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Algebraic Properties of Dual Toeplitz Operators on Harmonic Hardy Space over Polydisc

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Abstract In this paper, we introduce the harmonic Hardy space on $\mathbb{T}^n$ and study some algebraic properties of dual Toeplitz operator on the harmonic Hardy space on $\mathbb{T}^n$.

Keywords Hardy space; Toeplitz operator; spectrum; semi-commutative

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1. Introduction

Let $\mathbb{T}$ be the unit circle on the complex plane $\mathbb{C}$. For a fixed positive integer $n \geq 1$, the $\mathbb{C}^n$ and $\mathbb{T}^n$ are the cartesian products of n copies of $\mathbb{C}$ and $\mathbb{T}$, respectively. Denote by $z = (z_1, \ldots, z_n)$ the coordinates on $\mathbb{C}^n$. Let $\mathrm{d}\sigma$ be the normalized Haar measure on $\mathbb{T}^n$ and $L^2(\mathbb{T}^n)$ be the square integral functions with respect to $\mathrm{d}\sigma$. The Hardy space $H^2(\mathbb{T}^n)$ is the closure of analytic polynomials in $L^2(\mathbb{T}^n)$, that is

$$H^2(\mathbb{T}^n) = \mathrm{clos}\{p(z_1, \ldots, z_n) : p \text{ is anlytic polynomials}\}.$$

In the setting of classical Hardy space on $\mathbb{T}$, it is well known that $H^2(\mathbb{T}) + \overline{H^2(\mathbb{T})} = L^2(\mathbb{T})$, where $\overline{(\cdot)}$ is the complex conjugate. However, in the higher dimension ($n \geq 2$), the situation is completely different, indeed, $H^2(\mathbb{T}^n) + \overline{H^2(\mathbb{T}^n)}$ is much smaller than $L^2(\mathbb{T}^n)$. Denote

$$h^2(\mathbb{T}^n) = H^2(\mathbb{T}^n) + \overline{H^2(\mathbb{T}^n)}$$

and call it the harmonic Hardy space on $\mathbb{T}^n$. The reader may not confuse that $h^2(\mathbb{T}^n)$ does not contain all harmonic functions. In the whole paper, P denotes the orthogonal projection from $L^2(\mathbb{T}^n)$ onto $h^2(\mathbb{T}^n)$ and $Q = 1 - P$.

Let $L^\infty(\mathbb{T}^n)$ be the set of essentially bounded measurable functions on $\mathbb{T}^n$. For $\varphi \in L^\infty(\mathbb{T}^n)$, the Toeplitz operator T_φ on $h^2(\mathbb{T}^n)$ is defined by

$$T_\varphi f = P(\varphi f), \quad f \in h^2(\mathbb{T}^n).$$

The Toeplitz operators on analytic and harmonic function spaces have been widely studied [1].

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The Hankel operator and dual Toeplitz operator can also be defined as follows:

$$H_\varphi: \ h^2(\mathbb{T}^n) \longrightarrow (h^2(\mathbb{T}^n))^\perp$$
$$H_\varphi f = Q(\varphi f), \quad f \in h^2(\mathbb{T}^n).$$

$$S_\varphi: \ (h^2(\mathbb{T}^n))^\perp \longrightarrow (h^2(\mathbb{T}^n))^\perp$$
$$S_\varphi f = Q(\varphi f), \quad f \in h^2(\mathbb{T}^n)^\perp.$$

One can check that $H^*_{\overline{\varphi}} f = P(\varphi f), \ \ f \in h^2(\mathbb{T}^n)^\perp$, and

$$\|S_\varphi(f)\|_2 = \|Q(\varphi f)\|_2 \le \|\varphi f\|_2 \le \|\varphi\|_\infty \|f\|_2,$$

where $\|\cdot\|_\infty$ is the essential sup norm and $\|\cdot\|_2$ is the norm of $L^2(\mathbb{T}^n)$. The following algebraic properties of dual Toeplitz operators are also easy to check. For $\varphi, \psi \in L^\infty(\mathbb{T}^n)$, $\alpha, \beta \in \mathbb{C}$, we have

$$S^*_\varphi = S_{\overline{\varphi}}, \quad S_{\alpha\varphi+\beta\psi} = \alpha S_\varphi + \beta S_\psi.$$

For dual Toeplitz operator, Stroethoff and Zheng [2] studied algebraic and spectral properties of dual Toeplitz operators with general symbols on the Bergman space of the unit disk. Lu [3] characterized commuting dual Toeplitz operators on the Bergman space of the unit ball with pluriharmonic symbols. Lu and Shang [4] studied commutativity, essential commutativity and essential semi-commutativity of dual Toeplitz operators with general symbols on the polydisk. Lu and Yang [5] studied the properties of dual Toeplitz operators with general symbols on the weighted Bergman space of the unit ball. In the several-variable situation, the study of dual Toeplitz is much more complicated [6–8].

The Toeplitz operator, Hankel operator and dual Toeplitz operator have close relationships through the multiplication operators on $L^2(\mathbb{T}^n)$. Under the decomposition

$$L^2(\mathbb{T}^n) = h^2(\mathbb{T}^n) \oplus (h^2(\mathbb{T}^n))^\perp,$$

the multiplication operator M_φ, $\varphi \in L^\infty(\mathbb{T}^n)$ can be represented as follows

$$M_\varphi = \begin{pmatrix} T_\varphi & H^*_{\overline{\varphi}} \\ H_\varphi & S_\varphi \end{pmatrix}.$$

For $\varphi, \psi \in L^\infty(\mathbb{T}^n)$, the identity $M_\varphi M_\psi = M_{\varphi\psi} = M_\psi M_\varphi$ implies that

$$S_{\varphi\psi} = S_\varphi S_\psi + H_\varphi H^*_{\overline{\psi}} = S_\psi S_\varphi + H_\psi H^*_{\overline{\varphi}}. \tag{1}$$

Equation (1) will be used frequently.

2. Properties of spectrum

Characterizations of spectrum is one of important properties for bounded linear operators. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n)} \in \mathbb{Z}^n_+$, we denote $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$.

Lemma 2.1 *Let $\psi \in C(\mathbb{T}^n)$, where $C(\mathbb{T}^n)$ is the set of continuous functions on $\mathbb{T}^n$. Then for* *any $z \in \mathbb{T}^n$, we have*

$$\frac{1}{\lambda_m}\int_{\mathbb{T}^n}\psi(\zeta)\,\big|z^\alpha\frac{\langle\zeta,z\rangle}{n+1}\big|^2\,\big|1+z^\alpha\frac{\langle\zeta,z\rangle}{n+1}\big|^{2m}\mathrm{d}\sigma(\zeta)\longrightarrow\psi(z),\quad \text{as } m\to\infty,$$

where $\lambda_m=\int_{\mathbb{T}^n}|z^\alpha\frac{\langle\zeta,z\rangle}{n+1}|^2\,|1+z^\alpha\frac{\langle\zeta,z\rangle}{n+1}|^{2m}\mathrm{d}\sigma(\zeta)$, $\alpha=(2,0,0,\dots,0)$.

Proof Firstly, let us show that

$$\frac{1}{\lambda_m}\int_{\mathbb{T}^n\backslash V_z(\varepsilon)}\big|z^\alpha\frac{\langle\zeta,z\rangle}{n+1}\big|^2\,\big|1+z^\alpha\frac{\langle\zeta,z\rangle}{n+1}\big|^{2m}\mathrm{d}\sigma(\zeta)\ \longrightarrow\ 0,\quad \text{as } m\to\infty, \tag{2}$$

for any neighborhood of z with following form,

$$V_z(\varepsilon)=\big\{\zeta\in\mathbb{T}^n:\ \big|1-z^\alpha\frac{\langle\zeta,z\rangle}{n+1}\big|<\varepsilon\big\}.$$

On $\mathbb{T}^n\backslash V_z(\varepsilon)$,

$$\big|z^\alpha\frac{\langle\zeta,z\rangle}{n+1}\big(1+z^\alpha\frac{\langle\zeta,z\rangle}{n+1}\big)\big|\ \le\ \sup_{\zeta\in\mathbb{T}^n\backslash V_z(\varepsilon)}\big|1+z^\alpha\frac{\langle\zeta,z\rangle}{n+1}\big|=\rho<2, \tag{3}$$

whence

$$\int_{\mathbb{T}^n\backslash V_z(\varepsilon)}\big|z^\alpha\frac{\langle\zeta,z\rangle}{n+1}\big|^2\,\big|1+z^\alpha\frac{\langle\zeta,z\rangle}{n+1}\big|^{2m}\mathrm{d}\sigma(\zeta)\ \le\ \rho^{2m}\sigma(\mathbb{T}^n\backslash V_z). \tag{4}$$

If $0<\delta<2-\rho<1$, considering another neighborhood $V_z(\delta)$ of z, we can get

$$\lambda_m\ge\int_{V_z(\delta)}\big|z^\alpha\frac{\langle\zeta,z\rangle}{n+1}\big|^2\,\big|1+z^\alpha\frac{\langle\zeta,z\rangle}{n+1}\big|^{2m}\mathrm{d}\sigma(\zeta)\ge(1-\delta)^2(2-\delta)^{2m}\sigma(V_\delta). \tag{5}$$

From inequalities (3)–(5), we see that, as $m\to\infty$,

$$\frac{1}{\lambda_m}\int_{\mathbb{T}^n\backslash V_z(\varepsilon)}\big|z^\alpha\frac{\langle\zeta,z\rangle}{n+1}\big|^2\,\big|1+z^\alpha\frac{\langle\zeta,z\rangle}{n+1}\big|^{2m}\mathrm{d}\sigma(\zeta)\le c\big(\frac{\rho}{2-\delta}\big)^{2m}\ \longrightarrow\ 0.$$

Therefore,

$$\frac{1}{\lambda_m}\int_{V_z(\varepsilon)}\big|z^\alpha\frac{\langle\zeta,z\rangle}{n+1}\big|^2\,\big|1+z^\alpha\frac{\langle\zeta,z\rangle}{n+1}\big|^{2m}\mathrm{d}\sigma(\zeta)\ \longrightarrow\ 1,\quad \text{as } m\to\infty. \tag{6}$$

Since λ_m is independent of z in $\mathbb{T}^n$, we have

$$\begin{aligned}&\big|\frac{1}{\lambda_m}\int_{\mathbb{T}^n}\psi(\zeta)\,\big|z^\alpha\frac{\langle\zeta,z\rangle}{n+1}\big|^2\,\big|1+z^\alpha\frac{\langle\zeta,z\rangle}{n+1}\big|^{2m}\mathrm{d}\sigma(\zeta)-\psi(z)\big|\\
&\le\sup_{\zeta\in\mathbb{T}^n\backslash V_z(\varepsilon)}\big|\psi(\zeta)-\psi(z)\big|\frac{1}{\lambda_m}\int_{\mathbb{T}^n\backslash V_z(\varepsilon)}\big|z^\alpha\frac{\langle\zeta,z\rangle}{n+1}\big|^2\,\big|1+z^\alpha\frac{\langle\zeta,z\rangle}{n+1}\big|^{2m}\mathrm{d}\sigma(\zeta)+\\
&\quad\sup_{\zeta\in V_z(\varepsilon)}|\psi(\zeta)-\psi(z)|\frac{1}{\lambda_m}\int_{V_z(\varepsilon)}\big|z^\alpha\frac{\langle\zeta,z\rangle}{n+1}\big|^2\,\big|1+z^\alpha\frac{\langle\zeta,z\rangle}{n+1}\big|^{2m}\mathrm{d}\sigma(\zeta).\end{aligned}$$

The continuity of ψ with conclusion (2) and (6) yields the desired result. □

Lemma 2.2 *Let $\varphi\in L^\infty(\mathbb{T}^n)$. If S_φ is invertible in $(h^2(\mathbb{T}^n))^\perp$, then φ is invertible in $L^\infty(\mathbb{T}^n)$.*

Proof The assumption that S_φ is invertible implies that there is a constant $k>0$ satisfying $\|S_\varphi f\|\ge k\|f\|$, $f\in(h^2(\mathbb{T}^n))^\perp$. Considering the projection has norm 1, we can see

$$\|\varphi f\|\ge k\|f\|,\ f\in(h^2(\mathbb{T}^n))^\perp. \tag{7}$$

Particularly, for

$$f(z)=z^{\alpha}\frac{\langle\zeta,z\rangle}{n+1}\big(1+z^{\alpha}\frac{\langle\zeta,z\rangle}{n+1}\big)^{m},\quad \zeta\in\mathbb{T}^n,\ m\geq 1,\ \alpha=(2,0,0,\ldots,0),$$

which is clearly an element of $(h^2(\mathbb{T}^n))^{\perp}$, the inequality (7) yields

$$\int_{\mathbb{T}^n}|\varphi(z)|^2\big|z^{\alpha}\frac{\langle\zeta,z\rangle}{n+1}\big|^2\,\big|1+z^{\alpha}\frac{\langle\zeta,z\rangle}{n+1}\big|^{2m}\mathrm{d}\sigma(z)\geq k^2\lambda_m.$$

It follows that for any nonnegative $\psi\in C(\mathbb{T}^n)$, one has

$$\begin{aligned}&\frac{1}{\lambda_m}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n}|\varphi(z)|^2\psi(\zeta)\,\big|z^{\alpha}\frac{\langle\zeta,z\rangle}{n+1}\big|^2\,\big|1+z^{\alpha}\frac{\langle\zeta,z\rangle}{n+1}\big|^{2m}\mathrm{d}\sigma(z)\mathrm{d}\sigma(\zeta)\\&\geq k^2\int_{\mathbb{T}^n}\psi(\zeta)\mathrm{d}\sigma(\zeta)=k^2\int_{\mathbb{T}^n}\psi(z)\mathrm{d}\sigma(z).\end{aligned}$$

Hence, invoking Lemma 2.1, we obtain

$$\int_{\mathbb{T}^n}(|\varphi(z)|^2-k^2)\psi(z)\mathrm{d}\sigma(z)\geq 0,$$

which implies that $|\varphi(z)|\geq k>0$ a.e., in $\mathbb{T}^n$, hence $\varphi(z)$ is invertible in $L^{\infty}(\mathbb{T}^n)$. □

An immediate consequence is the following spectral inclusion theorem. But firstly, let us denote by $R(f)$ the essential range of the essentially bounded function f, and by $\sigma(T)$, $r(T)$ respectively the spectrum and the spectral radius of an operator T.

Theorem 2.3 *If φ is in $L^{\infty}(\mathbb{T}^n)$, then $R(\varphi)\subseteq\sigma(S_{\varphi})$.*

3. Semi-commuting dual Toeplitz operator

When will the product of two dual Toeplitz operators be semi-commutative? This question has been solved well in Hardy space and harmonic Bergman space [2–5]. But the crucial question is what are the conditions we need if the product of two dual Toeplitz operators with the special form we have discussed is a dual Toeplitz operator in harmonic Hardy space. The following several theorems have been observed.

Equation (1) suggests that S_{φ} and S_{ψ} commute if φ or ψ is constant, that is $H^*_{\overline{\varphi}}=0$ or $H^*_{\overline{\psi}}=0$. If a non-trivial linear combination of φ and ψ is constant, they do commute as well.

Lemma 3.1 *If $\varphi\in H^{\infty}(\mathbb{D}^n)$, then $H^*_{\overline{\varphi}}((h^2)^{\perp})\subseteq H^2$, $H^*_{\varphi}((h^2)^{\perp})\subseteq\overline{H^2}$.*

Proof Let $f=\sum_{k=1}^{\infty}\sum_{|\alpha|,|\beta|=k}a_{\alpha,\beta}z^{\alpha}\overline{z}^{\beta}$, the parameters $\alpha,\beta\in\mathbb{Z}^n_+$. To satisfy the condition $f\in(h^2)^{\perp}$, for any positive integer k, there exist i and j such that $\alpha_i>\beta_i$, $\alpha_j<\beta_j$. Let $H^*_{\overline{\varphi}}(f)=g+h$, g be analytic, and h be co-analytic, then we can get h is constant (If not, then $\alpha_i+c\leq\beta_i$, $c\geq 0$, this is a contradiction), which yields the first desired result. Similarly, the second result can be proved. □

Combining this lemma with Eq. (1), we get the following proposition.

Proposition 3.2 *If the symbols φ and ψ are both analytic or co-analytic, then the dual Toeplitz operators S_{φ} and S_{ψ} are commutative, i.e., $S_{\varphi}S_{\psi}=S_{\psi}S_{\varphi}$.*

Proof If φ and ψ are both analytic, by Lemma 3.1, for all $f\in (h^2(\mathbb{T}^n))^{\perp}$, there exists $g\in H^2$ such that $H_\varphi H^*_{\overline{\psi}}(f) = H_\varphi(g)$. Since the product of two analytic functions is still analytic, $H_\varphi(g) = Q(\varphi g)\equiv 0$, namely, $H_\psi H^*_{\overline{\varphi}}\equiv 0$, which is equivalent to the fact that $S_{\varphi\psi} = S_\varphi S_\psi = S_\psi S_\varphi$. The same result also can be proved when φ and ψ are both co-analytic through a similar method. □

Theorem 3.3 *Suppose that $\varphi = f + \overline{z}^{m_1}\overline{w}^{n_1}$, $\psi = g+\overline{z}^{m_2}\overline{w}^{n_2}$, where $f, g\in H^\infty(\mathbb{D}^2)$, and the parameters m_1, m_2, n_1, n_2 are non-zero positive integers. Then $S_\varphi S_\psi = S_{\varphi\psi}$ if and only if both f and g are constants.*

Proof By the definition of dual Toeplitz operator, we can get that

$$\begin{aligned} S_\varphi S_\psi &= S_f S_g + S_f S_{\overline{z}^{m_2}\overline{w}^{n_2}} + S_{\overline{z}^{m_1}\overline{w}^{n_1}} S_g + S_{\overline{z}^{m_1}\overline{w}^{n_1}} S_{\overline{z}^{m_2}\overline{w}^{n_2}} \\ &= S_{\varphi\psi} = S_{fg+f\cdot\overline{z}^{m_2}\overline{w}^{n_2}+\overline{z}^{m_1}\overline{w}^{n_1}\cdot g+\overline{z}^{m_1}\overline{w}^{n_1}\cdot\overline{z}^{m_2}\overline{w}^{n_2}}. \end{aligned} \tag{8}$$

Equation (1) and Proposition 3.2 can reduce Eq. (8) to the following formula:

$$\begin{aligned} &S_{f\cdot\overline{z}^{m_2}\overline{w}^{n_2}} - H_f H^*_{z^{m_2}w^{n_2}} + S_{\overline{z}^{m_1}\overline{w}^{n_1}\cdot g} - H_{\overline{z}^{m_1}\overline{w}^{n_1}} H^*_{\overline{g}} \\ &\quad= S_{f\cdot\overline{z}^{m_2}\overline{w}^{n_2}} + S_{\overline{z}^{m_1}\overline{w}^{n_1}\cdot g} \end{aligned} \tag{9}$$

which is equivalent to

$$H_f H^*_{z^{m_2}w^{n_2}} + H_{\overline{z}^{m_1}\overline{w}^{n_1}} H^*_{\overline{g}} = 0. \tag{10}$$

For the reason of the Eqs. (8)–(10), S_φ and S_ψ are semi-commutative if and only if Eq. (10) is set up on $(h^2(\mathbb{T}^2))^{\perp}$. Let

$$f=\sum_{i\ge 0, j\ge 0} a_{ij}z^i w^j,\ g=\sum_{k\ge 0,l\ge 0} b_{kl}z^k w^l,\ (z,w)\in\mathbb{T}^2.$$

Firstly, assume that f and g are both constants. It is clear that $H_f = H_{\overline{g}} = H^*_{\overline{g}} = 0$, so the Eq. (10) is set up, whence $S_\varphi S_\psi{=}S_{\varphi\psi}$. Now we assume that $S_\varphi S_\psi{=}S_{\varphi\psi}$. A little more computation gives that

$$H_f H^*_{z^{m_2}w^{n_2}}(z^\alpha\overline{w}^\beta) = \begin{cases} 0, \quad \alpha > m_2, \\ \sum\limits_{\substack{i>m_2-\alpha \\ 0\le j<\beta+n_2}} a_{ij} z^{\alpha-m_2+i}\overline{w}^{\beta+n_2-j} + \sum\limits_{\substack{0\le i<m_2-\alpha \\ j>\beta+n_2}} a_{ij}\overline{z}^{m_2-\alpha-i} w^{j-\beta-n_2}, \alpha\le m_2, \end{cases} \tag{11}$$

$$H_{\overline{z}^{m_1}\overline{w}^{n_1}} H^*_{\overline{g}}(z^\alpha\overline{w}^\beta) = \begin{cases} \sum\limits_{\substack{k\ge 0 \\ \beta\le l<\beta+n_1}} b_{kl} z^{\alpha-m_1+k}\overline{w}^{\beta+n_1-l}, \quad \alpha > m_1, \\ \sum\limits_{\substack{k>m_1-\alpha \\ \beta\le l<\beta+n_1}} b_{kl} z^{\alpha-m_1+k}\overline{w}^{\beta+n_1-l} + \sum\limits_{\substack{0\le k<m_1-\alpha \\ l>\beta+n_1}} b_{kl}\overline{z}^{m_1-\alpha-k} w^{l-\beta-n_1}, \alpha\le m_1, \end{cases} \tag{12}$$

$$H_f H^*_{z^{m_2}w^{n_2}}(\overline{z}^\alpha w^\beta) = \begin{cases} 0, \quad \beta > n_2, \\ \sum\limits_{\substack{i>m_2+\alpha \\ 0\le j<n_2-\beta}} a_{ij} z^{i-\alpha-m_2}\overline{w}^{n_2-\beta-j} + \sum\limits_{\substack{0\le i<m_2+\alpha \\ j>n_2-\beta}} a_{ij}\overline{z}^{m_2+\alpha-i} w^{\beta-n_2+j}, \beta\le n_2, \end{cases} \tag{13}$$

$$H_{\overline{z}^{m_1}\overline{w}^{n_1}}H_{\overline{g}}^*(\overline{z}^\alpha w^\beta)=\begin{cases}\sum\limits_{\substack{\alpha\le k<\alpha+m_1\\ l\ge 0}} b_{kl}\overline{z}^{\alpha+m_1-k}w^{\beta-n_1+l}, & \beta>n_1,\\ \sum\limits_{\substack{k>m_1+\alpha\\ 0\le l<n_1-\beta}} b_{kl}z^{k-\alpha-m_1}\overline{w}^{n_1-\beta-l}+\sum\limits_{\substack{\alpha\le k<\alpha+m_1\\ l>n_1-\beta}} b_{kl}\overline{z}^{\alpha+m_1-k}w^{\beta-n_1+l}, & \beta\le n_1.\end{cases} \tag{14}$$

We distinguish several cases.

Case 1 $m_1=m_2=m\ge 1$, $n_1=n_2=n\ge 1$. For $\beta>n$, since (13) + (14) = 0, we can get

$$0+\sum_{\substack{\alpha\le k<\alpha+m\\ l\ge 0}} b_{kl}\overline{z}^{\alpha+m-k}w^{\beta-n+l}=0.$$

By the linear independence and the arbitrariness of α, it follows

$$b_{kl}=0,\quad k>0,\ l\ge 0, \tag{15}$$

which means that g is only about w. For $\beta=n$, (13) + (14) = 0 implies

$$\sum_{\substack{0\le i<m+\alpha\\ j>0}} a_{ij}\overline{z}^{m+\alpha-i}w^j=0,$$

hence

$$a_{ij}=0,\quad i\ge 0,\ j>0, \tag{16}$$

which means that f is only about z.

If $\alpha=m$ in the case (11) + (12) = 0, together with the conclusion (16),

$$\sum_{\substack{i>0\\ j=0}} a_{i0}z^i\overline{w}^{\beta+n}=0,$$

hence

$$a_{i0}=0,\quad i\ge 1. \tag{17}$$

If $\alpha>m$ in the case (11) + (12) = 0, together with the conclusion (16), then by the same way, we can get

$$0+\sum_{\substack{k=0\\ \beta\le l<\beta+n}} b_{0l}z^{\alpha-m}\overline{w}^{\beta+n-l}=0,$$

so

$$b_{0l}=0,\quad l>0. \tag{18}$$

Considering the conclusions (15)–(18), both f and g are constants.

Case 2 $m_1=m_2=m\ge 1$, $n_2\ne n_1$. Without loss of generality, we can assume $n_2>n_1\ge 1$. For $\beta>n_2>n_1$, it follows from (13) + (14) = 0 that

$$0+\sum_{\substack{\alpha\le k<\alpha+m\\ l\ge 0}} b_{kl}\overline{z}^{\alpha+m-k}w^{\beta-n_1+l}=0,$$

so

$$b_{kl}=0,\quad k>0,\ l\ge 0. \tag{19}$$

For $\beta = n_2 > n_1$,

$$\sum_{\substack{0\le i<m+\alpha\\ j>0}} a_{ij}\overline{z}^{m+\alpha-i}w^j + \sum_{\substack{\alpha\le k<\alpha+m\\ l\ge 0}} b_{kl}\overline{z}^{\alpha+m-k}w^{\beta-n_1+l} = \sum_{\substack{0\le i<m+\alpha\\ j>0}} a_{ij}\overline{z}^{m+\alpha-i}w^j = 0,$$

which means that

$$a_{ij} = 0, \quad i \ge 0,\ j > 0. \tag{20}$$

We also need to consider the condition $(11) + (12) = 0$.

For $\alpha = m$, together with (19) and (20), the equation can be reduced to

$$\sum_{\substack{i>0\\ j=0}} a_{i0} z^i \overline{w}^{\beta+n_2} + 0 = 0,$$

hence

$$a_{i0} = 0, \quad i > 0. \tag{21}$$

For $\alpha > m \ge 1$,

$$b_{0l} = 0, \quad l > 0. \tag{22}$$

By the conclusions (19)–(22), the result that f and g are both constants can be observed immediately.

Case 3 $m_1 \ne m_2$, $n_1 = n_2 = n \ge 1$. It is easy to complete the proof by the similar way used in Case 2.

Case 4 $m_1 \ne m_2$, $n_1 \ne n_2$. Without loss of generality, we can assume that $m_2 > m_1 \ge 1$, $n_1 > n_2 \ge 1$. For $\alpha > m_2 > m_1$,

$$0 + \sum_{\substack{k\ge 0\\ \beta\le l<\beta+n_1}} b_{kl} z^{\alpha-m_1+k}\overline{w}^{\beta+n_1-l} = 0,$$

which means that

$$b_{kl} = 0, \quad k \ge 0,\ l > 0. \tag{23}$$

For $\alpha = m_2 > m_1$, together with (23), the following equation is checked easily:

$$0 + \sum_{\substack{i>0\\ 0\le j<\beta+n_2}} a_{ij} z^i \overline{w}^{\beta+n_2-j} = 0,$$

which implies that

$$a_{ij} = 0, \quad i > 0,\ j \ge 0. \tag{24}$$

Through a similar discussion of β, we can get

$$a_{0j} = 0,\ j > 0; \quad b_{k0} = 0,\ k > 0. \tag{25}$$

Therefore, the proof can be completed together with the conclusions (23)–(25). □

In Theorem 3.3, each parameter should be non-zero positive integer. Next theorem considers the case of $m_1 = m_2 = 0$.

Theorem 3.4 *Suppose that* $\varphi = f + \overline{w}^{n_1}$, $\psi = g + \overline{w}^{n_2}$, *where* $f, g \in H^\infty(\mathbb{D}^2)$, *and the parameters* n_1, n_2 *are non-zero positive integers. Then the following conclusions can be established:*

(i) *If* $n_1 = n_2$, $S_\varphi S_\psi = S_{\varphi\psi}$ *if and only if* f *and* g *are both only about* z, *and* $f + g = a + bz$;

(ii) *If* $n_1 \neq n_2$, $S_\varphi S_\psi = S_{\varphi\psi}$ *if and only if* $f = a_0 + a_1 z$, $g = b_0 + b_1 z$.

Proof Let $m_1 = m_2 = 0$ in the equation that $(11) + (12) = (13) + (14) = 0$. Then the theorem can be proved by a discussion of α and β as we have done before in this paper. □

References

[1] A. BROWN, P. R. HALMOS. *Algebraic properties of toeplitz operators.* J. Reine Angew. Math., 1964, **213**(1): 89–102.

[2] K. STROETHOFF, Dechao ZHENG. *Algebraic and spectral properties of dual Toeplitz operators.* Trans. Amer. Math. Soc., 2002, **354**(6): 2495–2520.

[3] Yufeng LU. *Commuting dual Toeplitz operators with pluriharmonic symbols.* J. Math. Anal. Appl., 2005, **302**(1): 738–749.

[4] Yufeng LU, Shuxia SHANG. *Commuting dual Toeplitz operators on the polydisk.* Acta Math. Sin. (Engl. Ser.), 2007, **23**(5): 857–868.

[5] Jingyu YANG, Yufeng LU. *Commuting dual Toeplitz operators on the harmonic Bergman space.* Sci. China Math., 2015, **58**(7): 1461–1472.

[6] Tao YU, Shiyue WU. *Algebraic properties of dual Toeplitz operators on the orthogonal complement of the Dirichlet space.* Acta Math. Sin. (Engl. Ser.), 2008, **24**(11): 1843–1852.

[7] Y. J. LEE. *Commuting dual Toeplitz operators on the polydisk.* J. Math. Anal. Appl., 2008, **341**(1): 738–749.

[8] G. HOCINE. *Dual Toeplitz operators on the sphere.* Acta Math. Sin. (Engl. Ser.), 2013, **29**(9): 1791–1808.