Journal of Mathematical Research with Applications
Nov., 2016, Vol. 36, No. 6, pp. 723–731
DOI:10.3770/j.issn:2095-2651.2016.06.012
Http://jmre.dlut.edu.cn

A Class of Complete Hypersurfaces Immersed in Semi-Riemannian Warped Product Spaces

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Abstract We deal with complete hypersurfaces immersed in a semi-Riemannian warped product of the type $\epsilon I \times_f M^n$, where M^n is a connected n-dimensional oriented Riemannian manifold. When the fiber M^n is complete with sectional curvature $-k \leq K_M$ for some positive constant k, under appropriate restrictions on the norm of the gradient of the height function h, we proceed with our technique in order to guarantee that complete hypersurface immersed in a semi-Riemannian warped product is a slice. Our approach is based on the well known generalized maximum principle and another suitable maximum principle at the infinity due to Yau.

Keywords semi-Riemannian manifold; warped product; complete hypersurface; height function

MR(2010) Subject Classification 53C42

1. Introduction

In this paper, we are interested in the study of the complete hypersurfaces immersed in a semi-Riemannian warped product of the type $\epsilon I \times_f M^n$, where M^n is a connected n-dimensional oriented Riemannian manifold, $I \subseteq \mathbb{R}$ is an open interval, $f : I \to \mathbb{R}$ is a positive smooth function, and $\epsilon = \pm 1$. Many authors have approached problems in this subject. Among them, we may cite Alias, Romero and Sanchez [1,2], Alias, Romero and Sanchez [3], Montiel [4,5], and Caballero, Romero and Rubio [6–8].

Some works have been attained on the study of hypersurfaces with constant mean curvature immersed in warped product spaces. In [9], Albujer, Camargo and de Lima have obtained some uniqueness results concerning complete spacelike hypersurfaces with constant mean curvature immersed in a Robertson-Walker spacetime (that is, a lorentzian warped product $-I \times_f M^n$ whose fiber M^n has constant sectional curvature). In [10], Aquino and de Lima investigated the rigidity of complete vertical graphs with constant mean curvature in a Riemannian warped product $I \times_f M^n$.

On the other hand, under suitable restrictions on the values of the mean curvature and the norm of the gradient of the height function, they obtained uniqueness theorems concerning such graphs. Aquino and de Lima [11] studied the unicity of complete hypersurfaces immersed in a semi-Riemannian warped product, which is supposed to obey a suitable convergence condition.

Received October 19, 2015; Accepted March 9, 2016
Supported by the National Natural Science Foundation of China (Nos. 11371076; 11431009).
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In this paper, our purpose is to establish new sufficient conditions to guarantee that a complete hypersurface immersed in a semi-Riemannian warped product of the type $\epsilon I \times_f M^n$ must be a slice. Inspired by the works of Aquino and de Lima in [11], we obtain Theorems 3.1, 3.2 and 3.3 by using the generalized maximum principle of Omori and Yau [12,13] when there are some suitable restrictions on the mean curvature and the norm of the gradient of the height function h.

2. Preliminaries

In this section, we introduce some basic notations and facts that will appear along the paper. let $\overline{M}^{n+1}$ be a connected semi-Riemannian manifold with metric $\overline{g} = \langle , \rangle$ of index $v \leq 1$, and semi-Riemannian connection $\overline{\nabla}$. Let $D(\overline{M})$ denote the ring of smooth functions $F : \overline{M}^{n+1} \to \mathbb{R}$ and TM the algebra of smooth vector fields on $\overline{M}^{n+1}$. For a vector field $X \in T\overline{M}^{n+1}$, let $\epsilon_X = \langle X, X\rangle$; X is a unit vector field if $\epsilon_X = \pm 1$, and timelike if $\epsilon_X = -1$.

In all that follows, we consider Riemannian immersion $\psi : \Sigma^n \to \overline{M}^{n+1}$, namely, immersion from a connected n-dimensional orientable differentiable manifold Σ^n into $\overline{M}^{n+1}$, such that the induced metric $g = \psi^*(\overline{g})$ turns Σ^n into a Riemannian manifold (in the Lorentzian case $v = 1$ and we refer to (Σ^n, g) as a spacelike hypersurface of $\overline{M}^{n+1}$), with Levi-Civita connection ∇. We orient Σ^n by the choice of a unit normal vector field N on it.

Let M^n be a connected n-dimensional orientable Riemannian manifold, $I \subseteq \mathbb{R}$ an interval, and $f : I \to \mathbb{R}$ a positive smooth function. In the product differentiable manifold $\overline{M}^{n+1} = \epsilon I \times_f M^n$, let π_I and π_M denote the projections onto the I and M factors, respectively. $\overline{M}$ with the metric $\langle v, w\rangle_p = \epsilon\langle (\pi_l)_* v, (\pi_l)_* w\rangle + f(p)^2 \langle (\pi_M)_* v, (\pi_M)_* w\rangle$, for all $p \in \overline{M}$ and all $v, w \in T_p\overline{M}$, where $\epsilon = \epsilon_{\partial t}$ and ∂t is the standard unit vector field tangent to I.

Let $\overline{\nabla}$ and ∇ denote the Levi-Civita connections in $\epsilon I \times_f M^n$ and Σ^n, respectively. Then the Gauss-Weingarten formulas for the spacelike hypersurface $\psi : \Sigma^n \to \epsilon I \times_f M^n$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y + \epsilon\langle AX, Y\rangle N, \quad AX = -\overline{\nabla}_X N,$$

for any $X \in T\Sigma^n$, where $A : T\Sigma^n \to T\Sigma^n$ is the shape operator of Σ^n with respect to its Gauss map N, and $T\Sigma^n$ denotes the Lie algebra of all tangential vector fields on Σ^n.

In this context, we consider two particular functions naturally attached to such a hypersurface Σ^n, namely, the (vertical) height function $h = (\pi_I)\,|_\Sigma$ and the angle function $\eta = \langle N, \partial_t\rangle$. The gradient of π_I on $\epsilon I \times_f M^n$ and the gradient of h on Σ^n are given by:

$$\overline{\nabla}\pi_I = \epsilon\langle \overline{\nabla}\pi_I, \partial_t\rangle \partial_t = \epsilon\partial_t, \quad \nabla h = (\overline{\nabla}\pi_I)^\top = \epsilon\partial_t^\top = \varepsilon\partial_t - \langle N, \partial_t\rangle N,$$

where $(\ \ )^\top$ denotes the tangential component of a vector field in $T\overline{M}^{n+1}$ along Σ^n. In particular, we get $|\nabla h|^2 = \epsilon(1 - \langle N, \partial_t\rangle^2)$, where $|\ \ |$ denotes the norm of a vector field on Σ^n.

On the other hand, the curvature tensor R of a spacelike hypersurface Σ^n is given by

$$R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where $[,]$ denotes the Lie bracket and $X, Y, Z \in T\Sigma^n$. We can describe the curvature tensor R of the hypersurface Σ^n in terms of the shape operator A and the curvature tensor $\overline{R}$ of $\mathbb{R} \times M^n$ by the so-called Gauss equation given by $R(X,Y)Z = (\overline{R}(X,Y)Z)^{\top} + \epsilon\langle AX, Z\rangle AY - \epsilon\langle AY, Z\rangle AX$, for every tangent vector fields $X, Y, Z \in T\Sigma^n$.

In the Lorentzian setting, the following result is a particular case of one obtained by Alias and Colares in [1].

Lemma 2.1 ([1]) *Let $\psi : \Sigma^n \to \epsilon I \times_f M^n$ be a Riemannain immersion. If $h = (\pi_I)\mid_\Sigma : \Sigma^n \to I$ is the height function of Σ^n, then*

$$L_r(h) = (\log f)'(\epsilon \mathrm{tr}\, P_r - \langle P_r \nabla h, \nabla h\rangle) + \mathrm{tr}\,(AP_r)\langle N, \partial_t\rangle. \tag{2.1}$$

In order to prove our results, we need the generalized maximum principle of Omori and Yau [12,13], which will be essential in order to establish our results.

Lemma 2.2 ([12,13]) *Let Σ^n be an n-dimensional complete Riemannain manifold whose Ricci curvature is bounded from below and $u : \Sigma^n \to \mathbb{R}$ be a smooth function which is bounded from above on Σ^n. Then there is a sequence of points $p_k \in \Sigma^n$ such that*

$$\lim_k u(p_k) = \sup_\Sigma u, \quad \lim_k |\nabla u(p_k)| = 0, \quad \lim_k \sup \triangle u(p_k) \le 0.$$

Lemma 2.3 ([12,13]) *Let Σ^n be an n-dimensional complete Riemannain manifold whose Ricci curvature is bounded from below and $u : \Sigma^n \to \mathbb{R}$ be a smooth function which is bounded from below on Σ^n. Then there is a sequence of points $p_k \in \Sigma^n$ such that*

$$\lim_k u(p_k) = \inf_\Sigma u, \quad \lim_k |\nabla u(p_k)| = 0, \quad \lim_k \inf \triangle u(p_k) \ge 0.$$

3. Main results

In this section, we will study the complete hypersurfaces immersed in a semi-Riemannian warped product of the type $\epsilon I \times_f M^n$, where M^n is a connected n-dimensional oriented Riemannian manifold. In what follows, $H_2 = \frac{2}{n(n-1)} S_2$ stands for the 2-mean curvature of the hypersurface Σ^n, that is, the mean value of the second elementary symmetric function S_2 on the eigenvalues of its Weingarten operator A.

Theorem 3.1 *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a Lorentzian warped product whose fiber M^n has sectional curvature $-k \le K_M$ for some positive constant k, and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface contained in a slab $[t_1, t_2] \times M^n$ of $\overline{M}^{n+1}$, suppose that the future mean curvature H is bounded, and one of the following conditions is satisfied:*

(1) $H \ge \max\{\frac{f'}{f}(h), 0\}$ and $|\nabla h| \le \inf_\Sigma (H - \frac{f'}{f}(h))$;

(2) $H \le \min\{\frac{f'}{f}(h), 0\}$ and $|\nabla h| \le \inf_\Sigma (\frac{f'}{f}(h) - H)$;

then Σ^n is a slice.

Proof First we claim the Ricci curvature of Σ^n is bounded from below. Let $X \in T\Sigma^n$ and $\{E_1, \ldots, E_n\}$ be a local orthonormal frame of $T\Sigma^n$. It follows from Gauss equation that

$$\mathrm{Ric}(X,X) = \sum_i \langle \overline{R}(X, E_i)X, E_i\rangle + nH\langle AX, X\rangle + \langle AX, AX\rangle$$

$$\geq \sum_i \langle \overline{R}(X, E_i)X, E_i\rangle - \frac{n^2H^2}{4}|X|^2, \tag{3.1}$$

$$\sum_i \langle \overline{R}(X, E_i)X, E_i\rangle = \sum_i \langle \overline{R}_M(X^*, E_i^*)X^*, E_i^*\rangle + (n-1)((\log f)')^2|X|^2 - (n-2)(\log f)''\langle X, \nabla h\rangle^2 - (\log f)''|\nabla h|^2|X|^2, \tag{3.2}$$

where R_M denotes the curvature tensor of M^n and $X^* = X + \langle X, \partial_t\rangle\partial_t$, $E_i^* = E_i + \langle E_i, \partial_t\rangle\partial_t$ are the projections of the tangent vector fields X and E_i onto M^n.

$$\begin{aligned}\sum_i \langle \overline{R}_M(X^*, E_i^*)X^*, E_i^*\rangle &= \sum_i f^2(|X^*|_M^2|E_i^*|_M^2 - \langle X^*, E_i^*\rangle^2)K_M(X^*, E_i^*)\\ &\geq \frac{-k}{f^2}((n-1)|X|^2 + |\nabla h|^2|X|^2 + (n-2)\langle X, \nabla h\rangle^2)\\ &\geq \frac{-k}{f^2}(n-1)(1+|\nabla h|^2)|X|^2. \end{aligned} \tag{3.3}$$

Substituting (3.3) into (3.2), we get

$$\begin{aligned}\sum_i \langle \overline{R}(X, E_i)X, E_i\rangle \geq & \frac{-k}{f^2}(n-1)(1+|\nabla h|^2)|X|^2 + (n-1)((\log f)')^2|X|^2 -\\ & (n-2)(\log f)''\langle X, \nabla h\rangle^2 - (\log f)''|\nabla h|^2|X|^2\\ = & \frac{-k}{f^2}(n-1)(1+|\nabla h|^2)|X|^2 + \frac{f'^2}{f^2}(n-1)|X|^2 + \frac{f'^2}{f^2}(n-2)\langle X, \nabla h\rangle^2 +\\ & \frac{f'^2}{f^2}|\nabla h|^2)|X|^2 - \frac{f''}{f}(n-2)\langle X, \nabla h\rangle^2 - \frac{f''}{f}|\nabla h|^2|X|^2\\ \geq & \frac{-k}{f^2}(n-1)(1+|\nabla h|^2)|X|^2 - \frac{|f''|}{f}(n-1)|\nabla h|^2|X|^2. \end{aligned} \tag{3.4}$$

From (3.4) and (3.3), we get

$$\mathrm{Ric}(X, X) \geq \frac{-k}{f^2}(n-1)(1+|\nabla h|^2)|X|^2 - \frac{|f''|}{f}(n-1)|\nabla h|^2|X|^2 - \frac{n^2H^2}{4}|X|^2.$$

Consequently, since Σ^n is supposed to be contained in a slab of $-I \times_f M^n$ and H is supposed to be bounded, from the restriction of the gradient of h, we conclude that the Ricci curvature of Σ^n is bounded from below.

(1) Let us suppose that the condition of item (1) is satisfied.

From (2.1) we have that $\triangle h = -\frac{f'}{f}(h)|\nabla h|^2 - n(\frac{f'}{f}(h) + H\langle N, \partial_t\rangle)$. Hence, we can appply Lemma 2.2 to the height function h obtaining a sequence of points $\{p_k\}$ in Σ^n, such that

$$\lim_k h(p_k) = \sup_\Sigma h, \quad \lim_k |\nabla h(p_k)| = 0, \quad \lim_k \sup \triangle h(p_k) \leq 0.$$

Consequently,

$$\begin{aligned} 0 \geq \lim_k \sup \triangle h(p_k) &= \lim_k \sup[-\frac{f'}{f}(h)|\nabla h|^2 - n(\frac{f'}{f}(h) + H\langle N, \partial_t\rangle)](p_k)\\ &\geq \lim_k \sup n(H - \frac{f'}{f}(h))(p_k) \geq 0. \end{aligned}$$

Then, we have that $\lim_k \sup(H - \frac{f'}{f}(h))(p_k) = 0$. Hence, $\inf_\Sigma(H - \frac{f'}{f}(h)) = 0$. Therefore, from our hypothesis $|\nabla h| \leq \inf_\Sigma(H - \frac{f'}{f}(h))$, we conclude that Σ^n is a slice of $-I \times_f M^n$.

(2) Let us suppose that the condition of item (2) is satisfied. Hence, we can appply Lemma 2.3 to the height function h obtaining a sequence of points $\{p_k\}$ in Σ^n, such that

$$\lim_k h(p_k) = \inf_\Sigma h, \quad \lim_k |\nabla h(p_k)| = 0, \quad \lim_k \inf \triangle h(p_k) \geq 0.$$

Consequently, we have that

$$\begin{aligned} 0 \leq \lim_k \inf \triangle h(p_k) &= \lim_k \inf[-\frac{f'}{f}(h)|\nabla h|^2 - n(\frac{f'}{f}(h) + H\langle N, \partial_t\rangle)](p_k) \\ &\leq \lim_k \inf n(H - \frac{f'}{f}(h))(p_k) \leq 0. \end{aligned}$$

Then, we have that $\lim_k \inf(H - \frac{f'}{f}(h))(p_k) = 0$. Hence, $\sup_\Sigma(H - \frac{f'}{f}(h)) = 0$. Then we have $\inf_\Sigma(\frac{f'}{f}(h) - H) = 0$. Therefore, from our hypothesis $|\nabla h| \leq \inf_\Sigma(\frac{f'}{f}(h) - H)$, we conclude that Σ^n is a slice of $-I \times_f M^n$. $\square$

Theorem 3.2 *Let $\overline{M}^{n+1} = I \times_f M^n$ be a Riemannian warped product whose fiber M^n has sectional curvature $-k \leq K_M$ for some positive constant k, and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface contained in a slab $[t_1, t_2] \times M^n$ of $\overline{M}^{n+1}$, with $f'(t) > 0$ for $t_1 \leq t \leq t_2$. Suppose that H_2 is bounded from below, and one of the following conditions is satisfied:*

(1) $0 < H \leq \frac{f'}{f}(h)$ *and* $|\nabla h| \leq \inf_\Sigma(\frac{f'}{f}(h) - H)$;

(2) $-\frac{f'}{f}(h) \leq H < 0$ *and* $|\nabla h| \leq \inf_\Sigma(\frac{f'}{f}(h) + H)$,

then Σ^n is a slice.

Proof First we claim the Ricci curvature of Σ^n is bounded from below. Let $X \in T\Sigma^n$ and $\{E_1, \ldots, E_n\}$ be a local orthonormal frame of $T\Sigma^n$. It follows from Gauss equation that

$$\begin{aligned} \mathrm{Ric}(X, X) &= \sum_i \langle \overline{R}(X, E_i)X, E_i\rangle + nH\langle AX, X\rangle - \langle AX, AX\rangle \\ &\geq \sum_i \langle \overline{R}(X, E_i)X, E_i\rangle - (n|H||A| + |H|^2)|X|^2, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \sum_i \langle \overline{R}(X, E_i)X, E_i\rangle &= \sum_i \langle \overline{R}_M(X^*, E_i^*)X^*, E_i^*\rangle - \frac{f''}{f}|X|^2 + \\ &((\log f)')^2(|\nabla h|^2 - (n-1))|X|^2 - (n-2)(\log f)''\langle X, \nabla h\rangle^2, \end{aligned} \tag{3.6}$$

where $X^* = X - \langle X, \partial_t\rangle\partial_t$, $E_i^* = E_i - \langle E_i, \partial_t\rangle\partial_t$ are the projections of the tangent vector fields X and E_i onto M^n.

$$\begin{aligned} \sum_i \langle \overline{R}_M(X^*, E_i^*)X^*, E_i^*\rangle &= f^2(|X^*|_M^2|E_i^*|_M^2 - \langle X^*, E_i^*\rangle^2)K_M(X^*, E_i^*) \\ &\geq \frac{-k}{f^2}((n-1)|X|^2 + |\nabla h|^2|X|^2 + (n-2)\langle X, \nabla h\rangle^2) \\ &\geq \frac{-k}{f^2}(n-1)(1 + |\nabla h|^2)|X|^2. \end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.6), we get

$$\begin{aligned}\sum_i\langle\overline{R}(X,E_i)X,E_i\rangle \geq&\frac{-k}{f^2}(n-1)(1+|\nabla h|^2)|X|^2+\frac{f'^2}{f^2}|\nabla h|^2)|X|^2+\\ &\frac{f'^2}{f^2}(n-2)\langle X,\nabla h\rangle^2-\frac{f''}{f}|X|^2-\frac{f'^2}{f^2}(n-1)|X|^2-\frac{f''}{f}(n-2)\langle X,\nabla h\rangle^2\\ \geq&\frac{-k}{f^2}(n-1)(1+|\nabla h|^2)|X|^2--\frac{|f''|}{f}(n-2)|\nabla h|^2|X|^2-\\ &\frac{f''}{f}|X|^2-\frac{f'^2}{f^2}(n-1))|X|^2. \end{aligned}\tag{3.8}$$

From (3.5) and (3.8), we get

$$\begin{aligned}\mathrm{Ric}(X,X)\geq&\frac{-k}{f^2}(n-1)(1+|\nabla h|^2)|X|^2-\frac{|f''|}{f}(n-2)|\nabla h|^2|X|^2-\\ &\frac{f''}{f}|X|^2-\frac{f'^2}{f^2}(n-1))|X|^2-(n|H||A|+|H|^2)|X|^2.\end{aligned}$$

Since Σ^n is supposed to be contained in a slab of $I\times_f M^n$ and H and H_2 are supposed to be bounded, from $|\nabla h|\leq\inf_\Sigma(\frac{f'}{f}(h)-H)$, we conclude that the Ricci curvature of Σ^n is bounded from below.

From (2.1) we have that $\triangle h=-\frac{f'}{f}(h)|\nabla h|^2+n(\frac{f'}{f}(h)+H\langle N,\partial_t\rangle)$. Since Σ^n is a complete spacelike hypersurface contained in a slab $[t_1,t_2]\times M^n$ of $\overline{M}^{n+1}$, we can apply Lemma 2.2 to the height function h obtaining a sequence of points $\{p_k\}$ in Σ^n, such that

$$\lim_k h(p_k)=\sup_\Sigma h,\quad \lim_k|\nabla h(p_k)|=0,\quad \limsup_k\triangle h(p_k)\leq 0.$$

(1) Suppose that the mean curvature H satisfies $0<H\leq\frac{f'}{f}(h)$, consequently, we have that

$$\begin{aligned}0\geq\limsup_k\triangle h(p_k)&=\limsup_k[-\frac{f'}{f}(h)|\nabla h|^2+n(\frac{f'}{f}(h)+H\langle N,\partial_t\rangle)](p_k)]\\ &\geq\limsup_k n(\frac{f'}{f}(h)-H)(p_k)\geq 0.\end{aligned}$$

Then, we get $\lim_k\sup(\frac{f'}{f}(h)-H)(p_k)=0$. Hence, $\inf_\Sigma(\frac{f'}{f}(h)-H)=0$. Therefore, from our hypothesis $|\nabla h|\leq\inf_\Sigma(\frac{f'}{f}(h)-H)$, we conclude that Σ^n is a slice of $I\times_f M^n$.

(2) Suppose that the mean curvature H satisfies $-\frac{f'}{f}(h)\leq H<0$, consequently, we have that

$$\begin{aligned}0\geq\limsup_k\triangle h(p_k)&=\limsup_k[-\frac{f'}{f}(h)|\nabla h|^2+n(\frac{f'}{f}(h)+H\langle N,\partial_t\rangle)](p_k)]\\ &\geq\limsup_k n(\frac{f'}{f}(h)+H)(p_k)\geq 0.\end{aligned}$$

Then, we get $\lim_k\sup(\frac{f'}{f}(h)+H)(p_k)=0$. Hence, $\inf_\Sigma(\frac{f'}{f}(h)+H)=0$. Therefore, from our hypothesis $|\nabla h|\leq\inf_\Sigma(\frac{f'}{f}(h)+H)$, we conclude that Σ^n is a slice of $I\times_f M^n$. □

Theorem 3.3 *Let $\overline{M}^{n+1}=I\times_f M^n$ be a Riemannian warped product whose fiber M^n has sectional curvature $-k\leq K_M$ for some positive constant k, and let $\psi:\Sigma^n\to\overline{M}^{n+1}$ be a* *complete spacelike hypersurface contained in a slab* $[t_1, t_2] \times M^n$ *of* $\overline{M}^{n+1}$, *with* $f'(t) < 0$ *for* $t_1 \le t \le t_2$. *Suppose that* H_2 *is bounded from below, and one of the following conditions is satisfied:*

(1) $\frac{f'}{f}(h) \le H < 0$ *and* $|\nabla h| \le \inf_\Sigma (H - \frac{f'}{f}(h))$;

(2) $0 < H \le -\frac{f'}{f}(h)$ *and* $|\nabla h| \le \inf\limits_\Sigma -(H + \frac{f'}{f}(h))$,

then Σ^n *is a slice.*

Proof From the proof of Theorem 3.2, we have that Ricci curvature of Σ^n is bounded from below. Since Σ^n is a complete spacelike hypersurface contained in a slab $[t_1, t_2] \times M^n$ of $\overline{M}^{n+1}$, we can apply Lemma 2.3 to the height function h obtaining a sequence of points $p_k \in \Sigma^n$, such that

$$\lim_k h(p_k) = \inf_\Sigma h, \quad \lim_k |\nabla h(p_k)| = 0, \quad \liminf_k \triangle h(p_k) \ge 0.$$

(1) Suppose that the mean curvature H satisfies $\frac{f'}{f}(h) \le H < 0$, consequently, we have

$$\begin{aligned} 0 &\le \liminf_k \triangle h(p_k) = \liminf_k [-\frac{f'^2}{f}(h)|\nabla h|^2 + n(\frac{f'}{f}(h) + H\langle N, \partial_t\rangle)](p_k)] \\ &\le \liminf_k n(\frac{f'}{f}(h) - H)(p_k) \le 0. \end{aligned}$$

Then, we have that $\lim_k \inf(\frac{f'}{f}(h) - H)(p_k) = 0$. Hence, $\sup_\Sigma(\frac{f'}{f}(h) - H) = 0$.

Therefore, we have $\inf_\Sigma(H - \frac{f'}{f}(h)) = 0$. From our hypothesis $|\nabla h| \le \inf_\Sigma(H - \frac{f'}{f}(h))$, we conclude that Σ^n is a slice of $I \times_f M^n$.

(2) Suppose that the mean curvature H satisfies $0 < H \le -\frac{f'}{f}(h)$, consequently, we have that

$$\begin{aligned} 0 &\le \liminf_k \triangle h(p_k) = \liminf_k [-\frac{f'^2}{f}(h)|\nabla h|^2 + n(\frac{f'}{f}(h) + H\langle N, \partial_t\rangle)](p_k)] \\ &\le \liminf_k n(\frac{f'}{f}(h) + H)(p_k) \le 0. \end{aligned}$$

Then, we have that $\lim_k \inf(\frac{f'}{f}(h) + H)(p_k) = 0$. Hence, $\sup\limits_\Sigma(\frac{f'}{f}(h) + H) = 0$.

Therefore, we have $\inf_\Sigma -(H + \frac{f'}{f}(h)) = 0$. From our hypothesis $|\nabla h| \le \inf_\Sigma -(H + \frac{f'}{f}(h))$, we conclude that Σ^n is a slice of $I \times_f M^n$. □

From Theorem 3.1, if $f'(t) > 0$ for $t_1 \le t \le t_2$, we get $H \ge \frac{f'}{f}(h)$. Then as a direct consequence of Theorem 3.1, we have the following Corollary 3.4.

Corollary 3.4 ([11, Theorem 4.2]) *Let* $\overline{M}^{n+1} = -I \times_f M^n$ *be a Lorentzian warped product whose fiber* M^n *has sectional curvature* K_M *satisfying the following convergence condition:*

$$K_M \ge \sup_I (ff'' - (f')^2).$$

Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ *be a complete spacelike hypersurface contained in a slab* $[t_1, t_2] \times M^n$ *of* $\overline{M}^{n+1}$, *with* $f'(t) > 0$ *for* $t_1 \le t \le t_2$. *Suppose that the future mean curvature* H *is bounded, satisfying:* $H \ge \frac{f'}{f}(h)$. *If* $|\nabla h| \le \inf_\Sigma(H - \frac{f'}{f}(h))$; *then* Σ^n *is a slice.*

From Theorem 3.2 we obtain the following result.

Corollary 3.5 ([11, Theorem 4.3]) *Let $\overline{M}^{n+1} = I \times_f M^n$ be a Lorentzian warped product whose fiber M^n has sectional curvature K_M satisfying the following convergence condition:*

$$K_M \geq \sup_I ((f')^2 - ff'').$$

Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface contained in a slab $[t_1, t_2] \times M^n$ of $\overline{M}^{n+1}$, with $f'(t) > 0$ for $t_1 \leq t \leq t_2$. Suppose that the angle function $\langle N, \partial_t \rangle$ is negative on Σ^n, the mean curvature H satisfies $0 < H \leq \frac{f'}{f}(h)$ and H_2 is bounded from below. If $|\nabla h| \leq \inf_\Sigma(\frac{f'}{f}(h) - H)$; then Σ^n is a slice.

As a direct consequence of Theorems 3.1, 3.2 and 3.3, we have the following corollary.

Corollary 3.6 *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a Lorentzian warped product whose fiber M^n has sectional curvature $-k \leq K_M$ for some positive constant k, and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface contained in a slab $[t_1, t_2] \times M^n$ of $\overline{M}^{n+1}$. Suppose that the future mean curvature H is bounded, and one of the following conditions is satisfied:*

(1) $H \geq \max\{\frac{f'}{f}(h), 0\}$ and $|\nabla h| \leq \alpha \inf_\Sigma(H - \frac{f'}{f}(h))^\beta$;

(2) $H \leq \min\{\frac{f'}{f}(h), 0\}$ and $|\nabla h| \leq \alpha \inf_\Sigma(\frac{f'}{f}(h) - H)^\beta$

for some positive constant α and β, then Σ^n is a slice.

Corollary 3.7 *Let $\overline{M}^{n+1} = I \times_f M^n$ be a Riemannian warped product whose fiber M^n has sectional curvature $-k \leq K_M$ for some positive constant k, and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface contained in a slab $[t_1, t_2] \times M^n$ of $\overline{M}^{n+1}$, with $f'(t) > 0$ for $t_1 \leq t \leq t_2$. Suppose that H_2 is bounded from below, and one of the following conditions is satisfied:*

(1) $0 < H \leq \frac{f'}{f}(h)$ and $|\nabla h| \leq \alpha \inf_\Sigma(\frac{f'}{f}(h) - H)^\beta$

(2) $-\frac{f'}{f}(h) \leq H < 0$ and $|\nabla h| \leq \alpha \inf_\Sigma(\frac{f'}{f}(h) + H)^\beta$

for some positive constant α and β, then Σ^n is a slice.

Corollary 3.8 *Let $\overline{M}^{n+1} = I \times_f M^n$ be a Riemannian warped product whose fiber M^n has sectional curvature $-k \leq K_M$ for some positive constant k, and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface contained in a slab $[t_1, t_2] \times M^n$ of $\overline{M}^{n+1}$, with $f'(t) < 0$ for $t_1 \leq t \leq t_2$. Suppose that H_2 is bounded from below, and one of the following conditions is satisfied:*

(1) $\frac{f'}{f}(h) \leq H < 0$ and $|\nabla h| \leq \alpha \inf_\Sigma(H - \frac{f'}{f}(h))^\beta$

(2) $0 < H \leq -\frac{f'}{f}(h)$ and $|\nabla h| \leq \alpha \inf_\Sigma -(H + \frac{f'}{f}(h))^\beta$

for some positive constants α and β, then Σ^n is a slice.

Acknowledgements We thank the referees for their time and comments.

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