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Density-Dependent Magnetohydrodynamic Equations with Velocity and Magnetic Fields in Besov Spaces of Negative Order

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Abstract In this paper, we consider the density-dependent magnetohydrodynamic equations with vacuum, and provide a regularity criterion involving the velocity and magnetic fields in Besov space of negative order, which improves [Jishan FAN, Fucai LI, G. NAKAMURA, Zhong TAN, Regularity criteria for the three-dimensional magnetohydrodynamic equations. J. Differential Equations, 2014, **256**(8): 2858–2875] in some sense. The method is to establish a new bilinear estimate.

Keywords density-dependent MHD equations; regularity criterion; Besov spaces

MR(2010) Subject Classification 35Q35; 76D03; 76N10

1. Introduction

This paper studies the following density-dependent magnetohydrodynamic (MHD) equations

$$\begin{aligned}
&\partial_t\rho+\operatorname{div}(\rho\boldsymbol{u})=0,\\
&\partial_t(\rho\boldsymbol{u})+\operatorname{div}(\rho\boldsymbol{u}\otimes\boldsymbol{u})-\triangle\boldsymbol{u}+\nabla\big(\pi+\frac{1}{2}|\boldsymbol{b}|^2\big)=(\boldsymbol{b}\cdot\nabla)\boldsymbol{b},\\
&\partial_t\boldsymbol{b}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}-(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}=\triangle\boldsymbol{b},\\
&\operatorname{div}\boldsymbol{u}=\operatorname{div}\boldsymbol{b}=0,
\end{aligned}\tag{1.1}$$

in $\big\{(t,x);\ t\in(0,\infty),x\in\mathbb{R}^3\big\}$ with prescribed initial data $(\rho,\boldsymbol{u},\boldsymbol{b})|_{t=0}=(\rho_0,\boldsymbol{u}_0,\boldsymbol{b}_0)$. Here, ρ is the density of the fluid, $\boldsymbol{u}=(u_1,u_2,u_3)$ is the velocity field of the charged fluid, $\boldsymbol{b}=(b_1,b_2,b_3)$ is the magnetic field induced by the motion of the charged fluid, and π is the pressure of the fluid.

The system (1.1) has attracted many authors' attention [1–8]. In case the initial density has a positive lower bound, the existence of a weak solution with finite energy in the whole space $\mathbb{R}^3$ and in the torus were established in [6] and [4], respectively; while the local existence of a unique strong solution and small date global existence were obtained in [1]. However, whether or not this local unique strong solution can exist globally is an outstanding open problem. In

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[8], a regularity criterion

$$\boldsymbol{u} \in L^{\frac{2}{1-r}}(0,T;\dot{\mathbb{X}}_r(\mathbb{R}^3)), \quad 0<r<1 \tag{1.2}$$

was established, that is, if (1.2) holds, then this strong solution can be extended smoothly beyond T. Here, $\dot{\mathbb{X}}_r(\mathbb{R}^3) = M(\dot{B}^r_{2,1}(\mathbb{R}^3), L^2(\mathbb{R}^3))$ is the multiplier space, whose elements f defines a bounded linear mapping of $\dot{B}^r_{2,1}(\mathbb{R}^3)$ (the Besov space, see Section 2 for details) into $L^2(\mathbb{R}^3)$ by pointwise multiplication, and thus the norm is given by the operator norm,

$$\|f\|_{\dot{X}_r} = \sup_{\|g\|_{\dot{B}^r_{2,1}}\leq 1} \|fg\|_{L^2}.$$

On the other hand, when the initial density contains vacuum, the local existence of a strong unique solution was established in [7] and [3]. Precisely, they show that if the initial data $\rho_0, \boldsymbol{u}_0$ and $\boldsymbol{b}_0$ satisfy

$$\begin{aligned} &0\leq \rho_0 \leq M<\infty, \quad \nabla\rho_0 \in L^2\cap L^q(\mathbb{R}^3), \quad 3<q\leq 6;\\ &\boldsymbol{u}_0,\ \boldsymbol{b}_0 \in H^2(\mathbb{R}^3), \quad \mathrm{div}\boldsymbol{u}_0 = \mathrm{div}\boldsymbol{b}_0 = 0, \end{aligned} \tag{1.3}$$

and the following compatibility condition

$$-\triangle \boldsymbol{u}_0 + \nabla\big(\pi_0 + \frac{1}{2}|\boldsymbol{b}_0|^2\big) - (\boldsymbol{b}_0\cdot\nabla)\boldsymbol{b}_0 = \sqrt{\rho_0}\boldsymbol{g}, \text{ for some } \boldsymbol{g}\in L^2(\mathbb{R}^3), \tag{1.4}$$

then there exists a positive $T^*\in(0,\infty]$ and a unique strong solution $\rho, \boldsymbol{u}, \boldsymbol{b}$ to the system (1.1) verifying the following properties

$$\begin{aligned} &0\leq\rho\leq M, \quad \nabla\rho,\ \partial_t\rho \in C([0,T^*];L^2\cap L^q(\mathbb{R}^3));\\ &\boldsymbol{u},\ \boldsymbol{b}\in C([0,T^*];H^2(\mathbb{R}^3))\cap L^2(0,T^*;W^{2,6}(\mathbb{R}^3));\\ &\sqrt{\rho}\partial_t\boldsymbol{u}\in L^\infty(0,T^*;L^2(\mathbb{R}^3)), \quad \partial_t\boldsymbol{u}\in L^2(0,T^*;H^1(\mathbb{R}^3));\\ &\partial_t\boldsymbol{b}\in L^\infty(0,T^*;L^2(\mathbb{R}^3))\cap L^2(0,T^*;H^1(\mathbb{R}^3)). \end{aligned} \tag{1.5}$$

In [5, Theorem 1.1], the regularity criterion (1.2) was extended to the system (1.1) with vacuum. And the motivation to the present paper is to improve the result in [6] from the multiplier spaces $\dot{\mathbb{X}}(\mathbb{R}^3)$ to be in the Besov spaces $\dot{B}^{-r}_{\infty,\infty}(\mathbb{R}^3)$ of negative order. Concisely, we obtain

Theorem 1.1 *Assume the initial data ρ_0, $\boldsymbol{u}_0$, $\boldsymbol{b}_0$ satisfy (1.3) and the compatibility condition (1.4). Let ρ, $\boldsymbol{u}$, $\boldsymbol{b}$ be the corresponding strong solution to the system (1.1) with the properties stated in (1.5). If*

$$\boldsymbol{u},\ \boldsymbol{b}\in L^{\frac{2}{1-r}}(0,T;\dot{B}^{-r}_{\infty,\infty}(\mathbb{R}^3)), \tag{1.6}$$

then the solution can be extended smoothly beyond T.

Remark 1.2 Observe that [9, Eq.(1.9)]

$$\dot{\mathbb{X}}(\mathbb{R}^3) = M(\dot{B}^r_{2,1}(\mathbb{R}^{)}, L^2(\mathbb{R}^3))\subset \dot{B}^{-r}_{\infty,\infty}(\mathbb{R}^3)),$$

we indeed improve Theorem 1.1 of [5] in some sense.

Remark 1.3 For the incompressible MHD system, Chen-Miao-Zhang [10] already established the regularity criterion involving the velocity field only. To see this, we only need to observe the following fact from [11]:

$$s < 0,\ p, q \geq 1 \Rightarrow \dot{B}^s_{p,q} \subset B^s_{p,q}.$$

Before proving Theorem 1.1 in Section 3, we shall first introduce the definition of Besov spaces and recall a bilinear estimates in Section 2. In the rest of the paper, we shall denote by C a generic constant which may change from line to line. For simplicity of presentation, we shall also omit the spatial domain $\mathbb{R}^3$ in the integrals and in the norm of a function, that is,

$$\int f\,\mathrm{d}x = \int_{\mathbb{R}^3} f\,\mathrm{d}x, \quad \|f\|_{L^2} = \|f\|_{L^2(\mathbb{R}^3)},$$

and etc.

2. Preliminaries

We first introduce the Littlewood-Paley decomposition. Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions. For $f \in \mathcal{S}(\mathbb{R}^3)$, its Fourier transform $\mathcal{F}f = \hat{f}$ is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} f(x) e^{-ix\cdot\xi}\,\mathrm{d}x.$$

Let us choose a non-negative radial function $\varphi \in \mathcal{S}(\mathbb{R}^3)$ such that

$$0 \leq \hat{\varphi}(\xi) \leq 1, \quad \hat{\varphi}(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq 1, \\ 0, & \text{if } |\xi| \geq 2, \end{cases}$$

and let

$$\psi(x) = \varphi(x) - 2^{-3}\varphi(x/2),\ \varphi_j(x) = 2^{3j}\varphi(2^j x),\ \psi_j(x) = 2^{3j}\psi(2^j x), \quad j \in \mathbb{Z}.$$

For $j \in \mathbb{Z}$, the Littlewood-Paley projection operators S_j and $\triangle_j$ are, respectively, defined by

$$S_j f = \varphi_j * f, \quad \triangle_j f = \psi_j * f.$$

Observe that $\triangle_j = S_j - S_{j-1}$. Also, it is easy to check that if $f \in L^2(\mathbb{R}^3)$, then

$$S_j f \to 0,\ \text{as } j \to -\infty; \quad S_j f \to f,\ \text{as } j \to \infty,$$

in the L^2 sense. By telescoping the series, we have the following Littlewood-Paley decomposition

$$f = \sum_{j=-\infty}^{\infty} \triangle_j f, \tag{2.1}$$

for all $f \in L^2(\mathbb{R}^3)$, where the summation is in the L^2 sense. Notice that

$$\triangle_j f = \sum_{l=j-2}^{j+2} \triangle_l \triangle_j f = \sum_{l=j-2}^{j+2} \psi_l * \psi_j * f,$$

we may use Young inequality to deduce that

$$\|\triangle_j f\|_{L^q} \leq C 2^{3j\left(\frac{1}{p}-\frac{1}{q}\right)} \|\triangle_j f\|_{L^p} \tag{2.2}$$

for $1 \leq p \leq q \leq \infty$, with C being a constant independent of f and j.

Let $s \in \mathbb{R}$; $p, q \in [1, \infty]$. The homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^3)$ and the homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(\mathbb{R}^3)$ are defined by the full dyadic decomposition as

$$\dot{B}^s_{p,q} = \left\{ f \in \mathcal{Z}'(\mathbb{R}^3);\ \|f\|_{\dot{B}^s_{p,q}} = \|\{2^{js}\|\triangle_j f\|_{L^p}\}_{j=-\infty}^{\infty}\|_{\ell^q} < \infty \right\},$$

$$\dot{F}^s_{p,q} = \left\{ f \in \mathcal{Z}'(\mathbb{R}^3);\ \|f\|_{\dot{F}^s_{p,q}} = \|\{2^{js}\|\triangle_j f\|_{\ell^q}\}_{j=-\infty}^{\infty}\|_{L^p} < \infty \right\},$$

where $\mathcal{Z}'(\mathbb{R}^3)$ is the dual space of

$$\mathcal{Z}(\mathbb{R}^3) = \left\{ f \in \mathcal{S}(\mathbb{R}^3);\ D^\alpha \hat{f}(0) = 0,\ \forall\, \alpha \in \mathbb{N}^3 \right\},$$

and for series $\{a_k\}$, we denote

$$\|\{a_k\}\|_{\ell^q} = \begin{cases} \left(\sum_k |a_k|^q\right)^{\frac{1}{q}}, & 1 \le q < \infty, \\ \sup_k |a_k|, & q = \infty. \end{cases}$$

It is well-known that (see [12] for example) for all $s \in \mathbb{R}$,

$$\dot{H}^s(\mathbb{R}^3) = \dot{B}^s_{2,2}(\mathbb{R}^3) = \dot{F}^s_{2,2}(\mathbb{R}^3), \quad \dot{B}^s_{\infty,\infty}(\mathbb{R}^3) = \dot{F}^s_{\infty,\infty}(\mathbb{R}^3), \tag{2.3}$$

and the gradient operator ∇ maps $\dot{B}^s_{p,q}(\mathbb{R}^3)$ to $\dot{B}^{s-1}_{p,q}$; moreover,

$$C_1 \|f\|_{\dot{B}^s_{p,q}} \le \|\nabla f\|_{\dot{B}^{s-1}_{p,q}} \le C_2 \|f\|_{\dot{B}^s_{p,q}} \tag{2.4}$$

for some positive constants C_1, C_2.

Also, Kozono-Shimada [13] proved the following bilinear estimates

$$\|f \cdot g\|_{\dot{F}^s_{p,q}} \le C\big(\|f\|_{\dot{F}^{s+\alpha}_{p_1,q}} \|g\|_{\dot{F}^{-\alpha}_{p_2,\infty}} + \|f\|_{\dot{F}^{-\beta}_{r_1,\infty}} \|g\|_{\dot{F}^{s+\beta}_{r_2,q}} \big), \tag{2.5}$$

where

$$s > 0,\ \alpha > 0,\ \beta > 0,\ \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}.$$

With (2.5), the following regularity criterion for the 3D incompressible Navier-Stokes equations

$$\boldsymbol{u} \in L^{\frac{2}{1-r}}(0, T; \dot{B}^{-r}_{\infty,\infty}), \quad 0 < r < 1$$

was proved in [13]. The main obstacle in utilizing (11) to system (1.1) is the strong coupling of the velocity field and the magnetic field, and this leads us to derive a new bilinear estimate, which corresponds to (2.5) with $s = 0$. Before stating the precise form, let us recall a refined Sobolev embedding theorem [14, Theorem 2.42]

$$\|f\|_{L^p} \le C \|f\|^{1-\frac{2}{p}}_{\dot{B}^{-r}_{\infty,\infty}} \|f\|^{\frac{2}{p}}_{\dot{H}^\beta}, \tag{2.6}$$

with $r > 0$, $\beta = r(\frac{p}{2} - 1)$, $2 < p < \infty$.

Now, our new bilinear estimate is as follows.

Lemma 2.1 *Let $0 < r \le 1$, $f \in \dot{B}^{-r}_{\infty,\infty} \cap \dot{H}^2$, $g \in \dot{B}^{-r}_{\infty,\infty} \cap \dot{H}^1 \cap \dot{H}^2$. Then there exists a constant $C = C(r)$ such that*

$$\|f \nabla g\|_{L^2} \le C \|(f, g)\|_{\dot{B}^{-r}_{\infty,\infty}} \|\nabla g\|^{1-r}_{L^2} \left\|(\nabla^2 f, \nabla^2 g)\right\|^r_{L^2}. \tag{2.7}$$

Proof By Hölder inequality, (2.6), and interpolation inequality,

$$\|f \nabla g\|_{L^2} \le \|f\|_{L^{2+\frac{4}{r}}} \|\nabla g\|_{L^{r+2}} \le C \|f\|^{\frac{2}{r+2}}_{\dot{B}^{-r}_{\infty,\infty}} \|f\|^{\frac{r}{r+2}}_{\dot{H}^2} \cdot \|\nabla g\|^{\frac{r}{r+2}}_{\dot{B}^{-1-r}_{\infty,\infty}} \|\nabla g\|^{\frac{2}{r+2}}_{\dot{H}^{\frac{r(r+1)}{2}}}$$

$$\leq C\left\|f\right\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{r+2}}\left\|\nabla^2 f\right\|_{L^2}^{\frac{r}{r+2}}\left\|g\right\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{r}{r+2}}\left\|\nabla g\right\|_{L^2}^{1-r}\left\|\nabla^2 g\right\|_{L^2}^{\frac{r(r+1)}{r+2}}. \quad \square$$

3. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1, which relies on establishing the a priori estimates (1.5) under the condition (1.6).

First, invoking the divergence-free condition $(1.1)_3$, we may rewrite $(1.1)_1$ as

$$\rho_t + (\boldsymbol{u}\cdot\nabla)\rho = 0.$$

The maximum principle then implies that

$$0 \leq \rho \leq M < \infty. \tag{3.1}$$

Next, taking the inner product of $(1.1)_2$ with $\boldsymbol{u}$, $(1.1)_3$ with $\boldsymbol{b}$ in $L^2(\mathbb{R}^3)$ respectively, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\boldsymbol{u}|^2\,\mathrm{d}x + \int |\nabla\boldsymbol{u}|^2\,\mathrm{d}x = \int [(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}]\cdot\boldsymbol{u}\,\mathrm{d}x, \tag{3.2}$$

as well as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\boldsymbol{b}|^2\,\mathrm{d}x + \int |\nabla\boldsymbol{b}|^2\,\mathrm{d}x = \int [(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}]\cdot\boldsymbol{b}\,\mathrm{d}x. \tag{3.3}$$

Summing up (3.2) and (3.3), and noticing that

$$\begin{aligned}&\int [(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}]\cdot\boldsymbol{u}\,\mathrm{d}x + \int [(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}]\cdot\boldsymbol{b}\,\mathrm{d}x = \int (\boldsymbol{b}\cdot\nabla)(\boldsymbol{b}\cdot\boldsymbol{u})\,\mathrm{d}x\\ &= \int (\nabla\cdot\boldsymbol{b})\cdot(\boldsymbol{b}\cdot\boldsymbol{u})\,\mathrm{d}x = 0,\end{aligned}$$

we get

$$\frac{1}{2}\int \rho|\boldsymbol{u}|^2 + |\boldsymbol{b}|^2\,\mathrm{d}x + \int |\nabla\boldsymbol{u}|^2 + |\nabla\boldsymbol{b}|^2\,\mathrm{d}x = 0.$$

Integrating in time over $(0,T)$ then yields

$$\sup_{0\leq t\leq T}\frac{1}{2}\int \rho|\boldsymbol{u}|^2 + |\boldsymbol{b}|^2\,\mathrm{d}x + \int_0^T\int |\nabla\boldsymbol{u}|^2 + |\nabla\boldsymbol{b}|^2\,\mathrm{d}x\,\mathrm{d}t \leq C. \tag{3.4}$$

Taking the inner product of $(1.1)_2$ with $\partial_t\boldsymbol{u}$ in $L^2(\mathbb{R}^3)$, and integrating by parts, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla\boldsymbol{u}|^2\,\mathrm{d}x + \int \rho|\partial_t\boldsymbol{u}|^2\,\mathrm{d}x\\
&= \int [(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}]\cdot\partial_t\boldsymbol{u}\,\mathrm{d}x - \int [(\rho\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\partial_t\boldsymbol{u}\,\mathrm{d}x\\
&= \sum_{i,j=1}^3\int b_j\partial_j b_i\partial_t u_i\,\mathrm{d}x - \int [(\rho\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\partial_t\boldsymbol{u}\,\mathrm{d}x\\
&= -\sum_{i,j=1}^3\int b_j b_i\partial_t\partial_j u_i\,\mathrm{d}x - \int [(\rho\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\partial_t\boldsymbol{u}\,\mathrm{d}x\\
&= -\sum_{i,j=1}^3\frac{\mathrm{d}}{\mathrm{d}t}\int b_j b_i\partial_j u_i\,\mathrm{d}x + \sum_{i,j=1}^3\int \partial_t b_j b_i\partial_j u_i\,\mathrm{d}x + \sum_{i,j=1}^3\int b_j\partial_t b_i\partial_j u_i\,\mathrm{d}x -\end{aligned}$$

$$\int [(\rho \boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\partial_t\boldsymbol{u}\,\mathrm{d}x.$$

Thus, by Hölder inequality,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla\boldsymbol{u}|^2\,\mathrm{d}x+\int \rho|\partial_t\boldsymbol{u}|^2\,\mathrm{d}x\\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int(\boldsymbol{b}\otimes\boldsymbol{b}):\nabla\boldsymbol{u}\,\mathrm{d}x+2\left\|\partial_t\boldsymbol{b}\right\|_{L^2}\left\||\boldsymbol{b}|\cdot|\nabla\boldsymbol{u}|\right\|_{L^2}+C\left\|\sqrt{\rho}\partial_t\boldsymbol{u}\right\|_{L^2}\left\||\boldsymbol{u}|\cdot|\nabla\boldsymbol{u}|\right\|_{L^2}\\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int(\boldsymbol{b}\otimes\boldsymbol{b}):\nabla\boldsymbol{u}\,\mathrm{d}x+\frac{1}{4}\left\|\partial_t\boldsymbol{b}\right\|_{L^2}^2+\frac{1}{2}\left\|\sqrt{\rho}\partial_t\boldsymbol{u}\right\|_{L^2}^2+C\left\||(\boldsymbol{u},\boldsymbol{b})|\cdot|\nabla(\boldsymbol{u},\boldsymbol{b})|\right\|_{L^2}^2. \quad (3.5)
\end{aligned}$$

Taking the inner product of $(1.1)_3$ with $\partial_t\boldsymbol{b}$ in $L^2(\mathbb{R}^3)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla\boldsymbol{b}|^2\,\mathrm{d}x+\int|\partial_t\boldsymbol{b}|^2\,\mathrm{d}x=-\int[(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}-(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}]\cdot\partial_t\boldsymbol{b}\,\mathrm{d}x\\
&\le\left\||(\boldsymbol{u},\boldsymbol{b})|\cdot|\nabla(\boldsymbol{u},\boldsymbol{b})|\right\|_{L^2}\left\|\partial_t\boldsymbol{b}\right\|_{L^2}\le\frac{1}{4}\left\|\partial_t\boldsymbol{b}\right\|_{L^2}^2+C\left\||(\boldsymbol{u},\boldsymbol{b})|\cdot|\nabla(\boldsymbol{u},\boldsymbol{b})|\right\|_{L^2}^2. \quad (3.6)
\end{aligned}$$

Gathering (3.5) and (3.6) together, and utilizing Lemma 2.1, (2.4) and interpolation inequality, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla(\boldsymbol{u},\boldsymbol{b})|^2\,\mathrm{d}x+\int\rho|\partial_t\boldsymbol{u}|^2+|\partial_t\boldsymbol{b}|^2\,\mathrm{d}x\\
&\le C\left\||(\boldsymbol{u},\boldsymbol{b})|\cdot|\nabla(\boldsymbol{u},\boldsymbol{b})|\right\|_{L^2}^2\\
&\le C\left\|(\boldsymbol{u},\boldsymbol{b})\right\|_{\dot{B}^{-r}_{\infty,\infty}}^2\left\|\nabla(\boldsymbol{u},\boldsymbol{b})\right\|_{L^2}^{2(1-r)}\left\|\triangle(\boldsymbol{u},\boldsymbol{b})\right\|_{L^2}^{2r}\\
&\le\varepsilon\left\|\triangle(\boldsymbol{u},\boldsymbol{b})\right\|_{L^2}^2+C\left\|(\boldsymbol{u},\boldsymbol{b})\right\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{1-r}}\left\|\nabla(\boldsymbol{u},\boldsymbol{b})\right\|_{L^2}^2. \quad (3.7)
\end{aligned}$$

To close the estimates, we need to get the bounds of $\left\|\triangle\boldsymbol{u}\right\|_{L^2}$ and $\left\|\triangle\boldsymbol{b}\right\|_{L^2}$. By $(1.1)_1$, we may rewrite $(1.1)_2$ as

$$-\triangle\boldsymbol{u}+\nabla(\pi+\frac{1}{2}|\boldsymbol{b}|^2)=(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}-(\rho\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\rho\partial_t\boldsymbol{u}, \quad (3.8)$$

and invoke the H^2-theory of the Stokes system [15] to deduce

$$\left\|\triangle\boldsymbol{u}\right\|_{L^2}\le C\left\||\boldsymbol{b}|\cdot|\nabla\boldsymbol{b}|\right\|_{L^2}+C\left\||\boldsymbol{u}|\cdot|\nabla\boldsymbol{u}|\right\|_{L^2}+C\left\|\sqrt{\rho}\partial_t\boldsymbol{u}\right\|_{L^2}. \quad (3.9)$$

On the other hand, by $(1.1)_3$,

$$\left\|\triangle\boldsymbol{b}\right\|_{L^2}\le\left\|\partial_t\boldsymbol{b}\right\|_{L^2}+\left\||\boldsymbol{u}|\cdot|\nabla\boldsymbol{b}|\right\|_{L^2}+\left\||\boldsymbol{b}|\cdot|\nabla\boldsymbol{u}|\right\|_{L^2}. \quad (3.10)$$

Summing up (3.9) and (3.10), and estimating as in (3.7), we find

$$\begin{aligned}
\left\|\triangle(\boldsymbol{u},\boldsymbol{b})\right\|_{L^2}&\le C\left\|\sqrt{\rho}\partial_t\boldsymbol{u}\right\|_{L^2}+C\left\|\partial_t\boldsymbol{b}\right\|_{L^2}+C\left\||(\boldsymbol{u},\boldsymbol{b})|\cdot|\nabla(\boldsymbol{u},\boldsymbol{b})|\right\|_{L^2}\\
&\le C\left\|\sqrt{\rho}\partial_t\boldsymbol{u}\right\|_{L^2}+C\left\|\partial_t\boldsymbol{b}\right\|_{L^2}+\frac{1}{2}\left\|\triangle(\boldsymbol{u},\boldsymbol{b})\right\|_{L^2}+C\left\|(\boldsymbol{u},\boldsymbol{b})\right\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{1}{1-r}}\left\|\nabla(\boldsymbol{u},\boldsymbol{b})\right\|_{L^2}.
\end{aligned}$$

Consequently,

$$\left\|\triangle(u,b)\right\|_{L^2}\le C\left\|\sqrt{\rho}\partial_t\boldsymbol{u}\right\|_{L^2}+C\left\|\partial_t\boldsymbol{b}\right\|_{L^2}+C\left\|(\boldsymbol{u},\boldsymbol{b})\right\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{1}{1-r}}\left\|\nabla(\boldsymbol{u},\boldsymbol{b})\right\|_{L^2}. \quad (3.11)$$

Putting (3.11) into (3.7), and taking ε sufficiently small, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla(\boldsymbol{u},\boldsymbol{b})|^2\,\mathrm{d}x+\int\rho|\partial_t\boldsymbol{u}|^2+|\partial_t\boldsymbol{b}|^2\,\mathrm{d}x\le C\left\|(\boldsymbol{u},\boldsymbol{b})\right\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{1-r}}\left\|\nabla(\boldsymbol{u},\boldsymbol{b})\right\|_{L^2}^2.$$

Applying Gronwall inequality then yields

$$\begin{aligned} &\|\nabla \boldsymbol{u}\|_{L^\infty(0,T;L^2)} \leq C; \quad \left\|\sqrt{\rho}\partial_t \boldsymbol{u}\right\|_{L^2(0,T;L^2)} \leq C; \\ &\|\boldsymbol{b}\|_{L^\infty(0,T;H^1)} \leq C; \quad\quad \|\partial_t \boldsymbol{b}\|_{L^2(0,T;L^2)} \leq C. \end{aligned} \tag{3.12}$$

With these uniform bounds at hand, it infers from (24) that

$$\|\nabla \boldsymbol{u}\|_{L^2(0,T;H^1)} \leq C, \quad \|\boldsymbol{b}\|_{L^2(0,T:H^2)} \leq C. \tag{3.13}$$

Up to now, we may just follow [5] to complete the proof of Theorem 1.1. □

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