/2},\ b,\ c,\ d,\ e, q^{-n})_k}{(q, a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d, aq/e, aq^{n+1})_k} q^k = \frac{(aq,\ aq/bc,\ aq/bd,\ aq/cd)_n}{(aq/b, aq/c, aq/d, aq/bcd)_n}, \tag{2.6}$$

where $a^2q = bcdeq^{-n}$.

3. Finite forms of (1.2)–(1.5)

In this section, we derive the finite forms of (1.2)–(1.5) using Jackson's q-analogue of Dougall's ${}_7F_6$ summation.

Theorem 3.1 *We have for* $|\frac{q}{bcd}| < 1$,

$$\sum_{k=-n}^{n} \frac{(b,\ c,\ d, q^{-n}, q^{1+n}/bcd)_k}{(q/b, q/c, q/d, q^{n+1}, bcdq^{-n})_k} q^k = \frac{(q,\ q/bc,\ q/bd,\ q/cd)_n}{(q/b, q/c, q/d, q/bcd)_n}. \tag{3.1}$$

Proof In (2.6) noting that $e = a^2q^{1+n}/bcd$ and then putting $a = 1$, we obtain

$$\begin{aligned} 1 &+ \sum_{k=1}^{n} \frac{(b,\ c,\ d, q^{-n}, q^{1+n}/bcd)_k}{(q/b, q/c, q/d, q^{n+1}, bcdq^{-n})_k} q^k + \sum_{k=1}^{n} \frac{(b,\ c,\ d, q^{-n}, q^{1+n}/bcd)_k}{(q/b, q/c, q/d, q^{n+1}, bcdq^{-n})_k} q^{2k} \\ &= \frac{(q,\ q/bc,\ q/bd,\ q/cd)_n}{(q/b, q/c, q/d, q/bcd)_n}. \end{aligned}$$

Changing k to $-k$ in the second sum on the left side of the above equation and using (2.1), we obtain (3.1) after some simplifications. □

Theorem 3.2 *We have for* $|\frac{q^2}{bcd}| < 1$,

$$\sum_{k=-(n+1)}^{n} \frac{(b,\ c,\ d, q^{-n}, q^{3+n}/bcd)_k}{(q^2/b, q^2/c, q^2/d, q^{n+2}, bcdq^{-n-1})_k} q^k = \frac{(q,\ q^2/bc,\ q^2/bd,\ q^2/cd)_n}{(q^2/b, q^2/c, q^2/d, q^2/bcd)_n}. \tag{3.2}$$

Proof In (2.6) noting that $e = a^2q^{1+n}/bcd$ and then putting $a = q$, we obtain

$$\sum_{k=0}^{n} \frac{(b,\ c,\ d, q^{-n}, q^{3+n}/bcd)_k}{(q^2/b, q^2/c, q^2/d, q^{n+2}, bcdq^{-n-1})_k} q^k - \sum_{k=0}^{n} \frac{(b,\ c,\ d, q^{-n}, q^{3+n}/bcd)_k}{(q^2/b, q^2/c, q^2/d, q^{n+2}, bcdq^{-n-1})_k} q^{3k+1}$$
$$= \frac{(q,\ q^2/bc,\ q^2/bd,\ q^2/cd)_n}{(q^2/b, q^2/c, q^2/d, q^2/bcd)_n}.$$

Changing k to $(-k-1)$ in the second sum on the left side of the above equation and using (2.1), we obtain (3.2) after some simplifications. □

Theorem 3.3 *We have for* $|\frac{q}{bcd}| < 1$,

$$\sum_{k=-n}^{n} \frac{(qw, b,\ c,\ d, q^{-n}, q^{1+n}/bcd)_k}{(w, q/b, q/c, q/d, q^{n+1}, bcdq^{-n})_k} q^k = \frac{(q,\ q/bc,\ q/bd,\ q/cd)_n}{(q/b, q/c, q/d, q/bcd)_n}. \tag{3.3}$$

Proof By reversing the summation index k to $-k$ in (3.1) and then using (2.1), we obtain, after some simplifications

$$\sum_{k=-n}^{n} \frac{(b,\ c,\ d, q^{-n}, q^{1+n}/bcd)_k}{(q/b, q/c, q/d, q^{n+1}, bcdq^{-n})_k} q^{2k} = \frac{(q,\ q/bc,\ q/bd,\ q/cd)_n}{(q/b, q/c, q/d, q/bcd)_n}.$$

Multiplying the above equation by w and then subtracting it from (3.1), we obtain (3.3). □

Theorem 3.4 *We have for* $|\frac{1}{bcdq}| < 1$,

$$\sum_{k=-n}^{n} \frac{(qu, qv, b,\ c,\ d, q^{-n-1}, q^{n-1}/bcd)_k}{(u, v, 1/b, 1/c, 1/d, q^{n+1}, bcdq^{-n+1})_k} q^k$$
$$= \frac{(q,\ 1/bc,\ 1/bd,\ 1/cd)_n}{(q/b, q/c, q/d, q^{-1}/bcd)_n} \frac{1 - 1/quv}{(1-1/u)(1-1/v)}. \tag{3.4}$$

Proof Changing k to $(k-1)$ in (3.2) and then replacing b, c and d by bq, cq and dq, respectively, we obtain

$$\sum_{k=-n}^{n+1} \frac{(b,\ c,\ d, q^{-n-1}, q^{n-1}/bcd)_k}{(1/b, 1/c, 1/d, q^{n+1}, bcdq^{-n+1})_k} q^k = \frac{-1}{q} \frac{(q,\ 1/bc,\ 1/bd,\ 1/cd)_n}{(q/b, q/c, q/d, q/bcd)_n}. \tag{3.5}$$

Changing k to $-(k-1)$ in the above equation, we obtain after some simplifications,

$$\sum_{k=-n}^{n} \frac{(b,\ c,\ d, q^{-n-1}, q^{n-1}/bcd)_k}{(1/b, 1/c, 1/d, q^{n+1}, bcdq^{-n+1})_k} q^{3k} = \frac{(q,\ 1/bc,\ 1/bd,\ 1/cd)_n}{(q/b, q/c, q/d, q^{-1}/bcd)_n}. \tag{3.6}$$

When the variable of finite form of ${}_3\psi_3$ series is situated between those of (3.5) and (3.6), there holds the following reduced formula

$$\sum_{k=-n}^{n} \frac{(b,\ c,\ d, q^{-n-1}, q^{n-1}/bcd)_k}{(1/b, 1/c, 1/d, q^{n+1}, bcdq^{-n+1})_k} q^{2k} = 0. \tag{3.7}$$

Considering the linear combination

$$\frac{(3.5)}{(1-u)(1-v)}+\frac{uv(3.6)}{(1-u)(1-v)}-\frac{(u+v)(3.7)}{(1-u)(1-v)}$$

and simplifying the result, we get (3.4). □

4. Some applications of identities (1.2) and (1.3)

In this section, we deduce some theta function identities from (1.2) and (1.3).

Corollary 4.1 *If $0<q<1$, then*

$$\frac{f^8(-q^2)}{f^4(-q^4)}=8\sum_{k=-\infty}^{\infty}\frac{(-1)^k q^{2k}}{(1+q^{2k})^3}, \tag{4.1}$$

$$\varphi^2(q)=1+4\sum_{k=1}^{\infty}\Big(\frac{q^k}{1+q^{2k}}\Big), \tag{4.2}$$

$$\frac{\varphi^2(q)\chi^4(-q)}{\chi^4(q)}=4\sum_{k=-\infty}^{\infty}\frac{(-q)^k}{(1+q^{2k})^2}, \tag{4.3}$$

$$\varphi^4(q)=8\sum_{k=-\infty}^{\infty}\frac{q^k}{(1+(-q)^k)^3}, \tag{4.4}$$

$$\psi^4(q)=\sum_{k=-\infty}^{\infty}\frac{q^k}{(1-q^{2k+1})^3}. \tag{4.5}$$

Proof Changing q to q^2 and then putting $b=c=d=-1$ in (1.2), we obtain (4.1). Changing q to q^2 and then putting $b=-1=d$ and $c=q$ in (1.2), we obtain (4.2). Changing q to q^2 and then putting $b=-q$ and $c=-1=d$ in (1.2), we obtain (4.3). Putting $b=c=d=-1$ in (1.2) and then changing q to $-q$, we obtain (4.4). Changing q to q^2 and then putting $b=c=d=q$ in (1.3), we obtain (4.5). □

Corollary 4.2 *We have,*

$$r_2(n)=4[d_1(n)-d_3(n)], \tag{4.6}$$

$$r_4(n)=8\sum_{d|n,\ 4\nmid d} d \tag{4.7}$$

and

$$t_4(n)=\sum_{d|2n+1} d. \tag{4.8}$$

Proof Expanding $(1+q^{2k})^{-1}$ in (4.2), we obtain,

$$\varphi^2(q)=1+4\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}(-1)^m q^{(2m+1)k}$$

which yields (4.6).

Equation (4.4) can be written as

$$\begin{aligned}\varphi^4(q) &= 1+8\Big[\sum_{k=1}^{\infty}\frac{q^k}{(1+(-q)^k)^3}+\sum_{k=1}^{\infty}\frac{q^{2k}(-1)^k}{(1+(-q)^k)^3}\Big]\\ &= 1+8\sum_{k=1}^{\infty}\frac{q^k}{[1+(-q)^k]^2}=1+8\sum_{k=1}^{\infty}\frac{kq^k}{1+(-q)^k}\\ &= 1+8\Big[\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}-\sum_{k=1}^{\infty}\frac{4kq^{4k}}{1-q^{4k}}\Big] \qquad (4.9)\end{aligned}$$

which yields (4.7).

Equation (4.5) can be written as

$$\psi^4(q)=\sum_{k=0}^{\infty}\frac{q^k(1+q^{2k+1})}{(1-q^{2k+1})^2}. \qquad (4.10)$$

Expanding $(1-q^{2k+1})^{-2}$ in (4.10), we obtain

$$\begin{aligned}\psi^4(q) &= \sum_{k=0}^{\infty}q^k(1+q^{2k+1})\sum_{m=0}^{\infty}(m+1)q^{m(2k+1)}\\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(m+1)q^{k(2m+1)+m}+\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(m+1)q^{k(2m+3)+m+1}. \qquad (4.11)\end{aligned}$$

Interchanging the order of summation on the right side of (4.11), we obtain,

$$\begin{aligned}\psi^4(q) &= \sum_{m=0}^{\infty}\frac{(m+1)q^m}{1-q^{2m+1}}+\sum_{m=0}^{\infty}\frac{(m+1)q^{m+1}}{1-q^{2m+3}}\\ &= \frac{1}{1-q}+\sum_{m=1}^{\infty}\frac{(m+1)q^m}{1-q^{2m+1}}+\sum_{m=1}^{\infty}\frac{mq^m}{1-q^{2m+1}}\\ &= \frac{1}{1-q}+\sum_{m=1}^{\infty}\frac{(2m+1)q^m}{1-q^{2m+1}}=\sum_{m=0}^{\infty}\frac{(2m+1)q^m}{1-q^{2m+1}}\end{aligned}$$

which yields (4.8). □

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