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Dunkl Multiplier Operators on a Class of Reproducing Kernel Hilbert Spaces

Fethi SOLTANI
Department of Mathematics, Faculty of Science, Jazan University, P. O. Box 277,
Jazan 45142, *Saudi Arabia*

Abstract We study some class of Dunkl multiplier operators; and we establish for them the Heisenberg-Pauli-Weyl uncertainty principle and the Donoho-Stark's uncertainty principle. For these operators we give also an application of the theory of reproducing kernels to the Tikhonov regularization on the Sobolev-Dunkl spaces.

Keywords Sobolev-Dunkl spaces; Dunkl multiplier operators; Heisenberg-Pauli-Weyl uncertainty principle; Donoho-Stark uncertainty principle; Tikhonov regularization; extremal functions

MR(2010) Subject Classification 42B10; 42B15; 46E35

1. Introduction

In this paper, we consider $\mathbb{R}^d$ with the Euclidean inner product $\langle .,. \rangle$ and norm $|y| := \sqrt{\langle y, y\rangle}$. For $\alpha \in \mathbb{R}^d\backslash\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α:

$$\sigma_\alpha x := x - \frac{2\langle \alpha, x\rangle}{|\alpha|^2}\alpha.$$

A finite set $\Re \subset \mathbb{R}^d\backslash\{0\}$ is called a root system, if $\Re \cap \mathbb{R}.\alpha = \{-\alpha, \alpha\}$ and $\sigma_\alpha \Re = \Re$ for all $\alpha \in \Re$. We assume that it is normalized by $|\alpha|^2 = 2$ for all $\alpha \in \Re$. For a root system $\Re$, the reflections σ_α, $\alpha \in \Re$, generate a finite group G. The Coxeter group G is a subgroup of the orthogonal group $O(d)$. All reflections in G correspond to suitable pairs of roots. For a given $\beta \in \mathbb{R}^d\backslash \bigcup_{\alpha\in\Re} H_\alpha$, we fix the positive subsystem $\Re_+ := \{\alpha \in \Re : \langle \alpha, \beta\rangle > 0\}$. Then for each $\alpha \in \Re$ either $\alpha \in \Re_+$ or $-\alpha \in \Re_+$.

Let $k, \ell : \Re \to \mathbb{C}$ be two multiplicity functions on $\Re$ (functions which are constants on the orbits under the action of G). As an abbreviation, we introduce the index $\gamma_k := \sum_{\alpha\in\Re_+} k(\alpha)$ and $\gamma_\ell := \sum_{\alpha\in\Re_+} \ell(\alpha)$.

Throughout this paper, we will assume that $k(\alpha), \ell(\alpha) \geq 0$ for all $\alpha \in \Re$, and $\gamma_\ell \geq \gamma_k$. Moreover, let w_k denote the weight function $w_k(x) := \prod_{\alpha\in\Re_+} |\langle \alpha, x\rangle|^{2k(\alpha)}$, for all $x \in \mathbb{R}^d$, which is G-invariant and homogeneous of degree $2\gamma_k$.

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E-mail address: fethisoltani10@yahoo.com

Let c_k be the Mehta-type constant given by

$$c_k := \Big(\int_{\mathbb{R}^d} e^{-|x|^2/2} w_k(x)\mathrm{d}x\Big)^{-1}. \tag{1.1}$$

We denote by μ_k the measure on $\mathbb{R}^d$ given by $\mathrm{d}\mu_k(x) := c_k w_k(x)\mathrm{d}x$; and by $L^p(\mu_k)$, $1 \le p \le \infty$, the space of measurable functions f on $\mathbb{R}^d$, such that

$$\|f\|_{L^p(\mu_k)} := \Big(\int_{\mathbb{R}^d} |f(x)|^p \mathrm{d}\mu_k(x)\Big)^{1/p} < \infty, \quad 1 \le p < \infty,$$
$$\|f\|_{L^\infty(\mu_k)} := \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} |f(x)| < \infty.$$

For $f \in L^1(\mu_k)$ the Dunkl transform is defined (see [1]) by

$$\mathcal{F}_k(f)(y) := \int_{\mathbb{R}^d} E_k(-ix, y) f(x)\mathrm{d}\mu_k(x), \quad y \in \mathbb{R}^d,$$

where $E_k(-ix, y)$ denotes the Dunkl kernel (for more details, see the next section).

Many uncertainty principles have already been proved for the Dunkl transform, namely by Rösler [2] and Shimeno [3] who established the Heisenberg-Pauli-Weyl inequality for the Dunkl transform, by showing that for every $f \in L^2(\mu_k)$,

$$\|f\|^2_{L^2(\mu_k)} \le \frac{2}{2\gamma_k + d} \| \,|x|f\|_{L^2(\mu_k)} \| \,|y|\mathcal{F}_k(f)\|_{L^2(\mu_k)}. \tag{1.2}$$

Recently, the author [4,5] proved general forms of the Heisenberg-Pauli-Weyl inequality for the Dunkl transform.

Let $s \in \mathbb{R}$. We consider the Sobolev type space's $H^s_{k\ell}$ consisting of all $f \in \mathcal{S}'(\mathbb{R}^d)$ (the space of tempered distributions) such that $\mathcal{F}_\ell(f)$ is a function and $(1 + |z|^2)^{s/2}\mathcal{F}_\ell(f) \in L^2(\mu_k)$. The space $H^s_{k\ell}$ is a Hilbert space when endowed with the inner product

$$\langle f, g\rangle_{H^s_{k\ell}} := \int_{\mathbb{R}^d} (1 + |z|^2)^s \mathcal{F}_\ell(f)(z)\overline{\mathcal{F}_\ell(g)(z)}\mathrm{d}\mu_k(z).$$

Let m be a function in $L^2(\mu_k)$. The Dunkl multiplier operators $T_{k,\ell,m}$, are defined for $f \in H^s_{k\ell}$ by

$$T_{k,\ell,m} f(a, x) := \mathcal{F}_k^{-1}\big(m(a.)\mathcal{F}_\ell(f)\big)(x), \quad (a, x) \in (0,\infty) \times \mathbb{R}^d.$$

These operators were studied in [6] where the author established some applications (Calderón's reproducing formulas, best approximation formulas, extremal functions ...). In particular, when $k = \ell$ these operators were studied in [7].

For $m \in L^2(\mu_k)$ verifying the admissibility condition $\int_0^\infty |m(ax)|^2 \frac{\mathrm{d}a}{a} = 1$, a.*e.* $x \in \mathbb{R}^d$, then the operators $T_{k,\ell,m}$ satisfy

$$\|T_{k,\ell,m} f\|_{L^2(\Omega_k)} = \|f\|_{H^0_{k\ell}}, \quad f \in H^0_{k\ell},$$

where Ω_k is the measure on $(0,\infty) \times \mathbb{R}^d$ given by $\mathrm{d}\Omega_k(a, x) := \frac{\mathrm{d}a}{a}\mathrm{d}\mu_k(x)$.

For the operators $T_{k,\ell,m}$ we establish a Heisenberg-Pauli-Weyl uncertainty principle. More precisely, we will show for $f \in H^0_{k\ell}$ that

$$\|f\|^2_{H^0_{k\ell}} \le \frac{2}{2\gamma_k + d} \| \,|y|\mathcal{F}_\ell(f)\|_{L^2(\mu_k)} \| \,|x|T_{k,\ell,m} f\|_{L^2(\Omega_k)},$$

provided $m \in L^2(\mu_k)$ satisfying $\int_0^\infty |m(ax)|^2 \frac{da}{a} = 1$, a.e. $x \in \mathbb{R}^d$.

Building on the techniques of Donoho-Stark [8], we show a continuous-time principle for the L^2 theory. Let E be a measurable subset of $\mathbb{R}^d$ and S be a measurable subset of $(0,\infty) \times \mathbb{R}^d$ and let $f \in H^s_{k\ell}$. If f is ε-concentrated on E and $T_{k,\ell,m}f$ is η-concentrated on S (see Section 4 for more details), then

$$\big(\mu_\ell(E)\big)^{1/2}\Big(\int\int_S \frac{d\Omega_k(a,x)}{a^{2(2\gamma_k+d)}}\Big)^{1/2} \geq \frac{(1-\eta-\varepsilon)}{2^{(\gamma_\ell-\gamma_k)/2}\|m\|_{L^1(\mu_k)}}\sqrt{\frac{c_k}{c_\ell}},$$

provided $m \in L^1 \cap L^2(\mu_k)$ satisfying $\int_0^\infty |m(ax)|^2 \frac{da}{a} = 1$, a.e. $x \in \mathbb{R}^d$.

Building on the ideas of [9–12], we give an application of the theory of reproducing kernels to the Tikhonov regularization, which gives the best approximation of the operator $T_{k,\ell,m}$ on the Sobolev-Dunkl spaces $H^s_{k\ell}$. More precisely, for all $\lambda > 0$, $g \in L^2(\Omega_k)$, the infimum

$$\inf_{f \in H^s_{k\ell}} \big\{\lambda \|f\|^2_{H^s_{k\ell}} + \|g - T_{k,\ell,m}f\|^2_{L^2(\Omega_k)}\big\},$$

is attained at one function $f^*_{\lambda,g}$, called the extremal function.

This paper is organized as follows. In Section 2 we define and study the Sobolev-Dunkl type spaces $H^s_{k\ell}$. In Section 3 we define and study the Dunkl multiplier operators $T_{k,\ell,m}$ on the spaces $H^s_{k\ell}$. In Section 4 we establish the Heisenberg-Pauli-Weyl uncertainty principle and the Donoho-Stark's uncertainty principle for the operators $T_{k,\ell,m}$. In the last section we give an application of the theory of reproducing kernels to the Tikhonov regularization for the operators $T_{k,\ell,m}$ on the Sobolev-Dunkl spaces $H^s_{k\ell}$.

2. Sobolev-Dunkl type spaces

The Dunkl operators $\mathcal{D}_j$; $j = 1,\dots,d$, on $\mathbb{R}^d$ associated with the finite reflection group G and multiplicity function k are given, for a function f of class C^1 on $\mathbb{R}^d$, by

$$\mathcal{D}_j f(x) := \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \Re_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle}.$$

For $y \in \mathbb{R}^d$, the initial problem $\mathcal{D}_j u(.,y)(x) = y_j u(x,y)$, $j = 1,\dots,d$, with $u(0,y) = 1$ admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $E_k(x,y)$ and called Dunkl kernel [13,14]. This kernel has a unique analytic extension to $\mathbb{C}^d \times \mathbb{C}^d$ (see [15]). In our case [1,13],

$$|E_k(\pm ix, y)| \leq 1, \quad x, y \in \mathbb{R}^d. \tag{2.1}$$

The Dunkl kernel gives rise to an integral transform, which is called Dunkl transform on $\mathbb{R}^d$, and was introduced by Dunkl in [1], where already many basic properties were established. Dunkl's results were completed and extended later by De Jeu [14]. The Dunkl transform of a function f in $L^1(\mu_k)$, is defined by

$$\mathcal{F}_k(f)(y) := \int_{\mathbb{R}^d} E_k(-ix, y) f(x) d\mu_k(x), \quad y \in \mathbb{R}^d.$$

We notice that $\mathcal{F}_0$ agrees with the Fourier transform $\mathcal{F}$ that is given by

$$\mathcal{F}(f)(y) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\langle x,y\rangle} f(x)\mathrm{d}x, \quad x \in \mathbb{R}^d.$$

Some of the properties of Dunkl transform $\mathcal{F}_k$ are collected below [1,14].

Theorem 2.1 *(i) $L^1 - L^\infty$-boundedness. For all $f \in L^1(\mu_k)$, $\mathcal{F}_k(f) \in L^\infty(\mu_k)$ and*

$$\|\mathcal{F}_k(f)\|_{L^\infty(\mu_k)} \leq \|f\|_{L^1(\mu_k)}.$$

(ii) Inversion theorem. Let $f \in L^1(\mu_k)$, such that $\mathcal{F}_k(f) \in L^1(\mu_k)$. Then

$$f(x) = \mathcal{F}_k(\mathcal{F}_k(f))(-x), \quad \text{a.e.} \quad x \in \mathbb{R}^d.$$

(iii) Plancherel theorem. The Dunkl transform $\mathcal{F}_k$ extends uniquely to an isometric isomorphism of $L^2(\mu_k)$ onto itself. In particular,

$$\|\mathcal{F}_k(f)\|_{L^2(\mu_k)} = \|f\|_{L^2(\mu_k)}.$$

(iv) The Dunkl transform $\mathcal{F}_k$ is a topological isomorphism from $\mathcal{S}(\mathbb{R}^d)$ onto itself, and from $\mathcal{S}'(\mathbb{R}^d)$ onto itself.

Let $s \in \mathbb{R}$. We define the Sobolev-Dunkl type space of order s, that will be denoted $H^s_{k\ell}$, as the set of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $\mathcal{F}_\ell(f)$ is a function and $(1+|z|^2)^{s/2}\mathcal{F}_\ell(f) \in L^2(\mu_k)$. The space $H^s_{k\ell}$ is endowed with the inner product

$$\langle f, g\rangle_{H^s_{k\ell}} := \int_{\mathbb{R}^d} \mathcal{F}_\ell(f)(z)\overline{\mathcal{F}_\ell(g)(z)}\mathrm{d}\mu_{k,s}(z),$$

and the norm

$$\|f\|_{H^s_{k\ell}} := \Big(\int_{\mathbb{R}^d} |\mathcal{F}_\ell(f)(z)|^2 \mathrm{d}\mu_{k,s}(z)\Big)^{1/2},$$

where $\mu_{k,s}$ is the measure on $\mathbb{R}^d$ given by

$$\mathrm{d}\mu_{k,s}(z) := (1+|z|^2)^s \mathrm{d}\mu_k(z).$$

The space $H^s_{k\ell}$ satisfies the following properties.

Lemma 2.2 *Let $s \in \mathbb{R}$. The space $H^s_{k\ell}$ is a Hilbert space.*

Proof Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of $H^s_{k\ell}$. From the definition of the norm $\|.\|_{H^s_{k\ell}}$, it is easy to see that $(\mathcal{F}_\ell(f_n))_{n\in\mathbb{N}}$ is a Cauchy sequence of $L^2(\mu_{k,s})$. Since $L^2(\mu_{k,s})$ is complete, there exists a function $g \in L^2(\mu_{k,s})$ such that

$$\lim_{n\to\infty} \|\mathcal{F}_\ell(f_n) - g\|_{L^2(\mu_{k,s})} = 0. \tag{2.2}$$

Then $g \in \mathcal{S}'(\mathbb{R}^d)$ and from Theorem 2.1 (iv), we obtain $f = (\mathcal{F}_\ell)^{-1}(g) \in \mathcal{S}'(\mathbb{R}^d)$. So, $\mathcal{F}_\ell(f) = g \in L^2(\mu_{k,s})$, which proves that $f \in H^s_{k\ell}$. Furthermore, using the relation (2.2), we obtain

$$\lim_{n\to\infty} \|f_n - f\|_{H^s_{k\ell}} = \lim_{n\to\infty} \|\mathcal{F}_\ell(f_n) - g\|_{L^2(\mu_{k,s})} = 0.$$

Hence, $H^s_{k\ell}$ is complete. □

Lemma 2.3 *Let $s \geq \gamma_\ell - \gamma_k$. The space $H^s_{k\ell}$ is continuously contained in $L^2(\mu_\ell)$ and*

$$\|f\|_{L^2(\mu_\ell)} \leq 2^{(\gamma_\ell-\gamma_k)/2}\sqrt{\frac{c_\ell}{c_k}}\|f\|_{H^s_{k\ell}}.$$

Proof Let $s \geq \gamma_\ell - \gamma_k$ and let $f \in H^s_{k\ell}$. Then

$$\|f\|^2_{L^2(\mu_\ell)} = \frac{c_\ell}{c_k}\int_{\mathbb{R}^d} |\mathcal{F}_\ell(f)(z)|^2 w_{\ell-k}(z)\mathrm{d}\mu_k(z).$$

By using the fact that $w_{\ell-k}(z) \leq 2^{\gamma_\ell-\gamma_k}|z|^{2(\gamma_\ell-\gamma_k)}$, we obtain

$$\|f\|^2_{L^2(\mu_\ell)} \leq 2^{\gamma_\ell-\gamma_k}\frac{c_\ell}{c_k}\int_{\mathbb{R}^d} \frac{|\mathcal{F}_\ell(f)(z)|^2}{(1+|z|^2)^{s-(\gamma_\ell-\gamma_k)}}\mathrm{d}\mu_{k,s}(z) \leq 2^{\gamma_\ell-\gamma_k}\frac{c_\ell}{c_k}\|f\|^2_{H^s_{k\ell}}.$$

This completes the proof. □

Lemma 2.4 *Let $s > 2\gamma_\ell - \gamma_k + d/2$. If $f \in H^s_{k\ell}$, then $\mathcal{F}_\ell(f) \in L^1(\mu_\ell)$ and*

$$\|\mathcal{F}_\ell(f)\|_{L^1(\mu_\ell)} \leq C_{k,\ell}\|f\|_{H^s_{k\ell}},$$

where

$$C_{k,\ell} = \Big(\frac{c_\ell}{c_k}\int_{\mathbb{R}^d} w_{\ell-k}(z)\mathrm{d}\mu_{\ell,-s}(z)\Big)^{1/2}. \tag{2.3}$$

Proof Let $s > 2\gamma_\ell - \gamma_k + d/2$ and let $f \in H^s_{k\ell}$. Then

$$\|\mathcal{F}_\ell(f)\|_{L^1(\mu_\ell)} = \frac{c_\ell}{c_k}\int_{\mathbb{R}^d} |\mathcal{F}_\ell(f)(z)|w_{\ell-k}(z)\mathrm{d}\mu_k(z).$$

Then by Hölder's inequality we obtain

$$\begin{aligned}\|f\|_{L^1(\mu_\ell)} &\leq \frac{c_\ell}{c_k}\Big(\int_{\mathbb{R}^d}\Big(w_{\ell-k}(z)\Big)^2\mathrm{d}\mu_{k,-s}(z)\Big)^{1/2}\|f\|_{H^s_{k\ell}}\\ &\leq \Big(\frac{c_\ell}{c_k}\int_{\mathbb{R}^d} w_{\ell-k}(z)\mathrm{d}\mu_{\ell,-s}(z)\Big)^{1/2}\|f\|_{H^s_{k\ell}}\\ &\leq C_{k,\ell}\|f\|_{H^s_{k\ell}},\end{aligned}$$

which yields the desired result. □

Remark 2.5 Let $s > 2\gamma_\ell - \gamma_k + d/2$. If $f \in H^s_{k\ell}$, then by Lemmas 2.3 and 2.4 the function $\mathcal{F}_\ell(f)$ belongs to $L^1 \cap L^2(\mu_\ell)$, and therefore

$$f(x) = \int_{\mathbb{R}^d} E_\ell(ix,z)\mathcal{F}_\ell(f)(z)\mathrm{d}\mu_\ell(z), \quad \text{a.e. } x \in \mathbb{R}^d.$$

3. Dunkl type multiplier operators

Let m be a function in $L^2(\mu_k)$. The Dunkl multiplier operators $T_{k,\ell,m}$, are defined for $f \in H^s_{k\ell}$ by

$$T_{k,\ell,m}f(a,x) := \mathcal{F}_k^{-1}\big(m(a.)\mathcal{F}_\ell(f)\big)(x), \quad (a,x) \in (0,\infty)\times\mathbb{R}^d. \tag{3.1}$$

The operators $T_{k,\ell,m}$ satisfy the following integral representation.

Lemma 3.1 *If $m \in L^1 \cap L^2(\mu_k)$ and $f \in L^1(\mu_\ell) \cap H^s_{k\ell}$, then*

$$T_{k,\ell,m} f(a,x) = \frac{1}{a^{2\gamma_k+d}} \int_{\mathbb{R}^d} W_{k\ell}(\frac{x}{a}, \frac{y}{a}, m) f(y) \mathrm{d}\mu_\ell(y), \quad (a,x) \in (0,\infty) \times \mathbb{R}^d,$$

where

$$W_{k\ell}(x,y,m) = \int_{\mathbb{R}^d} m(z) E_k(ix,z) E_\ell(-iy,z) \mathrm{d}\mu_k(z).$$

Proof From (3.1) and Theorem 2.1 (ii), we have

$$\begin{aligned} T_{k,\ell,m} f(a,x) &= \int_{\mathbb{R}^d} m(az) \mathcal{F}_\ell(f)(z) E_k(ix,z) \mathrm{d}\mu_k(z) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} m(az) f(y) E_k(ix,z) E_\ell(-iy,z) \mathrm{d}\mu_\ell(y) \mathrm{d}\mu_k(z). \end{aligned}$$

The result follows from Fubini-Tonnelli's theorem. □

We denote by Ω_k the measure on $(0,\infty) \times \mathbb{R}^d$ given by $\mathrm{d}\Omega_k(a,x) := \frac{\mathrm{d}a}{a} \mathrm{d}\mu_k(x)$; and by $L^2(\Omega_k)$, the space of measurable functions F on $(0,\infty) \times \mathbb{R}^d$, such that

$$\|F\|_{L^2(\Omega_k)} := \Big(\int_{\mathbb{R}^d} \int_0^\infty |F(a,x)|^2 \mathrm{d}\Omega_k(a,x) \Big)^{1/2} < \infty.$$

In the following, we give Plancherel formula for the operators $T_{k,\ell,m}$.

Theorem 3.2 *Let m be a function in $L^2(\mu_k)$ satisfying the admissibility condition*

$$\int_0^\infty |m(ax)|^2 \frac{\mathrm{d}a}{a} = 1, \quad a.e. \ x \in \mathbb{R}^d. \tag{3.2}$$

Then, for $f \in H^0_{k\ell}$, we have

$$\|T_{k,\ell,m} f\|_{L^2(\Omega_k)} = \|f\|_{H^0_{k\ell}}. \tag{3.3}$$

Proof From Fubini-Tonnelli's theorem, Theorem 2.1 (iii) and (3.2) we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \int_0^\infty |T_{k,\ell,m} f(a,x)|^2 \mathrm{d}\Omega_k(a,x) &= \int_0^\infty \int_{\mathbb{R}^d} |m(ay)|^2 |\mathcal{F}_\ell(f)(y)|^2 \mathrm{d}\mu_k(y) \frac{\mathrm{d}a}{a} \\ &= \int_{\mathbb{R}^d} |\mathcal{F}_\ell(f)(y)|^2 \Big(\int_0^\infty |m(ay)|^2 \frac{\mathrm{d}a}{a} \Big) \mathrm{d}\mu_k(y) \\ &= \int_{\mathbb{R}^d} |\mathcal{F}_\ell(f)(y)|^2 \mathrm{d}\mu_k(y) = \|f\|^2_{H^0_{k\ell}}. \end{aligned}$$

This gives the result. □

As applications, we give the following examples.

Example 3.3 Let the function m_t, $t > 0$, be defined by

$$m_t(x) := -\sqrt{8}\, t |x|^2 e^{-t|x|^2}, \quad x \in \mathbb{R}^d.$$

Then

(a) m_t belongs to $L^1 \cap L^2(\mu_k)$, and by (1.1), we have

$$\|m_t\|_{L^1(\mu_k)} = \sqrt{8}\, t \int_{\mathbb{R}^d} |x|^2 e^{-t|x|^2} \mathrm{d}\mu_k(x) = -\sqrt{8}\, t \frac{\partial}{\partial t} \Big(\int_{\mathbb{R}^d} e^{-t|x|^2} \mathrm{d}\mu_k(x) \Big) = \frac{\sqrt{2}(2\gamma_k + d)}{(\sqrt{2t})^{2\gamma_k+d}},$$

and

$$\|m_t\|^2_{L^2(\mu_k)} = 8t^2 \int_{\mathbb{R}^d} |x|^4 e^{-2t|x|^2} \mathrm{d}\mu_k(x) = 2t^2 \frac{\partial^2}{\partial t^2}\Big(\int_{\mathbb{R}^d} e^{-2t|x|^2}\mathrm{d}\mu_k(x)\Big)$$
$$= \frac{(2\gamma_k + d)(\gamma_k + d/2 + 1)}{(2\sqrt{t})^{2\gamma_k+d}}.$$

(b) m_t satisfies the admissibility condition (3.2), that is

$$\int_0^\infty |m_t(ax)|^2 \frac{\mathrm{d}a}{a} = 8t^2|x|^4 \int_0^\infty a^3 e^{-2t|x|^2a^2}\mathrm{d}a = 1.$$

Then the associated operators T_{k,ℓ,m_t} satisfy the formula (3.3).

We use Lemma 3.1, then for $f \in L^1(\mu_\ell) \cap H^s_{k\ell}$, we have

$$T_{k,\ell,m_t} f(a,x) = \frac{\sqrt{8}\,t}{a^{2\gamma_k+d}} \int_{\mathbb{R}^d} \frac{\partial}{\partial t}\big[h_{k\ell}(\frac{x}{a}, \frac{y}{a}, t)\big] f(y)\mathrm{d}\mu_\ell(y), \quad x \in \mathbb{R}^d, \tag{3.4}$$

where

$$h_{k\ell}(x,y,t) = \int_{\mathbb{R}^d} e^{-t|z|^2} E_k(ix,z)E_\ell(-iy,z)\mathrm{d}\mu_k(z).$$

If $k = \ell$, then h_{kk} is the Dunkl-type heat kernel [16,17] and this kernel is given by

$$h_{kk}(x,y,t) = \frac{1}{(2t)^{\gamma_k+d/2}} e^{-(|x|^2+|y|^2)/4t} E_k(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}).$$

Example 3.4 Let the function m_t, $t > 0$, be defined by

$$m_t(x) := -2t|x|e^{-t|x|}, \quad x \in \mathbb{R}^d.$$

Then

(a) m_t belongs to $L^1 \cap L^2(\mu_k)$, and

$$\|m_t\|_{L^1(\mu_k)} = 2t \int_{\mathbb{R}^d} |x|e^{-t|x|}\mathrm{d}\mu_k(x) = -2t\frac{\partial}{\partial t}\Big(\int_{\mathbb{R}^d} e^{-t|x|}\mathrm{d}\mu_k(x)\Big).$$

Since,

$$e^{-t|x|} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}} e^{-\frac{t^2}{4s}|x|^2}\mathrm{d}s, \tag{3.5}$$

by Fubini-Tonnelli's theorem and (1.1), we deduce that

$$\int_{\mathbb{R}^d} e^{-t|x|}\mathrm{d}\mu_k(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}}\Big(\int_{\mathbb{R}^d} e^{-\frac{t^2}{4s}|x|^2}\mathrm{d}\mu_k(x)\Big)\mathrm{d}s = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-s}}{\sqrt{s}}\big(\frac{\sqrt{2s}}{t}\big)^{2\gamma_k+d}\mathrm{d}s$$
$$= \frac{\Gamma(\gamma_k + \frac{d+1}{2})}{\sqrt{\pi}}\big(\frac{\sqrt{2}}{t}\big)^{2\gamma_k+d}.$$

Thus,

$$\|m_t\|_{L^1(\mu_k)} = \frac{2(2\gamma_k + d)\Gamma(\gamma_k + \frac{d+1}{2})}{\sqrt{\pi}}\big(\frac{\sqrt{2}}{t}\big)^{2\gamma_k+d}.$$

On the other hand,

$$\|m_t\|^2_{L^2(\mu_k)} = 4t^2 \int_{\mathbb{R}^d} |x|^2 e^{-2t|x|}\mathrm{d}\mu_k(x) = \frac{\partial^2}{\partial t^2}\Big(\int_{\mathbb{R}^d} e^{-2t|x|}\mathrm{d}\mu_k(x)\Big)$$
$$= \frac{\partial^2}{\partial t^2}\big(\frac{\Gamma(\gamma_k + \frac{d+1}{2})}{\sqrt{\pi}(\sqrt{2}\,t)^{2\gamma_k+d}}\big).$$

Thus,

$$\|m_t\|^2_{L^2(\mu_k)} = \frac{4(2\gamma_k+d)\Gamma(\gamma_k+\frac{d+3}{2})}{\sqrt{\pi}(\sqrt{2}\,t)^{2\gamma_k+d+2}}.$$

(b) m_t satisfies the admissibility condition (3.2), that is

$$\int_0^\infty |m_t(ax)|^2\frac{\mathrm{d}a}{a} = 4t^2|x|^2\int_0^\infty ae^{-2t|x|a}\mathrm{d}a = 1.$$

Then the associated operators T_{k,ℓ,m_t} satisfy the formula (3.3).

We use Lemma 3.1, then for $f\in L^1(\mu_\ell)\cap H^s_{k\ell}$, we have

$$T_{k,\ell,m_t}f(a,x) = \frac{2t}{a^{2\gamma_k+d}}\int_{\mathbb{R}^d}\frac{\partial}{\partial t}\big[p_{k\ell}(\frac{x}{a},\frac{y}{a},t)\big]f(y)\mathrm{d}\mu_\ell(y), \tag{3.6}$$

where

$$p_{k\ell}(x,y,t) = \int_{\mathbb{R}^d} e^{-t|z|}E_k(ix,z)E_\ell(-iy,z)\mathrm{d}\mu_k(z).$$

If $k=\ell$, then p_{kk} is the Dunkl-type Poisson kernel [18], and from (3.5) this kernel is given by

$$p_{kk}(x,y,t) = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-s}}{\sqrt{s}}h_{kk}(x,y,\frac{t^2}{4s})\mathrm{d}s.$$

4. Uncertainty principles

We can obtain the following inequality from the Heisenberg-Pauli-Weyl uncertainty principle.

Theorem 4.1 *Let m be a function in $L^2(\mu_k)$ satisfying the admissibility condition (3.2). Then, for $f\in H^0_{k\ell}$, we have*

$$\|f\|^2_{H^0_{k\ell}} \le \frac{2}{2\gamma_k+d}\|\,|y|\mathcal{F}_\ell(f)\|_{L^2(\mu_k)}\|\,|x|T_{k,\ell,m}f\|_{L^2(\Omega_k)}.$$

Proof Let $f\in H^s_{k\ell}$, $s\ge\gamma_\ell-\gamma_k$. Assume that $\|\,|y|\mathcal{F}_\ell(f)\|_{L^2(\mu_k)}<\infty$ and $\|\,|x|T_{k,\ell,m}f\|^2_{L^2(\Omega_k)}<\infty$. The inequality (1.2) leads to

$$\int_{\mathbb{R}^d}|T_{k,\ell,m}f(a,x)|^2\mathrm{d}\mu_k(x) \le \frac{2}{2\gamma_k+d}\Big(\int_{\mathbb{R}^d}|x|^2|T_{k,\ell,m}f(a,x)|^2\mathrm{d}\mu_k(x)\Big)^{1/2}\times \Big(\int_{\mathbb{R}^d}|y|^2\big|\mathcal{F}_k(T_{k,\ell,m}f(a,.))(y)\big|^2\mathrm{d}\mu_k(y)\Big)^{1/2}.$$

Integrating with respect to $\frac{\mathrm{d}a}{a}$ gives

$$\|T_{k,\ell,m}f\|^2_{L^2(\Omega_k)} \le \frac{2}{2\gamma_k+d}\int_0^\infty\Big(\int_{\mathbb{R}^d}|x|^2|T_{k,\ell,m}f(a,x)|^2\mathrm{d}\mu_k(x)\Big)^{1/2}\times \Big(\int_{\mathbb{R}^d}|y|^2\big|\mathcal{F}_k(T_{k,\ell,m}f(a,.))(y)\big|^2\mathrm{d}\mu_k(y)\Big)^{1/2}\frac{\mathrm{d}a}{a}.$$

From Theorem 3.2 and the Schwarz's inequality, we get

$$\|f\|^2_{H^0_{k\ell}} \le \frac{2}{2\gamma_k+d}\Big(\int_0^\infty\int_{\mathbb{R}^d}|x|^2|T_{k,\ell,m}f(a,x)|^2\mathrm{d}\mu_k(x)\frac{\mathrm{d}a}{a}\Big)^{1/2}\times \Big(\int_0^\infty\int_{\mathbb{R}^d}|y|^2\big|\mathcal{F}_k(T_{k,\ell,m}f(a,.))(y)\big|^2\mathrm{d}\mu_k(y)\frac{\mathrm{d}a}{a}\Big)^{1/2}.$$

But by (3.1), Fubini-Tonnelli's theorem and (3.2), we have

$$\int_0^\infty \int_{\mathbb{R}^d} |y|^2 \big|\mathcal{F}_k(T_{k,\ell,m}f(a,.))(y)\big|^2 \mathrm{d}\mu_k(y)\frac{\mathrm{d}a}{a} = \int_0^\infty \int_{\mathbb{R}^d} |y|^2|m(ay)|^2|\mathcal{F}_\ell(f)(y)|^2\mathrm{d}\mu_k(y)\frac{\mathrm{d}a}{a}$$
$$= \int_{\mathbb{R}^d} |y|^2|\mathcal{F}_\ell(f)(y)|^2\mathrm{d}\mu_k(y).$$

This yields the result and completes the proof of the theorem. □

Let E be a measurable subset of $\mathbb{R}^d$. We say that a function $f \in H^s_{k\ell}$, is ε-concentrated on E, if

$$\|f - \chi_E f\|_{H^s_{k\ell}} \leq \varepsilon\|f\|_{H^s_{k\ell}}, \tag{4.1}$$

where χ_E is the indicator function of the set E.

Let S be a measurable subset of $(0,\infty)\times\mathbb{R}^d$ and let $f \in H^s_{k\ell}$. We say that $T_{k,\ell,m}f$ is η-concentrated on S, if

$$\|T_{k,\ell,m}f - \chi_S T_{k,\ell,m}f\|_{L^2(\Omega_k)} \leq \eta\|T_{k,\ell,m}f\|_{L^2(\Omega_k)}. \tag{4.2}$$

Similarly as Theorem 4.1, we can obtain an inequality from the classical Donoho-Stark's uncertainty principle.

Theorem 4.2 *Let $f \in H^s_{k\ell}$, $s \geq \gamma_\ell - \gamma_k$ and let $m \in L^1 \cap L^2(\mu_k)$ satisfying (3.2). If f is ε-concentrated on E and $T_{k,\ell,m}f$ is η-concentrated on S, then*

$$\big(\mu_\ell(E)\big)^{1/2}\Big(\int\int_S \frac{\mathrm{d}\Omega_k(a,x)}{a^{2(2\gamma_k+d)}}\Big)^{1/2} \geq \frac{(1-\eta-\varepsilon)}{2^{(\gamma_\ell-\gamma_k)/2}\|m\|_{L^1(\mu_k)}}\sqrt{\frac{c_k}{c_\ell}}.$$

Proof Let $f \in H^s_{k\ell}$, $s \geq \gamma_\ell - \gamma_k$ and let $m \in L^1 \cap L^2(\mu_k)$. Assume that $\mu_\ell(E) < \infty$ and $\int\int_S \frac{\mathrm{d}\Omega_k(a,x)}{a^{2(2\gamma_k+d)}} < \infty$. From (4.1), (4.2) and Theorem 3.2 it follows that

$$\begin{aligned}
&\|T_{k,\ell,m}f - \chi_S T_{k,\ell,m}(\chi_E f)\|_{L^2(\Omega_k)} \\
&\leq \|T_{k,\ell,m}f - \chi_S T_{k,\ell,m}f\|_{L^2(\Omega_k)} + \|\chi_S T_{k,\ell,m}(f-\chi_E f)\|_{L^2(\Omega_k)} \\
&\leq \eta\|T_{k,\ell,m}f\|_{L^2(\Omega_k)} + \|T_{k,\ell,m}(f-\chi_E f)\|_{L^2(\Omega_k)} \\
&\leq \eta\|\mathcal{F}_\ell(f)\|_{L^2(\mu_k)} + \|f-\chi_E f\|_{H^s_{k\ell}} \leq (\eta+\varepsilon)\|f\|_{H^s_{k\ell}}.
\end{aligned}$$

Then the triangle inequality shows that

$$\begin{aligned}
\|T_{k,\ell,m}f\|_{L^2(\Omega_k)} &\leq \|\chi_S T_{k,\ell,m}(\chi_E f)\|_{L^2(\Omega_k)} + \|T_{k,\ell,m}f - \chi_S T_{k,\ell,m}(\chi_E f)\|_{L^2(\Omega_k)} \\
&\leq \|\chi_S T_{k,\ell,m}(\chi_E f)\|_{L^2(\Omega_k)} + (\eta+\varepsilon)\|f\|_{H^s_{k\ell}}.
\end{aligned}$$

But

$$\|\chi_S T_{k,\ell,m}(\chi_E f)\|_{L^2(\Omega_k)} = \Big(\int\int_S |T_{k,\ell,m}(\chi_E f)(a,x)|^2\mathrm{d}\Omega_k(a,x)\Big)^{1/2}.$$

Since $f \in H^s_{k\ell}$, by Lemma 2.3, the function f belongs to $L^2(\mu_\ell)$, and we have

$$\begin{aligned}
|T_{k,\ell,m}(\chi_E f)(a,x)| &\leq \|m(a.)\mathcal{F}_\ell(\chi_E f)\|_{L^1(\mu_k)} \leq \|m(a.)\|_{L^1(\mu_k)}\|\mathcal{F}_\ell(\chi_E f)\|_{L^\infty(\mu_\ell)} \\
&\leq \frac{1}{a^{2\gamma_k+d}}\|m\|_{L^1(\mu_k)}\|\chi_E f\|_{L^1(\mu_\ell)} \\
&\leq \frac{1}{a^{2\gamma_k+d}}\|m\|_{L^1(\mu_k)}\|f\|_{L^2(\mu_\ell)}\big(\mu_\ell(E)\big)^{1/2}.
\end{aligned}$$

Thus,

$$\|\chi_S T_{k,\ell,m}(\chi_E f)\|_{L^2(\Omega_k)} \le \|m\|_{L^1(\mu_k)}\|f\|_{L^2(\mu_\ell)}\big(\mu_\ell(E)\big)^{1/2}\Big(\int\int_S \frac{\mathrm{d}\Omega_k(a,x)}{a^{2(2\gamma_k+d)}}\Big)^{1/2}$$

and

$$\|T_{k,\ell,m}f\|_{L^2(\Omega_k)} \le \|m\|_{L^1(\mu_k)}\|f\|_{L^2(\mu_\ell)}\big(\mu_\ell(E)\big)^{1/2}\Big(\int\int_S \frac{\mathrm{d}\Omega_k(a,x)}{a^{2(2\gamma_k+d)}}\Big)^{1/2} + (\eta+\varepsilon)\|f\|_{H^s_{k\ell}}.$$

By applying Theorem 3.2, we obtain

$$\big(\mu_\ell(E)\big)^{1/2}\Big(\int\int_S \frac{\mathrm{d}\Omega_k(a,x)}{a^{2(2\gamma_k+d)}}\Big)^{1/2} \ge \frac{(1-\eta-\varepsilon)\|f\|_{H^s_{k\ell}}}{\|m\|_{L^1(\mu_k)}\|f\|_{L^2(\mu_\ell)}}.$$

Then Lemma 2.3 gives the desired result. □

Remark 4.3 If $S \subset \{(a,x) \in (0,\infty)\times\mathbb{R}^d : a \ge \delta\}$ for some $\delta > 0$, we suppose that $\alpha = \max\{\frac{1}{a} : (a,x) \in S$ for some $x \in \mathbb{R}^d\}$. Then by Theorem 4.2 we deduce that

$$\big(\mu_\ell(E)\big)^{1/2}\big(\Omega_k(S)\big)^{1/2} \ge \frac{(1-\eta-\varepsilon)}{\alpha^{2\gamma_k+d}2^{(\gamma_\ell-\gamma_k)/2}\|m\|_{L^1(\mu_k)}}\sqrt{\frac{c_k}{c_\ell}}.$$

5. Extremal functions

In this section, by using the theory of extremal function and reproducing kernel of Hilbert space [10,11,19] we study the extremal function associated to the Dunkl multiplier operators $T_{k,\ell,m}$. This function was studied firstly in [7] (when $k=\ell$), and some properties related to the dual Dunkl-Sonine operator of this function were given in [6].

Let $\lambda > 0$. We denote by $\langle .,.\rangle_{\lambda,H^s_{k\ell}}$ the inner product defined on the space $H^s_{k\ell}$ by

$$\langle f,g\rangle_{\lambda,H^s_{k\ell}} := \lambda\langle f,g\rangle_{H^s_{k\ell}} + \langle \mathcal{F}_\ell(f),\mathcal{F}_\ell(g)\rangle_{L^2(\mu_k)}, \tag{5.1}$$

and the norm $\|f\|_{\lambda,H^s_{k\ell}} := \sqrt{\langle f,f\rangle_{\lambda,H^s_{k\ell}}}$.

On $H^s_{k\ell}$ the two norms $\|.\|_{H^s_{k\ell}}$ and $\|.\|_{\lambda,H^s_{k\ell}}$ are equivalent. This $(H^s_{k\ell},\langle .,.\rangle_{\lambda,H^s_{k\ell}})$ is a Hilbert space with reproducing kernel given by the following theorem.

Lemma 5.1 *Let $\lambda > 0$, and let $s > 2\gamma_\ell - \gamma_k + d/2$. The space $(H^s_{k\ell},\langle .,.\rangle_{\lambda,H^s_{k\ell}})$ has the reproducing kernel*

$$K_s(x,y) = \frac{c_\ell}{c_k}\int_{\mathbb{R}^d} \frac{E_\ell(ix,z)E_\ell(-iy,z)}{1+\lambda(1+|z|^2)^s} w_{\ell-k}(z)\mathrm{d}\mu_\ell(z), \tag{5.2}$$

that is

(i) For all $y \in \mathbb{R}^d$, the function $x \to K_s(x,y)$ belongs to $H^s_{k\ell}$.

(ii) The reproducing property: for all $f \in H^s_{k\ell}$ and $y \in \mathbb{R}^d$,

$$\langle f, K_s(.,y)\rangle_{\lambda,H^s_{k\ell}} = f(y).$$

Proof (i) Let $y \in \mathbb{R}^d$ and $s > 2\gamma_\ell - \gamma_k + d/2$. From (2.1), the function

$$\Phi_y : z \to \frac{c_\ell}{c_k}\frac{E_\ell(-iy,z)}{1+\lambda(1+|z|^2)^s} w_{\ell-k}(z)$$

belongs to $L^1 \cap L^2(\mu_\ell)$. Then, the function K_s is well defined and by Theorem 2.1 (ii), we have

$$K_s(x, y) = \mathcal{F}_\ell^{-1}(\Phi_y)(x), \quad x \in \mathbb{R}^d. \tag{5.3}$$

Then by Theorem 2.1 (iii) and (2.1), we obtain

$$|\mathcal{F}_\ell(K_s(., y))(z)| \leq \frac{c_\ell}{c_k} \frac{w_{\ell-k}(z)}{\lambda(1 + |z|^2)^s},$$

and

$$\|K_s(., y)\|_{H^s_{k\ell}} \leq \frac{1}{\lambda} C_{k,\ell} < \infty.$$

This proves that for all $y \in \mathbb{R}^d$ the function $K_s(., y)$ belongs to $H^s_{k\ell}$.

(ii) Let $f \in H^s_{k\ell}$ and $y \in \mathbb{R}^d$. From (5.1) and (5.3), we have

$$\langle f, K_s(., y)\rangle_{\lambda, H^s_{k\ell}} = \int_{\mathbb{R}^d} E_\ell(iy, z)\mathcal{F}_\ell(f)(z)\mathrm{d}\mu_\ell(z),$$

and from Remark 2.5, we obtain the reproducing property:

$$\langle f, K_s(., y)\rangle_{\lambda, H^s_{k\ell}} = f(y).$$

This completes the proof of the theorem. □

The main result of this section can be stated as follows.

Theorem 5.2 *Let $s > 2\gamma_\ell - \gamma_k + d/2$ and let $m \in L^2(\mu_k)$ satisfy (3.2). For any $g \in L^2(\Omega_k)$ and for any $\lambda > 0$, there exists a unique function $f^*_{\lambda,g}$, such that the infimum*

$$\inf_{f \in H^s_{k\ell}} \left\{\lambda\|f\|^2_{H^s_{k\ell}} + \|g - T_{k,\ell,m}f\|^2_{L^2(\Omega_k)}\right\} \tag{5.4}$$

is attained. Moreover, the extremal function $f^*_{\lambda,g}$ is given by

$$f^*_{\lambda,g}(y) = \int_{\mathbb{R}^d}\int_0^\infty g(a, x)Q\big((a, x), y\big)\mathrm{d}\Omega_k(a, x),$$

where

$$Q\big((a, x), y\big) = \int_{\mathbb{R}^d} \frac{\overline{m(az)}E_k(-ix, z)E_\ell(iy, z)}{1 + \lambda(1 + |z|^2)^s}\mathrm{d}\mu_\ell(z).$$

Proof Let $\lambda > 0$. We denote by $\langle ., .\rangle_{\lambda, H^s_{k\ell}}$ the inner product defined on the space $H^s_{k\ell}$ by

$$\langle f, g\rangle_{\lambda, H^s_{k\ell}} := \lambda\langle f, g\rangle_{H^s_{k\ell}} + \langle T_{k,\ell,m}f, T_{k,\ell,m}g\rangle_{L^2(\Omega_k)}.$$

Since $m \in L^2(\mu_k)$ satisfies (3.2), by Theorem 3.2, the inner product $\langle ., .\rangle_{\lambda, H^s_{k\ell}}$ can be written

$$\langle f, g\rangle_{\lambda, H^s_{k\ell}} = \lambda\langle f, g\rangle_{H^s_{k\ell}} + \langle \mathcal{F}_\ell(f), \mathcal{F}_\ell(g)\rangle_{L^2(\mu_k)}.$$

Then, the existence and unicity of the extremal function $f^*_{\lambda,g}$ satisfying (5.4) is given as in the same of [9,20,21]. Especially, $f^*_{\lambda,g}$ is given by the reproducing kernel of $H^s_{k\ell}$ with $\|.\|_{\lambda, H^s_{k\ell}}$ norm as

$$f^*_{\lambda,g}(y) = \langle g, T_{k,\ell,m}(K_s(., y))\rangle_{L^2(\Omega_k)}, \tag{5.5}$$

where K_s is the kernel given by (5.2).

But by Theorem 2.1 (ii) and (5.3), we have

$$T_{k,\ell,m}(K_s(.,y))(a,x) = \int_{\mathbb{R}^d} m(az)\mathcal{F}_\ell(K_s(.,y))(z)E_k(ix,z)\mathrm{d}\mu_k(z)$$
$$= \int_{\mathbb{R}^d} m(az)\frac{E_k(ix,z)E_\ell(-iy,z)}{1+\lambda(1+|z|^2)^s}\mathrm{d}\mu_\ell(z).$$

This clearly yields the result. □

As application, we give the following examples.

Example 5.3 Let $s > 2\gamma_\ell - \gamma_k + d/2$, $\lambda > 0$ and $g \in L^2(\Omega_k)$.

(i) If $m_t(x) := -\sqrt{8}\,t|x|^2e^{-t|x|^2}$, then

$$f^*_{\lambda,g}(y) = \int_{\mathbb{R}^d}\int_0^\infty g(a,x)Q\big((a,x),y\big)\mathrm{d}\Omega_k(a,x),$$

where

$$Q\big((a,x),y\big) = -\sqrt{8}\,ta^2\int_{\mathbb{R}^d}\frac{|z|^2e^{-ta^2|z|^2}}{1+\lambda(1+|z|^2)^s}E_k(-ix,z)E_\ell(iy,z)\mathrm{d}\mu_\ell(z).$$

By (3.4), (5.5) and the fact that $K_s(y,z) = \overline{K_s(z,y)}$ we obtain

$$Q\big((a,x),y\big) = \frac{\sqrt{8}\,t}{a^{2\gamma_k+d}}\int_{\mathbb{R}^d}\frac{\partial}{\partial t}\big[h_{k\ell}(\frac{x}{a},\frac{z}{a},t)\big]K_s(y,z)\mathrm{d}\mu_\ell(z).$$

(ii) If $m_t(x) := -2t|x|e^{-t|x|}$, then

$$f^*_{\lambda,g}(y) = \int_{\mathbb{R}^d}\int_0^\infty g(a,x)Q\big((a,x),y\big)\mathrm{d}\Omega_k(a,x),$$

where

$$Q\big((a,x),y\big) = -2ta\int_{\mathbb{R}^d}\frac{|z|e^{-ta|z|}}{1+\lambda(1+|z|^2)^s}E_k(-ix,z)E_\ell(iy,z)\mathrm{d}\mu_\ell(z).$$

By (3.6) and (5.5) we deduce that

$$Q\big((a,x),y\big) = \frac{2t}{a^{2\gamma_k+d}}\int_{\mathbb{R}^d}\frac{\partial}{\partial t}\big[p_{k\ell}(\frac{x}{a},\frac{z}{a},t)\big]K_s(y,z)\mathrm{d}\mu_\ell(z).$$

Theorem 5.4 *Let $s > 2\gamma_\ell - \gamma_k + d/2$, $\lambda > 0$ and $g \in L^2(\Omega_k)$. The extremal function $f^*_{\lambda,g}$ satisfies:*

(i) $|f^*_{\lambda,g}(y)| \le \frac{C_{k,\ell}}{2\sqrt{\lambda}}\|g\|_{L^2(\Omega_k)}$,

where $C_{k,\ell}$ is the constant given by (2.3).

(ii) $\|f^*_{\lambda,g}\|^2_{L^2(\mu_\ell)} \le \frac{D_{k,\ell}}{\lambda}\|m\|^2_{L^2(\mu_k)}\int_{\mathbb{R}^d}\int_0^\infty |g(a,x)|^2\frac{e^{(|x|^2+a^2)/2}}{a^{2\gamma_k+d+1}}\mathrm{d}\Omega_k(a,x)$,

where

$$D_{k,\ell} = \sqrt{\pi}\frac{c_k}{c_\ell}2^{\gamma_\ell-\gamma_k-5/2}.$$

Proof (i) From (5.5) and Theorem 3.2, we have

$$|f^*_{\lambda,g}(y)| \le \|g\|_{L^2(\Omega_k)}\|T_{k,\ell,m}(K_s(.,y))\|_{L^2(\Omega_k)} \le \|g\|_{L^2(\Omega_k)}\|\mathcal{F}_\ell(K_s(.,y))\|_{L^2(\mu_k)}.$$

Then, by (5.3) we deduce

$$|f^*_{\lambda,g}(y)| \le \|g\|_{L^2(\Omega_k)}\Big(\frac{c_\ell}{c_k}\int_{\mathbb{R}^d}\frac{w_{\ell-k}(z)\mathrm{d}\mu_\ell(z)}{[1+\lambda(1+|z|^2)^s]^2}\Big)^{1/2}.$$

Using the fact that

$$\big[1+\lambda(1+|z|^2)^s\big]^2 \geq 4\lambda(1+|z|^2)^s, \tag{5.6}$$

we obtain the result.

(ii) We write

$$f^*_{\lambda,g}(y)=\int_{\mathbb{R}^d}\int_0^\infty \sqrt{a}e^{-(|x|^2+a^2)/4}\frac{e^{(|x|^2+a^2)/4}}{\sqrt{a}}g(a,x)Q((a,x),y)\mathrm{d}\Omega_k(a,x).$$

Applying Hölder's inequality, we obtain

$$|f^*_{\lambda,g}(y)|^2 \leq \sqrt{\frac{\pi}{2}}\int_{\mathbb{R}^d}\int_0^\infty |g(a,x)|^2\frac{e^{(|x|^2+a^2)/2}}{a}\big|Q((a,x),y)\big|^2\mathrm{d}\Omega_k(a,x).$$

Thus and from Fubini-Tonnelli's theorem, we get

$$\|f^*_{\lambda,g}\|^2_{L^2(\mu_\ell)} \leq \sqrt{\frac{\pi}{2}}\int_{\mathbb{R}^d}\int_0^\infty |g(a,x)|^2\frac{e^{(|x|^2+a^2)/2}}{a}\|Q((a,x),.)\|^2_{L^2(\mu_\ell)}\mathrm{d}\Omega_k(a,x).$$

Let $\Psi_x(z)=\frac{\overline{m(az)E_k(-ix,z)}}{1+\lambda(1+|z|^2)^s}$. Since $\Psi_x\in L^1\cap L^2(\mu_\ell)$, then

$$Q((a,x),y)=\mathcal{F}_\ell^{-1}(\Psi_x)(y).$$

Thus, by Theorem 2.1 (iii) we deduce that

$$\|Q((a,x),.)\|^2_{L^2(\mu_\ell)}=\int_{\mathbb{R}^d}|\mathcal{F}_\ell(Q((a,x),.))(z)|^2\mathrm{d}\mu_\ell(z)\leq\int_{\mathbb{R}^d}\frac{|m(az)|^2\mathrm{d}\mu_\ell(z)}{[1+\lambda(1+|z|^2)^s]^2}.$$

Then using the inequality (5.6), we obtain

$$\begin{aligned}\|Q((a,x),.)\|^2_{L^2(\mu_\ell)} &\leq \frac{1}{4\lambda}\frac{c_k}{c_\ell}\int_{\mathbb{R}^d}\frac{|m(az)|^2 w_{\ell-k}(z)}{(1+|z|^2)^s}\mathrm{d}\mu_k(z)\\ &\leq \frac{1}{\lambda}\frac{c_k}{c_\ell}2^{\gamma_\ell-\gamma_k-2}\int_{\mathbb{R}^d}\frac{|m(az)|^2|z|^{2(\gamma_\ell-\gamma_k)}}{(1+|z|^2)^s}\mathrm{d}\mu_k(z)\\ &\leq \frac{1}{\lambda}\frac{c_k}{c_\ell}2^{\gamma_\ell-\gamma_k-2}\int_{\mathbb{R}^d}|m(az)|^2\mathrm{d}\mu_k(z).\end{aligned}$$

Thus

$$\|Q((a,x),.)\|^2_{L^2(\mu_\ell)} \leq \frac{1}{\lambda}\frac{c_k}{c_\ell}\frac{2^{\gamma_\ell-\gamma_k-2}}{a^{2\gamma_k+d}}\|m\|^2_{L^2(\mu_k)}.$$

From this inequality we deduce the result. □

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