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A Unified Approach to Construct a Class of Daubechies Orthogonal Scaling Functions

Tianxiao HE[1,*], **Tung NGUYEN**[2]

1. *Department of Mathematics, Illinois Wesleyan University, Bloomington, IL* 61702-2900, *USA;*
2. *Department of Mathematics, University of Wisconsin-Madison, Madison, WI* 53706-1325, *USA*

Dedicated to Professor Renhong WANG on the Occasion of His Eightieth Birthday

Abstract We use Lorentz polynomials to give an efficient way to prove Daubechies' results on the existence of spline type orthogonal scaling functions and to evaluate a class of Daubechies scaling functions in a unified approach.

Keywords Daubechies scaling functions; MRA scaling function; B-spline; Lorentz polynomial

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1. Introduction and main results

Let $\phi \in L_2(\mathbb{R})$ be an MRA scaling function satisfying

$$\phi(x) = \sum_{k=-\infty}^{\infty} p_k \phi(2x-k), \tag{1.1}$$

i.e., ϕ is of a two-scale refinable property. By taking a Fourier transformation on both sides of (1.1) and denoting the Fourier transformation of ϕ by $\hat{\phi}(\xi) := \int_{-\infty}^{\infty} \phi(x) e^{-i\xi x} \mathrm{d}x$, we have

$$\hat{\phi}(\xi) = P(z)\hat{\phi}(\frac{\xi}{2}), \tag{1.2}$$

where

$$P(z) = \frac{1}{2} \sum_{k=-\infty}^{\infty} p_k z^k \text{ and } z = e^{-i\xi/2}. \tag{1.3}$$

Here, $P(z)$ is called the mask of the scaling function. Now, regarding the property that $\{\phi(x-k)\}$ must be an orthonormal basis, we have the following characterization theorem (see, for example, Chs. 2, 5 and 7 of Chui [1] and Ch. 3 of Hernándes and Weiss [2]).

Theorem 1.1 *Suppose the function ϕ satisfies the refinement relation $\phi(x) = \sum_{-\infty}^{\infty} p_k \phi(2x-k)$. Then (i) $\{\phi(x-k) : k \in \mathbb{Z}\}$ forms an orthonormal basis only if $|P(z)|^2 + |P(-z)|^2 = 1$ for $z \in \mathbb{C}$ with $|z| = 1$. (ii) Suppose $P(z)$ satisfies*

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* Corresponding author
E-mail address: the@iwu.edu (Tianxiao HE)

(a) $P(z) \in C^1$ *and is* 2π*-periodic;*

(b) $|P(z)|^2 + |P(-z)|^2 = 1$*;*

(c) $P(1) = 1$*;*

(d) $P(z) \neq 0$ *for all* $\xi \in [-\pi, \pi]$.

Then $\{\phi(x-k) : k \in \mathbb{Z}\}$ *forms an orthonormal basis.*

Denote the cardinal B-splines with integer knots in $\mathbb{N}_0$ by $B_n(x)$. It is well-known that $B_n(x)$ satisfy refinement relation (see for example, Wang [3])

$$B_n(x) = \sum_{j=0}^{n} \frac{1}{2^{n-1}} \binom{n}{j} B_n(2x - j) \tag{1.4}$$

and have masks $P_n(z)$ such that $\hat{B}_n(\xi) = P_n(z)\hat{B}_n(\xi/2)$, where $z = e^{-i\xi/2}$ and

$$P_n(z) = \frac{1}{2} \sum_{j=0}^{n} \frac{1}{2^{n-1}} \binom{n}{j} z^j = \frac{(1+z)^n}{2^n} = \big(\frac{1+z}{2}\big)^n. \tag{1.5}$$

It is clear that

$$\begin{aligned} |P_n(z)|^2 + |P_n(-z)|^2 &= \big|\frac{1+z}{2}\big|^{2n} + \big|\frac{1-z}{2}\big|^{2n} \\ &= \cos^{2n}(\xi/4) + \sin^{2n}(\xi/4) \leq \cos^2(\xi/4) + \sin^2(\xi/4) = 1. \end{aligned}$$

The equality happens only when $n = 1$. Therefore, except for the case of order one (i.e., $n = 1$), $B_n(x)$ are generally not orthogonal (indeed they are Riesz basis). To induce orthogonality, Daubechies [4,5] introduced a class of polynomial function factors $S(z)$. Hence, instead of $B_n(x)$, a scaling function $\phi_n(x)$, called spline type Daubechies scaling functions, with the mask $P_n(z)S_n(z)$ is considered so that

$$\phi_n(\xi) = P_n(z)S_n(z)\phi_n(\xi/2), \tag{1.6}$$

where $P_n(z)$ are defined as (1.5). We need to construct $S_n(z)$ such that the shift set of the new scaling function form an orthogonal basis. In other words, we need that $S_n(z)$ satisfy the following condition

$$|P_n(z)S_n(z)|^2 + |P_n(-z)S_n(-z)|^2 = 1. \tag{1.7}$$

Now we consider $S_n(z)$ of the following type: $S_n(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n$, $n \in \mathbb{N}$ and $a_i \in \mathbb{R}$, $i = 1, \ldots, n$. When $z = 1$, from equation (1.7) we have

$$\begin{aligned} 1 &= |P_n(1)S_n(1)|^2 + |P_n(-1)S_n(-1)|^2 \\ &= |P_n(1)|^2|S_n(1)|^2 + |P_n(-1)|^2|S_n(-1)|^2 \\ &= |S_n(1)|^2 + 0 = |S_n(1)|^2. \end{aligned}$$

Thus $S_n(1) = \sum_{i=1}^n a_i = \pm 1$. From Theorem 1.1 (ii), we further impose a restriction that $\sum_{i=1}^n a_i = 1$ in order to ensure the orthogonality of the scaling function.

Next, we set out to find the expressions and constructions of S_n. We have the following Lemma and leave the proof for next section.

Lemma 1.2 *Let $S_n(z)$ be defined as above. Then*

$$|S_n(z)|^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n-1} a_i a_{i+1}\cos(\xi/2) + 2\sum_{i=1}^{n-2} a_i a_{i+2}\cos(2\xi/2) + \cdots + 2a_1 a_n \cos((n-1)\xi/2).$$

From Lemma 1.2, if we write each $\cos(k\xi/2)$ as a polynomial of $\cos(\xi/2)$, then $|S_n(z)|^2 = Q_n(x)$ where $x = \cos(\xi/2)$. Obviously, $Q_n(x)$ has the degree of $n-1$. It is also easy to observe that $|S_n(-z)|^2 = Q_n(-x)$. Now equation (2.5) becomes

$$\begin{aligned} 1 &= |P_n(z)S_n(z)|^2 + |P_n(-z)S_n(-z)|^2 = \cos^{2n}(\xi/4)Q_n(x) + \sin^{2n}(\xi/4)Q_n(-x) \\ &= \Big(\frac{1+\cos(\xi/2)}{2}\Big)^n Q_n(x) + \Big(\frac{1-\cos(\xi/2)}{2}\Big)^n Q_n(-x) \\ &= \Big(\frac{1+x}{2}\Big)^n Q_n(x) + \Big(\frac{1-x}{2}\Big)^n Q_n(-x). \end{aligned}$$

So finally we get

$$\Big(\frac{1+x}{2}\Big)^n Q_n(x) + \Big(\frac{1-x}{2}\Big)^n Q_n(-x) = 1. \tag{1.8}$$

As a side note, (1.8) is equivalent to (6.1.7) of [4] or (4.9) of [5], but is of quite different form so that we may obtain a complete different solution of the equation shown below in (1.9) by using Lorentz polynomials, which yields an efficient proof of sufficiency for the orthogonality and a mechanical and elementary way to construct scaling functions ϕ_n.

Next, to show the existence of $Q(x)$ in the above equation, we make use of the polynomial extended Euclidean algorithm shown in Cormen et al. [6].

Lemma 1.3 (Polynomial extended Euclidean algorithm) *If a and b are two nonzero polynomials, then the extended Euclidean algorithm produces the unique pair of polynomials (s,t) such that $as + bt = \gcd(a,b)$, where $\deg(s) < \deg(b) - \deg(\gcd(a,b))$ and $\deg(t) < \deg(a) - \deg(\gcd(a,b))$.*

We notice that $\gcd((\frac{1+x}{2})^n, (\frac{1-x}{2})^n) = 1$, so by Lemma 1.3, there exists uniquely $Q(x)$ and $R(x)$ with degrees less than n such that $(\frac{1+x}{2})^n Q(x) + (\frac{1-x}{2})^n R(x) = 1$. If we replace x by $-x$ in the previous equation, we have $(\frac{1-x}{2})^n Q(-x) + (\frac{1+x}{2})^n R(-x) = 1$. Due to the uniqueness of the algorithm, we conclude that $R(x) = Q(-x)$. So we have showed the existence of a unique $Q(x) = Q_n(x)$ satisfying equation (1.8).

To construct $Q_n(x)$ explicitly, we use the Lorentz polynomials shown in Erdélyi and Szabados [7], Lorentz [8] and the following technique.

$$\begin{aligned} 1 =& \Big(\frac{1+x}{2} + \frac{1-x}{2}\Big)^{2n-1} = \sum_{i=0}^{2n-1} \binom{2n-1}{i} \Big(\frac{1+x}{2}\Big)^{2n-1-i}\Big(\frac{1-x}{2}\Big)^i \\ =& \Big(\frac{1+x}{2}\Big)^n \Big[\sum_{i=0}^{n-1} \binom{2n-1}{i}\Big(\frac{1+x}{2}\Big)^{n-1-i}\Big(\frac{1-x}{2}\Big)^i\Big] + \\ & \Big(\frac{1-x}{2}\Big)^n \Big[\sum_{i=0}^{n-1} \binom{2n-1}{i}\Big(\frac{1-x}{2}\Big)^{n-1-i}\Big(\frac{1+x}{2}\Big)^i\Big], \end{aligned}$$

where the polynomials presenting in the brackets are the Lorentz polynomials. We notice that the degrees of the two polynomials in the brackets are $n-1$, and because $Q_n(x)$ in equation (1.8) is unique, we can conclude that

$$Q_n(x) = \sum_{i=0}^{n-1} \binom{2n-1}{i} \left(\frac{1+x}{2}\right)^{n-1-i} \left(\frac{1-x}{2}\right)^{i}. \tag{1.9}$$

With the construction of $Q_n(x)$, we take a step further by showing the existence of $\sum a_i^2, \sum a_i a_{i+1}$, ... in Lemma 1.2.

It is well-known that the set $\{1, \cos(t), \cos(2t), \ldots, \cos((n-1)t)\}$ is linearly independent. As a result, $\{1, \cos(\xi/2), \cos(2\xi/2), \ldots, \cos((n-1)\xi/2)\}$ forms a basis of the space $P_{n-1}(x) = \{P(x) : x = \cos(\xi/2)$ and P is a polynomial of degree less than n. Based on this fact and the existence of $Q_n(x)$ in equation (1.8), it is obvious that the coefficients $\sum a_i^2, \sum a_i a_{i+1}, \ldots$ in Lemma 1.2 must exist uniquely.

We now survey our results in the following theorem and give its proof in next section.

Theorem 1.4 *Let $P_n(z)$ and $S_n(z)$ be defined as above. Then for $n = 1, 2, \ldots$, the spline type function $\phi_n(x)$ with the mask $P(z) = P_n(z)S_n(z)$, where $S_n(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n$, is a Daubechies scaling function that generates an orthogonal basis of V_0 in its MRA.*

Although we cannot give the explicit expression of $\phi_n(x)$, we may use the recursive method presented in Theorem 5.23 of [9] by Boggess and Narcowich to find an approximation of $\phi_n(x)$ with any accuracy. It is easy to see that all three conditions required by Theorem 5.23 are satisfied by $P_n(z)$: (i) $P_n(1) = 1$, (ii) $|P_n(z)|^2 + |P_n(-z)|^2 = 1$ ($|z| = 1$), and (iii) $P_n(z)| > 0$ ($\xi \in [-\pi, \pi]$). The examples for the cases $n = 3$ and 4 will be provided in Section 3 to demonstrate this procedure, while all proofs are given in next section.

In the next section, we give proofs of Lemma 1.2 and Theorem 1.4. The examples of the Daubechies scaling functions (the spline type scaling functions) for the cases of $n = 3$ and 4 are presented in Section 3.

2. Proofs

We now give the proof of Lemma 1.2.

Proof of Lemma 1.2 We have

$$\begin{aligned}
|S_n(z)|^2 =& |a_1(\cos(\xi/2) - i\sin(\xi/2)) + \cdots + a_n(\cos(n\xi/2) - i\sin(n\xi/2))|^2 \\
=& |(a_1\cos(\xi/2) + \cdots + a_n\cos(n\xi/2)) - i(a_1\sin(\xi/2) + \cdots + a_n\sin(n\xi/2))|^2 \\
=& (a_1\cos(\xi/2) + \cdots + a_n\cos(n\xi/2))^2 + (a_1\sin(\xi/2) + \cdots + a_n\sin(n\xi/2))^2 \\
=& a_1(\cos^2(\xi/2) + \sin^2(\xi/2)) + \cdots + a_n(\cos^2(n\xi/2) + \sin^2(n\xi/2)) + \\
& \sum_{i<j} 2a_i a_j(\cos(i\xi/2)\cos(j\xi/2) + \sin(i\xi/2)\sin(j\xi/2)) \\
=& \sum_{i=1}^{n} a_i^2 + \sum_{i<j} 2a_i a_j \cos((i-j)\xi/2)
\end{aligned}$$

$$=\sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n-1} a_i a_{i+1}\cos(\xi/2) + 2\sum_{i=1}^{n-2} a_i a_{i+2}\cos(2\xi/2) + \cdots + 2a_1 a_n \cos((n-1)\xi/2).$$

A similar procedure can be applied to find $|S_n(-z)|^2$

$$|S_n(-z)|^2 =\sum_{i=1}^{n} a_i^2 - 2\sum_{i=1}^{n-1} a_i a_{i+1}\cos(\xi/2) + 2\sum_{i=1}^{n-2} a_i a_{i+2}\cos(2\xi/2) + \cdots + (-1)^n 2a_1 a_n \cos((n-1)\xi/2).$$

We now prove Theorem 1.4 using our new mask expression (1.9).

Proof of Theorem 1.4 From Theorem 1.1 (ii), the sufficient conditions for the orthogonality of the scaling function are

(a) $P(z) \in C^1$ and is 2π-periodic;

(b) $|P(z)|^2 + |P(-z)|^2 = 1$;

(c) $P(1) = 1$;

(d) $P(z) \neq 0$ for all $\xi \in [-\pi, \pi]$.

From the construction of our $P(z) = P_n(z)S_n(z)$, the first two conditions are automatically satisfied. The third condition is also obvious: $P(1) = P_n(1)S_n(1) = (\frac{1+1}{2})^n \sum_{i=1}^{n} a_i = 1$ according to the construction of $S_n(z)$. Now we will prove that the final condition is fulfilled as well.

Indeed, if $\xi \in [-\pi, \pi]$, then firstly we have

$$\begin{aligned}|P_n(z)| =&|P_n(e^{-i\xi/2})| = |\frac{1+e^{-i\xi/2}}{2}|^n = |\frac{1+\cos(\xi/2) - i\sin(\xi/2)}{2}|^n\\ =&|\cos^2(\xi/4) - i\sin(\xi/4)\cos(\xi/4)|^n = \sqrt{\cos^4(\xi/4)+\cos^2(\xi/4)\sin^2(\xi/4)}^{\,n}\\ =&|\cos(\xi/4)|^n \geq |\cos(\pi/4)|^n > 0 \text{ for } \xi \in [-\pi, \pi].\end{aligned}$$

Secondly, from equation (1.9) we have

$$\begin{aligned}|S_n(z)|^2 = Q_n(x) &= \sum_{i=0}^{n-1} \binom{2n-1}{i} (\frac{1+x}{2})^{n-1-i} (\frac{1-x}{2})^i\\ &\geq \sum_{i=0}^{n-1} \binom{n-1}{i} (\frac{1+x}{2})^{n-1-i} (\frac{1-x}{2})^i\\ &= (\frac{1+x}{2} + \frac{1-x}{2})^{n-1} = 1.\end{aligned}$$

Thus $|S_n(z)| \geq 1$ and $|P(z)| = |P_n(z)||S_n(z)| \geq |\cos(\pi/4)|^n > 0$.

The remaining thing is to show that the newly constructed scaling function $\phi_n(x)$ with mask $P_n(z)S_n(z)$ is in $L_2(\mathbb{Z})$, where $S_n(z) = \sum_{j=1}^{n} a_j z^j$ and $n \in \mathbb{N}$. From He [10,11], we know that $\phi_n \in L_2(\mathbb{R})$ if

$$n\sum_{j=1}^{n} a_j^2 < 2^{2n-1}. \tag{2.1}$$

Recall from Lemma 1.2 that

$$Q_n(x)=Q_n(\cos(\xi/2))=\sum_{i=1}^{n}a_i^2+2\sum_{i=1}^{n-1}a_ia_{i+1}\cos(\xi/2)+\cdots+2a_1a_n\cos((n-1)\xi/2).$$

Taking the integration from 0 to 2π of both sides, we have

$$\begin{aligned}\int_0^{2\pi}Q_n(x)\mathrm{d}\xi&=\int_0^{2\pi}\Big(\sum_{i=1}^{n}a_i^2+2\sum_{i=1}^{n-1}a_ia_{i+1}\cos(\xi/2)+\cdots+2a_1a_n\cos((n-1)\xi/2)\Big)\mathrm{d}\xi\\&=2\pi\sum_{i=1}^{n}a_i^2.\end{aligned}$$

On the other hand, from the expression of $Q_n(x)$ in (1.9) we have

$$\begin{aligned}\int_0^{2\pi}Q_n(x)\mathrm{d}\xi&=\int_0^{2\pi}\sum_{i=0}^{n-1}\binom{2n-1}{i}\Big(\frac{1+x}{2}\Big)^{n-1-i}\Big(\frac{1-x}{2}\Big)^{i}\mathrm{d}\xi\\&=\int_0^{2\pi}\sum_{i=0}^{n-1}\binom{2n-1}{i}\Big(\frac{1+\cos(\xi/2)}{2}\Big)^{n-1-i}\Big(\frac{1-\cos(\xi/2)}{2}\Big)^{i}\mathrm{d}\xi.\end{aligned}$$

Combining the two equations above, we have

$$\sum_{i=1}^{n}a_i^2=\frac{1}{2\pi}\int_0^{2\pi}\sum_{i=0}^{n-1}\binom{2n-1}{i}\Big(\frac{1+\cos(\xi/2)}{2}\Big)^{n-1-i}\Big(\frac{1-\cos(\xi/2)}{2}\Big)^{i}\mathrm{d}\xi.\tag{2.2}$$

Now we need to show that the expression on the right hand side of (2.2) is smaller than $\frac{2^{2n-1}}{n}$. First of all, it is easy to see that for $0\le i\le n-1$

$$\binom{2n-1}{i}<\binom{2n-1}{n-1}=\frac{1}{2}\binom{2n}{n}.$$

Applying this inequality into (2.2) yields

$$\begin{aligned}&\frac{1}{2\pi}\sum_{i=0}^{n-1}\binom{2n-1}{i}\int_0^{2\pi}\Big(\frac{1+\cos(\xi/2)}{2}\Big)^{n-1-i}\Big(\frac{1-\cos(\xi/2)}{2}\Big)^{i}\mathrm{d}\xi\\&<\frac{1}{4\pi}\binom{2n}{n}\sum_{i=0}^{n-1}\int_0^{2\pi}\Big(\cos\frac{\xi}{4}\Big)^{2(n-1-i)}\Big(\sin\frac{\xi}{4}\Big)^{2i}\mathrm{d}\xi\\&=\frac{1}{\pi}\binom{2n}{n}\sum_{i=0}^{n-1}\int_0^{\pi/2}(\cos x)^{2(n-1-i)}(\sin x)^{2i}\mathrm{d}x\end{aligned}\tag{2.3}$$

for $x=\xi/4$. Now let

$$A=\sum_{i=0}^{n-1}\int_0^{\pi/2}(\cos x)^{2(n-1-i)}(\sin x)^{2i}\mathrm{d}x.$$

We can express A as the sumation of two terms $A=A_1+A_2$, where

$$\begin{aligned}A_1&=\sum_{i=0}^{[\frac{n-1}{2}]}\int_0^{\pi/2}(\cos x)^{2(n-1-i)}(\sin x)^{2i}\mathrm{d}x,\\A_2&=\sum_{i=[\frac{n-1}{2}]+1}^{n-1}\int_0^{\pi/2}(\cos x)^{2(n-1-i)}(\sin x)^{2i}\mathrm{d}x.\end{aligned}$$

For $0 \le i \le [\frac{n-1}{2}], 2(n-1-i) > 2i$, the term A_1 becomes

$$\begin{aligned} A_1 &= \sum_{i=0}^{[\frac{n-1}{2}]} \int_0^{\pi/2} (\cos x)^{2(n-1-2i)} (\sin x \cos x)^{2i} \mathrm{d}x \\ &= \sum_{i=0}^{[\frac{n-1}{2}]} \frac{1}{4^i} \int_0^{\pi/2} (\cos x)^{2(n-1-2i)} (\sin 2x)^{2i} \mathrm{d}x \\ &\le \sum_{i=0}^{[\frac{n-1}{2}]} \frac{1}{4^i} \int_0^{\pi/2} (\cos x)^{2(n-1-2i)} \mathrm{d}x. \end{aligned}$$

For $[\frac{n-1}{2}] + 1 \le i \le n-1, 2(n-1-i) < 2i$, the term A_2 becomes

$$\begin{aligned} A_2 &= \sum_{i=[\frac{n-1}{2}]+1}^{n-1} \int_0^{\pi/2} (\sin x \cos x)^{2(n-1-i)} (\sin x)^{2(2i-n+1)} \mathrm{d}x \\ &= \sum_{i=[\frac{n-1}{2}]+1}^{n-1} \frac{1}{4^{n-1-i}} \int_0^{\pi/2} (\sin x)^{2(2i-n+1)} (\sin 2x)^{2(n-1-i)} \mathrm{d}x \\ &\le \sum_{i=[\frac{n-1}{2}]+1}^{n-1} \frac{1}{4^{n-1-i}} \int_0^{\pi/2} (\sin x)^{2(2i-n+1)} \mathrm{d}x \\ &\le \sum_{i=0}^{[\frac{n-1}{2}]} \frac{1}{4^i} \int_0^{\pi/2} (\sin x)^{2(n-1-2i)} \mathrm{d}x. \end{aligned}$$

Next, we make use of the following well-known result

$$\begin{aligned} \int_0^{\pi/2} (\sin x)^{2n} \mathrm{d}x = \int_0^{\pi/2} (\cos x)^{2n} \mathrm{d}x &= \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2} \\ &= \frac{(2n)!}{[(2n)!!]^2} \frac{\pi}{2} = \frac{\pi}{2} \frac{(2n)!}{4^n (n!)^2} = \frac{\pi}{2} \frac{1}{4^n} \binom{2n}{n}. \end{aligned}$$

Next, we try to find the upper bound for $\binom{2n}{n}$. Based on Stirling estimation, we have the following inequalities

$$\binom{2n}{n} \le \frac{4^n}{\sqrt{3n+1}} \tag{2.4}$$

and

$$\binom{2n}{n} \le \frac{4^n}{\sqrt{\pi n}} (1 + \frac{1}{12n-1}). \tag{2.5}$$

Using (2.4) on A_1 and A_2 yields

$$\begin{aligned} A = A_1 + A_2 &\le 2 \sum_{i=0}^{[\frac{n-1}{2}]} \frac{1}{4^i} \frac{1}{\sqrt{3(n-1-2i)+1}} \frac{\pi}{2} \\ &= \pi \sum_{i=0}^{[\frac{n-1}{2}]} \frac{1}{3^i} \frac{1}{(\frac{4}{3})^i \sqrt{3(n-1-2i)+1}}. \end{aligned}$$

We consider the denominator of the fraction, and let

$$f(x) = (\frac{4}{3})^{2x}[3(n-1-2x)+1], \quad x \in [0, \frac{n-1}{2}].$$

Surveying the function, we have $f(x)$ attains minimum at 0, or

$$f(x) = (\frac{4}{3})^{2x}[3(n-1-2x)+1] > f(0) = 3(n-1)+1.$$

Thus, we have

$$A \le \pi \sum_{i=0}^{[\frac{n-1}{2}]} \frac{1}{3^i} \frac{1}{\sqrt{3(n-1)+1}} \le \frac{\pi}{\sqrt{3n-2}} \sum_{i=0}^{\infty} \frac{1}{3^i} = \frac{3\pi}{2\sqrt{3n-2}}. \tag{2.6}$$

Finally, combining (2.2)–(2.6), we have

$$\sum_{i=1}^{n} a_i^2 \le \frac{1}{\pi}\binom{2n}{n} A \le \frac{1}{\pi}\frac{4^n}{\sqrt{\pi n}}(1+\frac{1}{12n-1})\frac{3\pi}{2\sqrt{3n-2}}.$$

For $n \ge 17$, we can easily verify that the right hand side is less than $\frac{2^{2n-1}}{n}$. Using Mathematica for direct calculation of the case $n \le 16$, we find that the inequality in Theorem 3.1 holds. Thus, it holds for every integer n, and we have shown that $\phi \in L_2(\mathbb{R})$ and the proof is completed. □

3. Examples

It is easy to find that $S_1(z) = z$ and $\phi_n(x)$ is the Haar function. For $n = 2$,

$$S_2(z) = \frac{1+\sqrt{3}}{2}z + \frac{1-\sqrt{3}}{2}z^2$$

and the corresponding $\phi_2(x)$ is the Daubechies D_2 scaling function.

As examples, we consider the efficiency of the computation in using mask expression (1.9) to construct Daubechies scaling functions (the spline type scaling functions) for the cases of $n = 3$ and 4. According to the previous sections, we will construct the Daubechies D_3 scaling function $\phi_3(x)$ from the third order B-spline function $B_3(x)$ using our expression (1.9).

In order to construct the function $\phi_3(x)$, we start with its mask $P_3(z)S_3(z)$, where $P_3(z) = (\frac{1+z}{2})^3$ is the mask of the third order B-spline. Let $S_3(z) = a_1z + a_2z^2 + a_3z^3$. Then by Lemma 1.2, we have

$$\begin{aligned} Q_3(x) = |S_3(z)|^2 &= (a_1^2+a_2^2+a_3^2) + 2(a_1a_2+a_2a_3)\cos(\xi/2) + 2a_1a_3\cos(\xi) \\ &= (a_1^2+a_2^2+a_3^2) + 2(a_1a_2+a_2a_3)\cos(\xi/2) + 2a_1a_3(2\cos^2(\xi/2)-1) \\ &= (a_1^2+a_2^2+a_3^2-2a_1a_3) + 2(a_1a_2+a_2a_3)\cos(\xi/2) + 4a_1a_3\cos^2(\xi/2) \\ &= (a_1^2+a_2^2+a_3^2-2a_1a_3) + 2(a_1a_2+a_2a_3)x + 4a_1a_3x^2 \end{aligned}$$

where $x = \cos(\xi/2)$ and $z = e^{-i\xi/2}$.

On the other hand, by equation (1.9) we have

$$Q_3(x) = \sum_{i=0}^{2}\binom{5}{i}(\frac{1+x}{2})^{2-i}(\frac{1-x}{2})^i$$

$$= \left(\frac{1+x}{2}\right)^2 + 5\left(\frac{1+x}{2}\right)\left(\frac{1-x}{2}\right) + 10\left(\frac{1-x}{2}\right)^2$$
$$= \frac{3}{2}x^2 - \frac{9}{2}x + 4.$$

Thus, from the above equations, we have the following system of equations

$$\begin{cases} a_1^2 + a_2^2 + a_3^2 - 2a_1a_3 = 4, \\ 2(a_1a_2 + a_2a_3) = -\dfrac{9}{2}, \\ 4a_1a_3 = \dfrac{3}{2}. \end{cases} \tag{3.1}$$

Simplifying (3.1), we get

$$\begin{cases} a_1^2 + a_2^2 + a_3^2 = \dfrac{19}{4}, \\ a_1a_2 + a_2a_3 = -\dfrac{9}{4}, \\ a_1a_3 = \dfrac{3}{8}. \end{cases} \tag{3.2}$$

From this system, we have $(a_1 + a_2 + a_3)^2 = a_1^2 + a_2^2 + a_3^2 + 2(a_1a_2 + a_2a_3 + a_1a_3) = 1$. Without loss of generality, consider the case $a_1 + a_2 + a_3 = 1$. Combining this with $a_1a_2 + a_2a_3 = -\frac{9}{4}$ and $a_1a_3 = \frac{3}{8}$, we have the following solution

$$\begin{cases} a_1 = \dfrac{1}{4}(1 + \sqrt{10} - \sqrt{5 + 2\sqrt{10}}), \\ a_2 = \dfrac{1}{2}(1 - \sqrt{10}), \\ a_3 = \dfrac{1}{4}(1 + \sqrt{10} + \sqrt{5 + 2\sqrt{10}}). \end{cases} \tag{3.3}$$

We verify the condition for ϕ_3 to be in $L_2(\mathbb{R})$

$$a_1^2 + a_2^2 + a_3^2 = \frac{19}{4} < \frac{32}{3} = \frac{2^{2\cdot 3-1}}{3}.$$

Thus ϕ_3 is indeed in $L_2(\mathbb{R})$. Now we will attempt to construct ϕ_3 explicitly. It has the mask

$$\begin{aligned} P_3(z)S_3(z) &= \left(\frac{1+z}{2}\right)^3(a_1z + a_2z^2 + a_3z^3) \\ &= 0.0249x - 0.0604x^2 - 0.095x^3 + 0.325x^4 + 0.571x^5 + 0.2352x^6. \end{aligned}$$

Hence, we have the refinement equation

$$\begin{aligned} \phi_3(x) =&0.0498\phi_3(2x-1) - 0.121\phi_3(2x-2) - 0.191\phi_3(2x-3)+ \\ &0.650\phi_3(2x-4) + 1.141\phi_3(2x-5) + 0.4705\phi_3(2x-6). \end{aligned} \tag{3.4}$$

Again, we construct the Daubechies D_4 orthogonal scaling function $\phi_4(x)$ from the forth order B-spine $B_4(x)$ with the mask $P_4(z) = (\frac{1+z}{2})^4$. Now we examine the mask $P_4(z)S_4(z)$ of $\phi_4(x)$ where we define $S_4(z) = a_1z + a_2z^2 + a_3z^3 + a_4z^4$. By Lemma 1.2, we have

$$\begin{aligned} Q_4(x) =&|S_4(z)|^2 = (a_1^2 + a_2^2 + a_3^2 + a_4^2) + 2(a_1a_2 + a_2a_3 + a_3a_4)\cos(\xi/2)+ \\ &2(a_1a_3 + a_2a_4)\cos(\xi) + 2a_1a_4\cos(3\xi/2) \\ =&(a_1^2 + a_2^2 + a_3^2 + a_4^2) + 2(a_1a_2 + a_2a_3 + a_3a_4)\cos(\xi/2)+ \end{aligned}$$

$$
\begin{aligned}
&2(a_1a_3+a_2a_4)(2\cos^2(\xi/2)-1)+2a_1a_4(4\cos^3(\xi/2)-3\cos(\xi/2))\\
=&(a_1^2+a_2^2+a_3^2+a_4^2-2a_1a_3-2a_2a_4)+\\
&(2a_1a_2+2a_2a_3+2a_3a_4-6a_1a_4)\cos(\xi/2)+\\
&(4a_1a_3+4a_2a_4)\cos^2(\xi/2)+8a_1a_4\cos^3(\xi/2)\\
=&(a_1^2+a_2^2+a_3^2+a_4^2-2a_1a_3-2a_2a_4)+\\
&(2a_1a_2+2a_2a_3+2a_3a_4-6a_1a_4)x+\\
&(4a_1a_3+4a_2a_4)x^2+8a_1a_4x^3
\end{aligned}
$$

where $x=\cos(\xi/2)$ and $z=e^{-i\xi/2}$.

Using equation (1.9), we get another expression for $Q_4(x)$

$$
\begin{aligned}
Q_4(x)&=\sum_{i=0}^{3}\binom{7}{i}(\frac{1+x}{2})^{3-i}(\frac{1-x}{2})^i\\
&=(\frac{1+x}{2})^3+7(\frac{1+x}{2})^2(\frac{1-x}{2})+21(\frac{1+x}{2})(\frac{1-x}{2})^2+35(\frac{1-x}{2})^3\\
&=8-\frac{29}{2}x+10x^2-\frac{5}{2}x^3.
\end{aligned}
$$

Thus, from the above equations, we have the following system of equations

$$
\begin{cases}
a_1^2+a_2^2+a_3^2+a_4^2-2a_1a_3-2a_2a_4=8,\\
2a_1a_2+2a_2a_3+2a_3a_4-6a_1a_4=-\dfrac{29}{2},\\
4a_1a_3+4a_2a_4=10,\\
8a_1a_4=-\dfrac{5}{2}.
\end{cases}
\tag{3.5}
$$

Simplifying (3.5), we have the following system

$$
\begin{cases}
a_1^2+a_2^2+a_3^2+a_4^2=13,\\
a_1a_2+a_2a_3+a_3a_4=-\dfrac{131}{16},\\
a_1a_3+a_2a_4=\dfrac{5}{2},\\
a_1a_4=-\dfrac{5}{16}.
\end{cases}
\tag{3.6}
$$

Solving for this system of equations yields 8 solutions. One of the numerical solutions is

$$
\begin{cases}
a_1=2.6064,\\
a_2=-2.3381,\\
a_3=0.8516,\\
a_4=-0.1199.
\end{cases}
\tag{3.7}
$$

We verify the condition for ϕ_4 to be in $L_2(\mathbb{R})$

$$a_1^2+a_2^2+a_3^2+a_4^2=13<32=\frac{2^{2\cdot 4-1}}{4}.$$

Thus ϕ_4 is indeed in $L_2(\mathbb{R})$. Now we will attempt to construct ϕ_4 explicitly. It has the mask

$$P_4(z)S_4(z)=(\frac{1+z}{2})^4(a_1z+a_2z^2+a_3z^3+a_4z^4)$$

$$
\begin{aligned}
&= 0.1629z + 0.5055z^2 + 0.4461z^3 - 0.0198z^4 - 0.1323z^5 + 0.0218z^6 + \\
&\quad 0.0233z^7 - 0.0075z^8.
\end{aligned}
$$

Hence, we have the refinement equation

$$
\begin{aligned}
\phi_4(x) =& 0.3258\phi_4(2x-1) + 1.011\phi_4(2x-2) + 0.8922\phi_4(2x-3) - 0.0396\phi_4(2x-4) - \\
& 0.2646\phi_4(2x-5) + 0.0436\phi_4(2x-6) + 0.0466\phi_4(2x-7) - 0.015\phi_4(2x-8). \quad (3.8)
\end{aligned}
$$

The above examples demonstrate the efficiency of the computation of the Daubechies scaling functions by using expression (1.9).

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