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BMO Estimates for Quasilinear Elliptic Equations with BMO Coefficients

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Abstract We establish a global limiting case of nonlinear Calderón-Zygmund theory to quasilinear elliptic equations $\operatorname{div}\mathcal{A}(x,Du)=\operatorname{div}(|F|^{p-2}F)$ under the BMO smallness of the nonlinearity, that is $|F|^{p-2}F\in \mathrm{BMO}$ implies that $Du\in\mathrm{BMO}$.

Keywords Calderón-Zygmund theory; Quasilinear elliptic equations; BMO space

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1. Introduction and main results

The most classical instance of Calderón-Zygmund theory, going back to Calderón and Zygmund [1,2], occurs when considering the Poisson equation

$$\Delta u=\operatorname{div} F.$$

By representation formula involving the so called fundamental solution, a priori estimate yields for each $0<\gamma<\infty$,

$$\|Du\|_{L^\gamma}\le C\|F\|_{L^\gamma}, \tag{1.1}$$

where C only depends on γ. This means that integrability of F transfers to integrability of Du. Iwaniec [3] extended the above estimations (1.1) to p-Laplacian equations

$$\Delta_p u=\operatorname{div}(|F|^{p-2}F), \tag{1.2}$$

where $\Delta_p u$ is the p-Laplacian of u defined by $\Delta_p u:=\operatorname{div}(|Du|^{p-2}Du)$, and showed that, mostly called nonlinear Calderoón-Zygmund theory, $F\in L^\gamma(\mathbb{R}^N,\mathbb{R}^N)$ implies $Du\in L^\gamma(\mathbb{R}^N,\mathbb{R}^N)$ for every $\gamma\ge p$.

Recently, Phuc [4], using some comparison lemmas obtained by Duzaar and Mingione [5,6], established global L^γ boundedness of $\mathcal{A}$-superharmonic function in $\mathbb{R}^N$ with $\gamma>\max\{1,p-1\}$

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provided that $\int_{\mathbb{R}^N} |Du|^\gamma dx < \infty$, and in the same paper, Phuc also gave a simple proof of main results of [7].

It would be interesting to extend above nonlinear Calderoón-Zygmund theory to p-Laplacian system

$$\Delta_p \mathbf{u} = \mathrm{div}_p(|\mathbf{F}|^{p-1}\mathbf{F}), \tag{1.3}$$

where $\Delta_p \mathbf{u}$ is the p-Laplacian of $\mathbf{u} = (u_1, u_2, \ldots, u_m) \in [W_1^p(\Omega)]^m$. Indeed, Dibenedetto and Manfredi [8] improved the results of scalar equations (1.2) to p-Laplacian system (1.3). They showed, similarly to scalar p-Laplacian equations,

$$\|D\mathbf{u}\|_{L^\gamma(\mathbb{R}^N)} \le C\|\mathbf{F}\|_{L^\gamma(\mathbb{R}^N)},$$

provided that $\mathbf{u} \in \left[W_1^p(\mathbb{R}^N)\right]^m$ is a weak solution of (1.3) with $\mathbf{F} = (F_1, F_2, \ldots, F_m) \in [W_1^p(\mathbb{R}^N)]^m$ and $\gamma \ge p$, where constant C depends only upon N, p and γ. The case $|F|^{p-2}F \in [\mathrm{BMO}(\mathbb{R}^N)]^m$ was settled in the same paper. In this case, they obtained

$$\|D\mathbf{u}\|_{\mathrm{BMO}(\mathbb{R}^N)} \le C\||\mathbf{F}|^{p-2}\mathbf{F}\|_{\mathrm{BMO}(\mathbb{R}^N)}, \tag{1.4}$$

where C depends only upon N, p. Recently, limiting BMO estimates (1.4) was extended to certain general elliptic systems in [9],

$$-\mathrm{div}(\mathcal{A}(D\mathbf{u})) = -\mathrm{div}\mathbf{F},$$

where $\mathcal{A}(D\mathbf{u})$ is defined by

$$\mathcal{A}(D\mathbf{u}) = \varphi'(|D\mathbf{u}|)\frac{D\mathbf{u}}{|D\mathbf{u}|},$$

for a suitable N-function φ. For further interesting extension of nonlinear Calderoón-Zygmund theory to nonlinear elliptic equations with discontinuous coefficients and measure data, we refer for instance to [10–19] and the references therein.

It is worth pointing out that nonlinear Calderoón-Zygmund theory is closely related to solvability and a priori estimates in L^γ as long as $\gamma > p-1$, usually referred to as an estimate below the natural growth exponent, to Dirichlet problem (1.2) with homogeneous boundary condition in bounded domain. Unfortunately, such interesting estimate still remains a certainly difficult open problem [20].

The aim of this paper is to establish limiting nonlinear Calderoón-Zygmund theory to degenerate quasilinear equations in divergence form with p-growth of the type

$$-\mathrm{div}\mathcal{A}(x, Du) = \mathrm{div}(|F|^{p-2}F), \tag{1.5}$$

where the nonlinearity $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ in (1.5) is assumed to be at least measurable in the coefficients x, C^1-regular in the gradient variable $\xi \in \mathbb{R}^n \setminus \{0\}$ and satisfying the following growth, ellipticity and continuity assumptions:

$$|\mathcal{A}(x,\xi)| + |\mathcal{A}_\xi(x,\xi)|(|\xi|^2+s^2)^{\frac{1}{2}} \le L(|\xi|^2+s^2)^{\frac{(p-1)}{2}}, \tag{1.6}$$

$$\nu(|\xi|^2+s^2)^{\frac{(p-1)}{2}}\lambda^2 \le \langle \mathcal{A}_\xi(x,\xi)\lambda, \lambda\rangle, \tag{1.7}$$

fore very $(\lambda, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}$ a.e. $x \in \Omega$. Here ν and L are positive structural constants. Here and in the rest of the paper we are assuming that ν, L, s are fixed parameters such that $0 < \nu \le L$ and $s \ge 0$. Note that the parameter $s \ge 0$ is used to distinguish the case of degenerate ellipticity ($s = 0$) from the nondegenerate one ($s > 0$). Moreover, when $p \ge 2$, assumption (1.7) immediately implies that, there exists constant C, depending on p, N and ν, such that

$$C|\xi - \eta|^p \le \langle \mathcal{A}(x,\xi) - \mathcal{A}(x,\eta), \xi - \eta \rangle. \tag{1.8}$$

A typical example of such a nolinearity $\mathcal{A}$, satisfying (1.6) and (1.7), is given by $\mathcal{A}(\xi) = |\xi|^{p-2}\xi$.

For the purpose of this paper we also require that the nonlinearity $\mathcal{A}$ satisfy a smallness condition of BMO-type in the x-variable. We call such a condition the (δ_0, R_0)-BMO condition.

Definition 1.1 *We say that $\mathcal{A}(x,\xi)$ satisfies a (δ_0, R_0)-BMO condition for some $\delta_0, R_0 > 0$ with exponent $\alpha > 0$, if*

$$[\mathcal{A}]_\alpha^{R_0} = \sup_{y \in \mathbb{R}^N, 0 < r \le R_0} \left(\fint_{B_r(y)} \Upsilon^\alpha(\mathcal{A}, B_r(y))(x) \mathrm{d}x \right)^{\frac{1}{\alpha}} \le \delta_0,$$

where

$$\Upsilon(\mathcal{A}, B_r(y))(x) = \sup_{\xi \in \mathbb{R}^N \setminus \{0\}} \frac{|\mathcal{A}(x,\xi) - \overline{\mathcal{A}}_{B_r(y)}(\xi)|}{|\xi|^{p-1}},$$

with $\overline{\mathcal{A}}_{B_r(y)}(\xi)$ denoting the average of $\mathcal{A}(x,\xi)$ over the ball $B_r(y)$, i.e.,

$$\overline{\mathcal{A}}_{B_r(y)}(\xi) = \fint_{B_r(y)} \mathcal{A}(x,\xi) \mathrm{d}x = \frac{1}{|B_r(y)|} \int_{B_r(y)} \mathcal{A}(x,\xi) \mathrm{d}x.$$

Let us state the main result of this paper.

Theorem 1.2 *Suppose that $u \in W_1^p(\mathbb{R}^N)$ is a weak solution of (1.5), $p > 2$ and*

$$|F|^{p-2}F \in \mathrm{BMO}(\mathbb{R}^N), \tag{1.9}$$

then

$$Du \in \mathrm{BMO}(\mathbb{R}^N). \tag{1.10}$$

Remark 1.3 It is interesting to note that $F \in \mathrm{BMO}(\mathbb{R}^N)$ does not imply $F^\kappa \in \mathrm{BMO}(\mathbb{R}^N)$ for positive κ. An example in one variable is given by $\log|x|$.

Let us briefly outline the strategy by describing the organization of the paper. In Section 2, we derive a few comparison lemmas allowing to treat with low regularity coefficients. The proof of Theorems 1.2 will be given in Section 3.

2. Preparations

2.1. General notation

In the following, we denote by C a general constant larger (or equal) than one, possibly varying from line to line, to indicate a dependence of C on the real parameters N, p, ν, L, we shall write $C = C(N, p, \nu, L)$. We also denote by $B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ the open ball with center x_0 and radius $r > 0$. When not important, or clear from the context, we shall omit the center and denote the open ball as $B_r = B(x_0, r)$. Unless otherwise stated, different balls in the same context will have the same center.

The space of functions of bounded mean oscillation, called BMO space, naturally arises as the class of functions whose deviation from their means over cubes is bounded. A function is said to be of bounded mean oscillation if its mean oscillation over all balls is bounded. Precisely, given a locally integrable function f on $\mathbb{R}^N$ and a measurable set Ω in $\mathbb{R}^N$, denote by

$$(f)_\Omega = \fint_\Omega f(x)\mathrm{d}x,$$

the mean (or average) of f over Ω.

Definition 2.1 *For f a locally integrable function on $\mathbb{R}^N$, set*

$$\|f\|_{\mathrm{BMO}} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - (f)_Q| \mathrm{d}x,$$

where the supremum is taken over all balls Q in $\mathbb{R}^N$. The function f is called of bounded mean oscillation if $\|f\|_{\mathrm{BMO}} < \infty$.

Note that $\|\cdot\|_{\mathrm{BMO}}$ is not a norm. The problem is that if $\|f\|_{\mathrm{BMO}} = 0$, this does not imply that $f = 0$, but that f is a constant. Moreover, every constant function C satisfies $\|C\|_{\mathrm{BMO}} = 0$. Consequently, functions f and $f + C$ have the same BMO norms whenever C is a constant. In the sequel, we keep in mind that elements of BMO whose difference is a constant are identified. Although $\|\cdot\|_{\mathrm{BMO}}$ is only a seminorm, we occasionally refer to it as a norm when there is no possibility of confusion.

The following important L^p characterization of BMO norms will be used in the sequel.

Proposition 2.2 *For all $1 < p < \infty$ we have*

$$c\|f\|_{\mathrm{BMO}} \le \sup_Q \frac{1}{|Q|} \int_Q |f(x) - (f)_Q|^p \mathrm{d}x \le C\|f\|_{\mathrm{BMO}},$$

where two constants c, C depend only upon N and p.

2.2. Comparison results

The proof of limiting nonlinear Calderoón-Zygmund theory (1.9) and (1.10) is done via comparison to a suitable homogeneous problem, which provides good reference estimates. This comparison will be performed within two steps. First we shall compare the original inhomogeneous problem (1.5) to the associated homogeneous problem

$$\begin{cases} -\mathrm{div}\, \mathcal{A}(x, Dw) = 0, & x \in B_{2r}(x_0), \\ w = u, & x \in \partial B_{2r}(x_0), \end{cases} \tag{2.1}$$

where $w \in u + W_0^{1,p}(B_{2r}(x_0))$ is the unique solution to the Dirichlet problem (2.1) for a fixed ball $B_{2r} \equiv B_{2r}(x_0) \subset \Omega$ with suitably small radius $2r$. Subsequently this homogeneous problem

(2.1) shall be compared to a $\overline{\mathcal{A}}_{B_r(x_0)}(\xi)$-harmonic functions

$$\begin{cases} -\mathrm{div}\,\overline{\mathcal{A}}_{B_r}(Dv) = 0, & x \in B_r(x_0), \\ v = w, & x \in \partial B_r(x_0). \end{cases} \tag{2.2}$$

Existence and uniqueness of w and v are guaranteed by standard monotonicity arguments, which can be done also in the generalized Sobolev space $W^{1,p}(\Omega)$.

In this section we start recalling a few crucial comparison estimates between the original solution of (1.5) and solutions to homogeneous boundary value problems (2.1). By a well-known version of Gehring's lemma applied to the function w defined as (2.1), yields the following reverse Hölder type inequality.

Lemma 2.3 *Let $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ and w be as in (2.1). Then there exists a constant $\theta_0 = \theta_0(N, p, \nu, L) > 1$ such that for any $t \in (0, p]$ the reverse Hölder type inequality*

$$\Big(\fint_{B_{\rho/2}(z)} |Dw|^{\theta_0 p} \mathrm{d}x\Big)^{\frac{1}{\theta_0 p}} \le C\Big(\fint_{B_\rho(z)} |Dw|^t \mathrm{d}x\Big)^{\frac{1}{t}},$$

holds for all balls $B_\rho(z) \subset B_{2r}(x_0)$ for a constant C depending only on N, p, ν, L, t.

We now come to the decay estimate below the natural growth exponent.

Lemma 2.4 *With $p > 2$, let $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ be a solution of (1.5) and let w be as in (2.1). Then there is a constant $C = C(N, p, \nu, L)$ such that*

$$\Big(\fint_{B_{2r}(x_0)} |Du - Dw|^q \mathrm{d}x\Big)^{\frac{1}{q}} \le C\Big(\fint_{B_{2r}(x_0)} [|F|^{p-2}F - (|F|^{p-1})_{B_{2r}}]^{\frac{p}{p-1}} \mathrm{d}x\Big)^{\frac{1}{p}}, \tag{2.3}$$

holds with $0 < q \le p$ and $0 < 2r < R_0$, where R_0 appears in Definition 1.1.

Proof According to (1.5) and (2.1), we know that

$$\int_{B_{2r}} \langle \mathcal{A}(x, Du) - \mathcal{A}(x, Dw), D\varphi\rangle \mathrm{d}x = \int_{B_{2r}} [|F|^{p-2}F - (|F|^{p-1})_{B_{2r}}]|D\varphi| \mathrm{d}x, \tag{2.4}$$

for every $\varphi \in W^{1,p}_0(B_{2r})$. Take the testing function $\varphi = u - w$ in (2.4). Thus, the fact that both u and w are solutions, together with Hölder's inequality, implies that

$$\begin{aligned} \int_{B_{2r}} |Du - Dw|^p \mathrm{d}x &\le C \int_{B_{2r}} \langle \mathcal{A}(x, Du) - \mathcal{A}(x, Dw), Du - Dw\rangle \mathrm{d}x \\ &= C \int_{B_{2r}} [|F|^{p-2}F - (|F|^{p-1})_{B_{2r}}]|Du - Dw| \mathrm{d}x \\ &\le C\Big(\int_{B_{2r}} [|F|^{p-2}F - (|F|^{p-1})_{B_{2r}}]^{\frac{p}{p-1}} \mathrm{d}x\Big)^{\frac{p-1}{p}} \Big(\int_{B_{2r}} |Du - Dw|^p \mathrm{d}x\Big)^{\frac{1}{p}}, \end{aligned}$$

which leads to

$$\fint_{B_{2r}} |Du - Dw|^p \mathrm{d}x \le C \fint_{B_{2r}} [|F|^{p-2}F - (|F|^{p-1})_{B_{2r}}]^{\frac{p}{p-1}} \mathrm{d}x.$$

Consider $0 < q < p$ now. Using the previous inequality and Hölder's inequality again we get that

$$\Big(\fint_{B_{2r}} |Du - Dw|^q \mathrm{d}x\Big)^{\frac{1}{q}} \le C\Big(\fint_{B_{2r}} |Du - Dw|^p \mathrm{d}x\Big)^{\frac{1}{p}}$$
$$\le C\Big(\fint_{B_{2r}} [|F|^{p-2}F - (|F|^{p-1})_{B_{2r}}]^{\frac{p}{p-1}} \mathrm{d}x\Big)^{\frac{1}{p}}.$$

Thus we arrive at (2.3). □

Lemma 2.5 *Let also w and v be as in (2.1) and (2.2), respectively. Then there is a constant $C = C(N, p, \nu, L)$ such that*

$$\Big(\fint_{B_r} |Dw - Dv|^q \mathrm{d}x\Big)^{\frac{1}{q}} \le C\delta_0^{\frac{1}{p-1}} \fint_{B_{2r}} |Du| \mathrm{d}x + C\delta_0^{\frac{1}{p-1}} \Big(\fint_{B_{2r}} [|F|^{p-2}F - (|F|^{p-1})_{B_{2r}}]^{\frac{p}{p-1}} \mathrm{d}x\Big)^{\frac{1}{p}} \tag{2.5}$$

holds with $0 < q \le p$ and $0 < 2r < R_0$.

Proof Indeed, according to the fact that both v and w are solutions, we get

$$\fint_{B_r} |Dw - Dv|^p \mathrm{d}x \le c \fint_{B_r} \langle \overline{\mathcal{A}}_{B_r}(Dw) - \overline{\mathcal{A}}_{B_r}(Dv), Dw - Dv\rangle \mathrm{d}x$$
$$= C \fint_{B_r} \langle \overline{\mathcal{A}}_{B_r}(Dw) - \mathcal{A}(x, Dw), Dw - Dv\rangle \mathrm{d}x$$
$$\le C \fint_{B_r} \Upsilon(\mathcal{A}, B_r(y))|Dw|^{p-1}|Dw - Dv| \mathrm{d}x.$$

Using Hölder's inequality with exponents

$$(\alpha, \frac{\theta p}{p-1}, p) := (\frac{\theta p}{(\theta-1)(p-1)}, \frac{\theta p}{p-1}, p),$$

we find

$$\fint_{B_r} |Dw - Dv|^p \mathrm{d}x \le C\Big(\fint_{B_r} \Upsilon^\alpha(\mathcal{A}, B_r(y)) \mathrm{d}x\Big)^{\frac{1}{\alpha}} \Big(\fint_{B_r} |Dw|^{\theta p} \mathrm{d}x\Big)^{\frac{p-1}{\theta p}} \Big(\fint_{B_r} |Dw - Dv|^p \mathrm{d}x\Big)^{\frac{1}{p}}$$
$$\le C\delta_0 \Big(\fint_{B_r} |Dw|^{\theta p} \mathrm{d}x\Big)^{\frac{p-1}{\theta p}} \Big(\fint_{B_r} |Dw - Dv|^p \mathrm{d}x\Big)^{\frac{1}{p}},$$

which gives

$$\Big(\fint_{B_r} |Dw - Dv|^p \mathrm{d}x\Big)^{\frac{1}{p}} \le C\delta_0^{\frac{1}{p-1}} \Big(\fint_{B_r} |Dw|^{\theta p} \mathrm{d}x\Big)^{\frac{1}{\theta p}}.$$

On the other hand, by Lemma 2.3 with $t = 1$ and triangle inequality, we know that

$$\Big(\fint_{B_r} |Dw|^{\theta p} \mathrm{d}x\Big)^{\frac{1}{\theta p}} \le C \fint_{B_{2r}} |Dw| \mathrm{d}x \le C \fint_{B_{2r}} |Du - Dw| \mathrm{d}x + C \fint_{B_{2r}} |Du| \mathrm{d}x. \tag{2.6}$$

This fact combined with (2.3) leads to estimate (2.5) with $q = p$. In a similar way as the proof of Lemma 2.4, we can prove that (2.5) holds with $0 < q < p$. □

The next result encodes the Höder continuity properties of Dv in an integral way (see [6] for a proof).

Lemma 2.6 *Let $v \in W_0^{1,p}(B_r)$ be a weak solution to (2.2) under the assumptions (1.6) and (1.7). Then there exist constants $\beta \in (0,1]$ and $C \geq 1$, both depending only on N, p, υ, L, such that the estimate*

$$\fint_{B_\varrho(x_0)} |Dv - (Dv)_{B_\varrho(z)}| \mathrm{d}x \leq C(\frac{\varrho}{\tau})^\beta \fint_{B_\tau(x_0)} |Dv - (Dv)_{B_\tau(z)}| \mathrm{d}x \tag{2.7}$$

holds whenever $0 < \varrho \leq \tau \leq r < R_0/2$.

The next Lemma follows easily by (2.3), (2.5) and triangle inequality.

Lemma 2.7 *Let u and v be as in (1.5) and (2.2) with $2r < R_0$, respectively. Then*

$$\begin{aligned} &\Big(\fint_{B_r(x_0)} |Du - Dv|^q \mathrm{d}x\Big)^{\frac{1}{q}} \\ &\leq C\delta_0^{\frac{1}{p-1}} \fint_{B_{2r}(x_0)} |Du| \mathrm{d}x + C\delta_0^{\frac{1}{p-1}} \Big(\fint_{B_{2r}(x_0)} [|F|^{p-2}F - (|F|^{p-1})_{B_{2r}}]^{\frac{p}{p-1}} \mathrm{d}x\Big)^{\frac{1}{p}}. \end{aligned} \tag{2.8}$$

The next result shows the regularity properties of v in decay estimates for a suitable excess functionals of the gradient.

Lemma 2.8 *Let $u \in W_{\mathrm{loc}}^{1,p}(\mathbb{R}^N)$ be a weak solution to (1.5) and $\mathcal{A}$ satisfy small (δ_0, R_0)-BMO condition with exponent $\alpha = \frac{\theta p}{(\theta-1)(p-1)}$ for some $\theta \in (1, \theta_0]$, where $\theta_0 > 1$ is as in Lemma 2.3. Then there are constants $\sigma = \sigma(N, p, \nu, L) \in (0,1]$ and $c = c(N, p, \nu, L) \geq 1$ such that*

$$\begin{aligned} \fint_{B_\varrho} |Du - (Du)_{B_\varrho}| \mathrm{d}x \leq{}& C(\frac{\varrho}{r})^\beta \fint_{B_r} |Du - (Du)_{B_r}| \mathrm{d}x + \\ & C\delta_0^{\frac{1}{p-1}} (\frac{r}{\varrho})^n \Big[\fint_{B_{2r}} |Du| \mathrm{d}x + \Big(\fint_{B_{2r}} [|F|^{p-2}F - (|F|^{p-1})_{B_{2r}}]^{\frac{p}{p-1}} \mathrm{d}x\Big)^{\frac{1}{p}}\Big], \end{aligned} \tag{2.9}$$

holds with $0 < \varrho < r$, where β appears in Lemma 2.6.

Proof According to Lemmas 2.6, 2.7 and triangle inequality, we have

$$\begin{aligned} \fint_{B_\varrho} |Du - (Du)_{B_\varrho}| \mathrm{d}x &\leq 2\fint_{B_\varrho} |Du - (Dv)_{B_\varrho}| \mathrm{d}x \\ &\leq 2\fint_{B_\varrho} |Du - Dv| \mathrm{d}x + 2\fint_{B_\varrho} |Dv - (Dv)_{B_\varrho}| \mathrm{d}x \\ &\leq 2(\frac{r}{\varrho})^n \fint_{B_r} |Du - Dv| \mathrm{d}x + C(\frac{\varrho}{r})^\beta \fint_{B_r} |Dv - (Dv)_{B_r}| \mathrm{d}x, \end{aligned} \tag{2.10}$$

where β appears in Lemma 2.6, and we use the elementary property given by the following, for every $\gamma \in \mathbb{R}^N$,

$$\fint_{B_\varrho} |Du - (Du)_{B_\varrho}| \mathrm{d}x \leq 2\fint_{B_\varrho} |Du - \gamma| \mathrm{d}x, \quad \text{for every} \quad \gamma \in \mathbb{R}^N.$$

Recall that

$$\fint_{B_r} |Dv - (Dv)_{B_r}| \mathrm{d}x \leq \fint_{B_r} |Du - Dv| \mathrm{d}x + \fint_{B_r} |Du - (Du)_{B_r}| \mathrm{d}x. \tag{2.11}$$

Together with (2.8), (2.10) and (2.11), it follows

$$\begin{aligned}
&\fint_{B_\varrho} |Du-(Du)_{B_\varrho}|\mathrm{d}x\\
&\le C(\frac{\varrho}{r})^\beta \fint_{B_r} |Du-(Du)_{B_r}|\mathrm{d}x + C[(\frac{r}{\varrho})^n + (\frac{\varrho}{r})^\beta] \fint_{B_R} |Du-Dv|\mathrm{d}x\\
&\le C(\frac{\varrho}{r})^\beta \fint_{B_r} |Du-(Du)_{B_r}|\mathrm{d}x+\\
&\quad C\delta_0^{\frac{1}{p-1}}(\frac{r}{\varrho})^n\Big[\fint_{B_{2r}}|Du|\mathrm{d}x + \Big(\fint_{B_{2r}}[|F|^{p-2}F-(|F|^{p-1})_{B_{2r}}]^{\frac{p}{p-1}}\mathrm{d}x\Big)^{\frac{1}{p}}\Big].
\end{aligned}$$

Therefore, (2.9) holds as desired. □

3. Proof of Theorem 1.2

In this section, We are now ready to prove the main theorem of this paper by preliminary lemmas obtained in Section 2.

Proof of Theorem 1.2 Let $0<\varepsilon<1$ which is to be chosen. According to Lemma 2.8, we have

$$\begin{aligned}
&\fint_{B_{\varepsilon r}} |Du-(Du)_{B_{\varepsilon r}}|\mathrm{d}x\\
&\le C\varepsilon^\beta \fint_{B_r}|Du-(Du)_{B_r}|\mathrm{d}x + C\delta_0^{\frac{1}{p-1}}\varepsilon^{-n}\fint_{B_{2r}}|Du|\mathrm{d}x+\\
&\quad\Big(\fint_{B_{2r}}[|F|^{p-2}F-(|F|^{p-1})_{B_{2r}}]^{\frac{p}{p-1}}\mathrm{d}x\Big)^{\frac{1}{p}}\\
&\le (C\varepsilon^\beta + C\delta_0^{\frac{1}{p-1}}\varepsilon^{-n})\fint_{B_r}|Du-(Du)_{B_r}|\mathrm{d}x+\\
&\quad C\delta_0^{\frac{1}{p-1}}\varepsilon^{-n}\Big(\fint_{B_{2r}}[|F|^{p-2}F-(|F|^{p-1})_{B_{2r}}]^{\frac{p}{p-1}}\mathrm{d}x\Big)^{\frac{1}{p}}.
\end{aligned}\tag{3.1}$$

Choose ε such that $C\varepsilon^\beta<1/4$ and then let δ_0 be small enough so that $C\delta_0^{\frac{1}{p-1}}\varepsilon^{-n}<1/4$. With this choice of ε and δ_0, we take the supremum over $r>0$ in (3.1) to find (1.10) holds. □

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