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On Φ-τ-Supplement Subgroups of Finite Groups

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Abstract Let τ be a subgroup functor and H a p-subgroup of a finite group G. Let $\bar{G} = G/H_G$ and $\bar{H} = H/H_G$. We say that H is Φ-τ-supplement in G if $\bar{G}$ has a subnormal subgroup $\bar{T}$ and a τ-subgroup $\bar{S}$ contained in $\bar{H}$ such that $\bar{G} = \bar{H}\bar{T}$ and $\bar{H} \cap \bar{T} \leq \bar{S}\Phi(\bar{H})$. In this paper, some new characterizations of hypercyclically embedability and p-nilpotency of a finite group are obtained based on the assumption that some primary subgroups are Φ-τ-supplement in G.

Keywords Sylow subgroups; subnormal subgroups; subgroup functor; p-nilpotent group; Φ-τ-supplement

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1. Introduction

Throughout this paper, all groups considered are finite and G always denotes a group and p denotes a prime. All unexplained notation and terminology are standard, as in [1,2].

A chief factor L/K of G is called a Frattini (non-Frattini) chief factor if $L/K \leq \Phi(G/K)$ (resp., $L/K \not\leq \Phi(G/K)$). For a class of groups $\mathfrak{F}$, a chief factor L/K of G is said to be $\mathfrak{F}$-central in G if $L/K \rtimes G/C_G(L/K) \in \mathfrak{F}$. A normal subgroup N of G is said to be $\mathfrak{F}$-hypercentral ($\mathfrak{F}\Phi$-hypercentral) in G if either $N = 1$ or every chief factor (every non-Frattini chief factor) of G below N is $\mathfrak{F}$-central in G. Let $Z_{\mathfrak{F}}(G)$ and $Z_{\mathfrak{F}\Phi}(G)$ denote the $\mathfrak{F}$-hypercentre (resp., $\mathfrak{F}\Phi$-hypercentre) of G, respectively, that is, the product of all $\mathfrak{F}$-hypercentral ($\mathfrak{F}\Phi$-hypercentral) normal subgroups of G. In this paper, we use $\mathfrak{U}$ to denote the classes of all supersoluble groups. It is well known that $\mathfrak{U}$ is a saturated formation.

A function τ which assigns each group G to a set of subgroups $\tau(G)$ of G is called a subgroup functor [3] if $1 \in \tau(G)$ and $\theta(\tau(G)) = \tau(\theta(G))$ for any isomorphism $\theta : G \to G^*$. If $H \in \tau(G)$, then we say that H is a τ-subgroup of G.

Recall that a subgroup H of G is S-quasinormal in G if H permutes with every Sylow subgroup of G. A subgroup H of G is said to be s-semipermutable in G (see [4]) if $HG_p = G_pH$ for any Sylow p-subgroup G_p of G with $(p, |H|) = 1$; weakly s-permutable in G (see [5]) if G has a subnormal subgroup T and an s-permutable subgroup S contained in H such that $G = HT$

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and $H \cap T \leq S$; weakly SS-permutable in G (see [6]) if G has a subnormal subgroup T and an SS-permutable subgroup S contained in H such that $G = HT$ and $H \cap T \leq S$; weakly s-semipermutable in G (see [7]) if G has a subnormal subgroup T and an s-semipermutable subgroup S contained in H such that $G = HT$ and $H \cap T \leq S$; weakly s-supplemently embedded in G (see [8]) if G has a subnormal subgroup T and an S-quasinormal embedded subgroup S contained in H such that $G = HT$ and $H \cap T \leq S$; Π-normal in G (see [9]) if G has a subnormal subgroup T such that $G = HT$ and $H \cap T \leq S$, where S is a subgroup of G contained in H and S satisfied Π-property; $S\Phi$-supplemented [10] in G if there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq \Phi(H)$. Naturally, it is necessary to unify the above-mentioned generalized normal subgroups and discuss the influence on the structure of a finite group by connecting these subgroups with Frattini subgroup of G. Hence we give the following notion.

Definition 1.1 *Let τ be a subgroup functor and H a p-subgroup of a finite group G. Let $\bar{G} = G/H_G$ and $\bar{H} = H/H_G$. We say that H is Φ-τ-supplement in G if $\bar{G}$ has a subnormal subgroup $\bar{T}$ and a τ-subgroup $\bar{S}$ contained in $\bar{H}$ such that $\bar{G} = \bar{H}\bar{T}$ and $\bar{H} \cap \bar{T} \leq \bar{S}\Phi(\bar{H})$.*

By [11, Examples 1.5, 1.7 and 1.9] and [12, Examples 4.6 and 4.9], we know the above mentioned p-subgrops are Φ-τ-supplement in G. Now we introduce some properties of subgroup functors (also, see [11, Definition 1.3]) which will be used in our results. If τ is a subgroup functor, then we say that τ is:

(1) Inductive if for any group G, whenever $H \in \tau(G)$ is a p-group and $N \trianglelefteq G$, then $HN/N \in \tau(G/N)$.

(2) Hereditary if for any group G, whenever $H \in \tau(G)$ is a p-group and $H \leq E \leq G$, then $H \in \tau(E)$.

(3) Regular (resp., quasiregular) if for any group G, whenever $H \in \tau(G)$ is a p-group and N is a minimal normal subgroup (resp., an abelian minimal normal subgroup) of G, then $|G : N_G(H \cap N)|$ is a power of p.

(4) Φ-regular (resp., Φ-quasiregular) if for any primitive group G, whenever $H \in \tau(G)$ is a p-group and N is a minimal normal subgroup (resp., an abelian minimal normal subgroup) of G, then $|G : N_G(H \cap N)|$ is a power of p.

2. Preliminaries

In the following section, we will introduce some lemmas used in this paper.

Lemma 2.1 *Let H be a p-subgroup of G and τ an inductive subgroup functor. Suppose that H is Φ-τ-supplement in G.*

(1) If $N \trianglelefteq G$ and either $N \leq H$ or $(|H|, |N|) = 1$, then HN/N is Φ-τ-supplement in G/N.

(2) If τ is hereditary and $H \leq K \leq G$, then H is Φ-τ-supplement in K.

Proof Let $\bar{G} = G/H_G$ and $\bar{H} = H/H_G$. Since H is Φ-τ-supplement G, $\bar{G}$ has a subnormal subgroup $\bar{T}$ and a τ-subgroup $\bar{S}$ contained in $\bar{H}$ such that $\bar{G} = \bar{H}\bar{T}$ and $\bar{H} \cap \bar{T} \leq \bar{S}\Phi(\bar{H})$.

(1) Let $\widehat{G} = G/(HN)_G$, $\widehat{HN} = HN/(HN)_G$, $\widehat{T} = T(HN)_G/(HN)_G$ and $\widehat{S} = S(HN)_G/(HN)_G$. Clearly, $H_G \leq (HN)_G$. Then $\widehat{S} \in \tau(\widehat{G})$ for τ is inductive. It is easy to see that $\widehat{T}$ is subnormal

in $\widehat{G}$ and $\widehat{G} = \widehat{HN}\widehat{T}$. Since $(|N|, |H|) = 1$, $(|NH \cap T : T \cap N|, |NH \cap T : T \cap H|) = 1$. Hence $(NH \cap T) = (N \cap T)(H \cap T)$. It follows that $\widehat{HN} \cap \widehat{T} = HN/(HN)_G \cap T(HN)_G/(HN)_G = (H \cap T)(HN)_G/(HN)_G \le (S(HN)_G/(HN)_G)\Phi(HN/(HN)_G)) = \widehat{S}\Phi(\widehat{HN})$. Therefore, HN/N is Φ-τ-supplement in G/N.

(2) It is easy to see that $H_G \le H_K$. Let $\widetilde{K} = K/H_K$, $\widetilde{H} = H/H_K$, $\widetilde{T} = TH_K/H_K \cap K/H_K$ and $\widetilde{S} = SH_K/H_K$. Since τ is hereditary and inductive, $\widetilde{S} \in \tau(\widetilde{K})$. Clearly, $\widetilde{T}$ is subnormal in $\widetilde{K}$ and $\widetilde{K} = \widetilde{H}\widetilde{T}$. It is easy to see that $\widetilde{H} \cap \widetilde{T} = H/H_K \cap TH_K/H_K = (H \cap T)H_K/H_K \le (SH_K/H_K)\Phi(H/H_K) = \widetilde{S}\Phi(\widetilde{H})$. Hence H is Φ-τ-supplement in K. □

Lemma 2.2 [12, Lemma 2.6] *Let $\mathfrak{F}$ be a nonempty solubly saturated formation and P a normal subgroup of G. If $P/\Phi(P) \le Z_{\mathfrak{F}}(G/\Phi(P))$, then $P \le Z_{\mathfrak{F}}(G)$.*

The next lemma is clear.

Lemma 2.3 *Let p be a prime divisor of $|G|$ with $(|G|, p-1) = 1$.*

(1) If G has a cyclic Sylow p-subgroup, then G is p-nilpotent.

(2) If N is a normal subgroup of G such that $|N|_p \le p$ and G/N is p-nilpotent, then G is p-nilpotent.

Let P be a p-group. If P is not a non-abelian 2-group, then we use $\Omega(P)$ to denote the subgroup $\Omega_1(P)$. Otherwise, $\Omega(P) = \Omega_2(P)$.

Lemma 2.4 ([11, Lemma 4.4]) *Let $\mathfrak{F}$ be a saturated formation, P a normal p-subgroup of G and C a Thompson critical subgroup of P (see [13, p.186]). If $C \le Z_{\mathfrak{F}}(G)$ or $\Omega(C) \le Z_{\mathfrak{F}}(G)$, then $P \le Z_{\mathfrak{F}}(G)$.*

Lemma 2.5([14, Lemma 2.10]) *Let C be a Thompson critical subgroup of a nontrivial p-group P.*

(1) If p is odd, then the exponent of $\Omega_1(C)$ is p.

(2) If P is an abelian 2-group, then the exponent of $\Omega_1(C)$ is 2.

(3) If $p = 2$, then the exponent of $\Omega_2(C)$ is at most 4.

Lemma 2.6 ([15, Theorem B]) *Let $\mathfrak{F}$ be any formation and E a normal subgroup of G. If $F^*(E) \le Z_{\mathfrak{F}}(G)$, then $E \le Z_{\mathfrak{F}}(G)$.*

3. Main results

In this section, we will give the main conclusions of this paper.

Proposition 3.1 *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and τ a Φ-quasiregular (resp., quasiregular) inductive subgroup functor. Suppose that P is a normal p-subgroup of G and every maximal subgroup of P is Φ-τ-supplement in G. Then $P \le Z_{\mathfrak{F}\Phi}(G)$ (resp., $P \le Z_{\mathfrak{F}}(G)$).*

Proof Suppose that the theorem is false and let (G, P) be a counterexample with $|G| + |P|$ minimal. Then:

(1) G has a unique minimal normal subgroup N contained in P, $P/N \leq Z_{\mathfrak{F}\Phi}(G/N)$ (resp., $P/N \leq Z_{\mathfrak{F}}(G/N)$) and $P \cap Z_{\mathfrak{F}\Phi}(G) = 1$ (resp., $P \cap Z_{\mathfrak{F}}(G) = 1$).

Let N be any minimal normal subgroup of G contained in P. Clearly, by Lemma 2.1(1), $(G/N, P/N)$ satisfies the hypothesis, and so the choice of (G, P) yields that $P/N \leq Z_{\mathfrak{F}\Phi}(G/N)$ (resp., $P/N \leq Z_{\mathfrak{F}}(G/N)$). If $P \cap Z_{\mathfrak{F}\Phi}(G) > 1$ (resp., $P \cap Z_{\mathfrak{F}}(G) > 1$), without loss of generality, we may assume that $N \leq P \cap Z_{\mathfrak{F}\Phi}(G)$ (resp., $N \leq P \cap Z_{\mathfrak{F}}(G)$). It induces that $P \leq Z_{\mathfrak{F}\Phi}(G)$ (resp., $P \leq Z_{\mathfrak{F}}(G)$), a contradiction. Thus $P \cap Z_{\mathfrak{F}\Phi}(G) = 1$ (resp., $P \cap Z_{\mathfrak{F}}(G) = 1$). Suppose that G has a minimal normal subgroup R contained in P such that $N \neq R$. With a similar discussion as above, we have that $P/R \leq Z_{\mathfrak{F}\Phi}(G/R)$ (resp., $P/R \leq Z_{\mathfrak{F}}(G/R)$). First, assume that $NR/R \nleq \Phi(G/R)$. Then, in the above two cases, we have $NR/R \leq Z_{\mathfrak{F}}(G/R)$. Now we assume that $NR/R \leq \Phi(G/R)$. If $P \cap Z_{\mathfrak{F}\Phi}(G) = 1$, then $P \cap \Phi(G) = 1$. By [1, Chap. A, Lemma 9.1], $NR \leq P \cap \Phi(G)R = R$, a contradiction. Hence we only consider τ is quasiregular. Then $P/N \leq Z_{\mathfrak{F}}(G/N)$, and so $NR/R \leq Z_{\mathfrak{F}}(G/R)$. From G-isomorphism $R \cong NR/R$, we have $N \leq Z_{\mathfrak{F}}(G)$, which is impossible. Thus N is the unique minimal normal subgroup of G contained in P.

(2) $\Phi(P) \neq 1$.

If $\Phi(P) = 1$, then P is elementary abelian. Let N_1 be a maximal subgroup of N such that N_1 is normal in some Sylow p-subgroup of G, say G_p. Then $P_1 = N_1 S$ is a maximal subgroup of P, where S is a complement of N in P. Obviously, $(P_1)_G = 1$ and $\Phi(P_1) = 1$. Therefore by hypothesis, G has a subnormal subgroup T and a τ-subgroup S contained in P_1 such that $G = P_1 T$ and $P_1 \cap T \leq S$. Then $G = PT$ and $P = P \cap P_1 T = P_1(P \cap T)$. It is easy to see that $1 \neq P \cap T \trianglelefteq G$. Hence $N \leq P \cap T$, and so $P_1 \cap N \leq P_1 \cap T \leq S$. It follows that $N_1 = P_1 \cap N = S \cap N$. If $N \nleq \Phi(G)$, then G has a maximal subgroup M such that $G = N \rtimes M$. Clearly by (1), $P \cap M_G = 1$. By hypothesis, $|G : N_G(N_1 M_G)| = |G : N_G((S \cap N)M_G)| = |G : N_G(SM_G \cap NM_G)|$ is a power of p. This implies that $N_1 M_G \trianglelefteq G$ and so $N_1 = N_1 M_G \cap P \trianglelefteq G$, a contradiction. We may, therefore, assume that $N \leq \Phi(G)$. If $P/N \leq Z_{\mathfrak{F}\Phi}(G/N)$, then $P \leq Z_{\mathfrak{F}\Phi}(G)$, a contradiction. Hence we only consider that τ is quasiregular. It follows that $|G : N_G(N_1)| = |G : N_G(S \cap N)|$ is a power of p. Thus $N_1 \trianglelefteq G$, a contradiction too. Therefore $\Phi(P) \neq 1$.

(3) The final contradiction.

By (1) and (2), $N \leq \Phi(P)$. This induces $P/\Phi(P) \leq Z_{\mathfrak{F}\Phi}(G/\Phi(P))$ (resp., $P/\Phi(P) \leq Z_{\mathfrak{F}}(G/\Phi(P))$) and so $P \leq Z_{\mathfrak{F}\Phi}(G)$ (resp., $P \leq Z_{\mathfrak{F}}(G)$) by Lemma 2.2. The final contradiction ends the proof. □

Theorem 3.2 *Let E be a normal subgroup of G and P a Sylow p-subgroup of E such that $(|E|, p-1) = 1$. Suppose that τ is a Φ-regular inductive subgroup functor and every τ-subgroup of G contained in P is subnormally embedded in G. If every maximal subgroup of P is Φ-τ-supplement in G, then E is p-nilpotent.*

Proof Suppose that the theorem is false and let (G, E) be a counterexample with $|G| + |E|$ minimal. We now proceed via the following steps:

(1) $O_{p'}(E) = 1$.

Suppose that $O_{p'}(E) \neq 1$. Let $M/O_{p'}(E)$ be a maximal subgroup of $PO_{p'}(E)/O_{p'}(E)$. Then

$M = P_1 O_{p'}(E)$ for some maximal subgroup P_1 of P. By the Lemma 2.1(1) and the hypothesis, $P_1 O_{p'}(E)/O_{p'}(E)$ is Φ-τ-supplement $E/O_{p'}(E)$. This shows that $(G/O_{p'}(E), E/O_{p'}(E))$ satisfies the hypothesis of the theorem. The choice of (G, E) implies that $E/O_{p'}(E)$ is p-nilpotent, and so E is p-nilpotent, a contradiction. Hence $O_{p'}(E) = 1$.

(2) G has a unique minimal normal subgroup N contained in E, E/N is p-nilpotent and $G = NM$, where M is a maximal subgroup of G.

Let N be a minimal normal subgroup of G contained in E and H/N be a maximal subgroup PN/N. Then there exists a maximal subgroup P_1 of P such that $H = P_1N$ and $P_1 \cap N = P \cap N$. Set $\bar{G} = G/(P_1)_G$ and $\bar{P}_1 = P_1/(P_1)_G$. By the hypothesis, $\bar{G}$ has a subnormal subgroup $\bar{T}$ and a τ-subgroup $\bar{S}$ contained in $\bar{P}_1$ such that $\bar{G} = \bar{P}_1\bar{T}$ and $\bar{P}_1 \cap \bar{T} \leq \bar{S}\Phi(\bar{P}_1)$, where $\bar{S} = S/(P_1)_G$ and $\bar{T} = T/(P_1)_G$. Let $\widehat{G} = G/(P_1N)_G$, $\widehat{P_1N} = P_1N/(P_1N)_G$, $\widehat{T} = T(P_1N)_G/(P_1N)_G$ and $\widehat{S} = S(P_1N)_G/(P_1N)_G$. Since $(|P_1N \cap T : P_1 \cap T|, |P_1N \cap T : N \cap T|) = 1$, $P_1N \cap T = (P_1 \cap T)(N \cap T)$. By using a similar discussion as in the proof of Lemma 2.1(1), we have that H/N is Φ-τ-supplement in G/N. This shows that $(G/N, E/N)$ satisfies the hypothesis of the theorem. The choice of (G, E) implies that E/N is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, N is the unique minimal normal subgroup of G contained in E and $N \nleq \Phi(G)$. Then there exists a maximal subgroup M of G such that $G = NM$.

(3) $O_p(E) = 1$.

Suppose that $O_p(E) \neq 1$. Then by (2), $N \leq O_p(E)$ and $G = N \rtimes M$. Since $O_p(G) \leq C_G(N)$, $O_p(G) \cap M$ is normal in G and so $O_p(E) \cap M$ is normal in G. If $O_p(E) \cap M \neq 1$, then $N \leq O_p(E) \cap M$, a contradiction. Thus $O_p(E) \cap M = 1$. It follows that $O_p(E) = O_p(E) \cap NM = N$ and it is easy to see that $C_E(N) = N$. Denote $K = M \cap E$. Then $E = N \rtimes K$. Let K_p be a Sylow p-subgroup of K such that $P = NK_p$ and M_p a Sylow p-subgroup of M containing K_p. Then $G_p = NM_p$ is a Sylow p-subgroup of G. Let N_1 be a maximal subgroup of N such that N_1 is normal in G_p. Then $G_1 = N_1M_p$ is a maximal subgroup of G_p, $P_1 = N_1K_p$ is a maximal subgroup of P and $P = NP_1$. If $(P_1)_G \neq 1$, then by (2), $N \leq P_1$ and so $P = P_1$, a contradiction. Hence $(P_1)_G = 1$. By the hypothesis, G has a subnormal subgroup T and a τ-subgroup S contained in P_1 such that $G = P_1T$ and $P_1 \cap T \leq S\Phi(P_1)$.

Since τ is a Φ-regular inductive subgroup functor, $|G/M_G : N_{G/M_G}(SM_G/M_G \cap NM_G/M_G)|$ is a power of p. If $SM_G \cap NM_G \neq M_G$, then $(SM_G/M_G \cap NM_G/M_G)^{G/M_G} = (SM_G/M_G \cap NM_G/M_G)^{G_pM_G/M_G} \leq G_1M_G/M_G$ and so $N \leq G_1M_G$. Hence $N = N \cap G_1M_G = N \cap N_1M_pM_G = N_1$, a contradiction. Thus $SM_G \cap NM_G = M_G$. Obviously, $SN \cap M_G = 1$ because $E \cap M_G = 1$. Hence $SM_G \cap NM_G = (S \cap N)M_G = M_G$ and so $S \cap N \leq M_G \cap N = 1$. Assume that $S \neq 1$. Since S is subnormally embedded in G, there exists a subnormal subgroup V of G such that S is a Sylow p-subgroup of V. Without loss of generality, we may assume that $V \leq E$. Let L be a minimal subnormal subgroup of G contained in V. Since $O_{p'}(L)$ is subnormal in G, $O_{p'}(L) = 1$ by (1). By (2), we know that E is p-soluble and so L is p-soluble. This follows that $L = O_p(L) \leq O_p(E) = N$. It implies that $L \cap S = 1$, which is impossible. Hence $S = 1$. Since $E = E \cap P_1T = P_1(E \cap T)$, $O^p(E) \leq E \cap T$ and so $N \leq T$ by (2). It implies that $P_1 \cap N \leq \Phi(P_1)$. This deduces that $P_1 = P_1 \cap NK_p = K_p(P_1 \cap N) = K_p$. Hence $N_1 = P_1 \cap N = K_p \cap N = 1$, and thereby $|N| = p$. By (2) and Lemma 2.3(2), we have that E is

p-nilpotent, a contradiction. Therefore $O_p(E) = 1$.

(4) $N \cap P < P$.

If not, then $P \le N$. If $N < E$, then the choice of the (G, E) shows N is p-nilpotent. Then by (1), N is a p-group, which contradicts (3). Hence $E = N$. Let P_1 be a maximal subgroup of P. Obviously, $(P_1)_G = 1$. Hence G has a subnormal subgroup T and a τ-subgroup S contained in P_1 such that $G = P_1T$ and $P_1 \cap T \le S\Phi(P_1)$. Assume that $S \ne 1$. Since τ is Φ-regular and inductive, $|G : N_G(SM_G)|$ is a power of p. It follows that $N \le S^G M_G = S^{G_p} M_G \le G_p M_G$, where G_p is a Sylow p-subgroup of G containing P. Then $N = N \cap G_p M_G = N \cap G_p$ because $N \cap M_G = 1$. It follows that N is a p-group. This contradicts (3). Hence $S = 1$. It is easy to see that $N \le O^p(G) \le T$. It follows that $P_1 = \Phi(P_1)$, a contradiction. Hence $N \cap P < P$.

(5) Final contradiction.

By (4), P has a maximal subgroup P_1 such that $N \cap P \le P_1$. Clearly, $(P_1)_G = 1$. By hypothesis, G has a subnormal subgroup T and a τ-subgroup S contained in P_1 such that $G = P_1T$ and $P_1 \cap T \le S\Phi(P_1)$.

We show that $S = 1$. Assume that $S \ne 1$. By (2), $SN \cap M_G = 1$. Thus $SM_G \cap NM_G = (S \cap N)M_G$. Since τ is Φ-regular and inductive, $|G : N_G(SM_G \cap NM_G)|$ is a power of p. If $S \cap N \ne 1$, then $N \le (S \cap N)^G M_G = (S \cap N)^{G_p} M_G \le G_p M_G$, where G_p is a Sylow p-subgroup of G contained in P. It follows that $N = N \cap G_p M_G = N \cap G_p$, that is, N is a p-group, which contradics (3). Thus $S \cap N = 1$. By using a similar dicussion as in (3), let S be a Sylow p-subgroup of a subnormal subgroup V of G and L a minimal subnormal subgroup of G contained in V. By (1) and (3), L is a nonabelian simple group. It is easy to see that $L \cap N = 1$ or $L \le N$. If $L \cap N = 1$, then by (2), $L \cong LN/N \le E/N$ is p-nilpotent, which is impossible. If $L \le N$, then $S \cap L = 1$. It implies that L is a p'-group, a contradiction. Hence $S = 1$.

Clearly, $N \le T$ and so $N \cap P \le N \cap P_1 \le \Phi(P)$. Then by [16, Chap. IV, Satz 4.7], N is p-nilpotent, a contradiction too. The proof is completed. □

Proposition 3.3 *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups, E be a normal subgroup of G and τ a regular inductive subgroup functor. Suppose that every τ-subgroup of G contained in E is subnormally embedded in G and every maximal subgroup of every noncyclic Sylow subgroup of E is Φ-τ-supplement in G. Then $E \le Z_{\mathfrak{F}}(G)$.*

Proof Suppose that the theorem is false and let (G, E) be a counterexample with $|G| + |E|$ minimal. Let p be the smallest prime divisor of $|E|$ and P a Sylow p-subgroup of X. If P is cyclic, then E is p-nilpotent by Lemma 2.3(1). Now assume that P is not cyclic. Then by Theorem 3.2, E is p-nilpotent. Let V be the normal p-complement of E. Then V is normal in G. If $V = 1$, then by Proposition 3.1, $E \le Z_{\mathfrak{F}}(G)$, a contradiction. Hence $V \ne 1$. Then it is easy to see that (G, V) satisfies the hypothesis, so $V \le Z_{\mathfrak{F}}(G)$. On the other hand, by Lemma 2.1(1), we know that $(G/V, E/V)$ satisfies the hypothesis. The choice of (G, E) implies that $E/V \le Z_{\mathfrak{F}}(G/V)$. It implies that $E \le Z_{\mathfrak{F}}(G)$, a contradiction too. □

Proposition 3.4 *Let τ be a quasiregular inductive subgroup functor and P a normal p-subgroup of G. If every cyclic subgroup of P of prime order or order 4 (when P is a non-abelian 2-group)* *is Φ-τ-supplement in G, then $P \leq Z_{\mathfrak{U}}(G)$.*

Proof Suppose that the theorem is false and let (G, P) be a counterexample with $|G| + |P|$ minimal. Then:

(1) G has a unique normal subgroup N contained in P such that P/N is a chief factor of G, $N \leq Z_{\mathfrak{U}}(G)$ and $|P/N| > p$.

Let P/N be a chief factor of G. Clearly, (G, N) satisfies the hypothesis. The choice of (G, P) implies that $N \leq Z_{\mathfrak{U}}(G)$. If $|P/N| = p$, then $P/N \leq Z_{\mathfrak{U}}(G/N)$, and so $P \leq Z_{\mathfrak{U}}(G)$, a contradiction. Hence $|P/N| > p$. Now assume that P/R is a chief factor of G such that $N \neq R$. Then with a similar argument as above, we have that $R \leq Z_{\mathfrak{U}}(G)$. It follows that $P = NR \leq Z_{\mathfrak{U}}(G)$, a contradiction. Therefore, N is the unique normal subgroup of G such that P/N is a chief factor of G.

(2) The exponent of P is p or 4 (when P is a non-abelian 2-group).

Let C be a Thompson critical subgroup of P. If $\Omega(C) < P$, then $\Omega(C) \leq N \leq Z_{\mathfrak{U}}(G)$ by (1), and so $P \leq Z_{\mathfrak{U}}(G)$ by Lemma 2.4, which is impossible. Thus $P = \Omega(C)$. Then by Lemma 2.5, the exponent of P is p or 4 (when P is a non-abelian 2-group).

(3) The final contradiction.

Since $P/N \cap Z(G_p/N) > 1$, where G_p is a Sylow p-subgroup of G, there exists a subgroup V/N of order p contained in $P/N \cap Z(G_p/N)$. Let $x \in V \backslash N$ and $H = \langle x \rangle$. Then $V = HN$. By (2), $|H| = p$ or 4 (when P is a non-abelian 2-group). If $V \trianglelefteq G$, then $P = V$ by (1), and so $|P/N| = p$, a contradiction. Hence V is not normal in G. Clearly by (1), $H_G \leq V_G = N$. By the hypothesis, G/H_G has a subnormal subgroup T/H_G and a τ-subgroup S/H_G contained in H/H_G such that $G = HT$ and $(H/H_G) \cap (T/H_G) \leq (S/H_G)\Phi(H/H_G)$. Assume that $S/H_G = H/H_G$. Since τ is a quasiregular inductive subgroup functor, SN/N is a τ-subgoup of G/N and $|G : N_G(V)| = |G : N_G(HN)|$ is a power of p. It follows that $V \trianglelefteq G$, a contradiction. Therefore, we assume that $S/H_G \neq H/H_G$. Then $H/H_G \cap T/H_G \leq \Phi(H/H_G)$. Obviously, $H_G \neq H$. Hence $H \cap T \leq \Phi(H)$. In this case, $P \cap T < P$, and so $(P \cap T)^G = (P \cap T)^P < P$. This means from (1) that $(P \cap T)^G \leq N$, and so $P = H(P \cap T) = HN = V$. The final contradiction completes the proof of the theorem. $\square$

Theorem 3.5 *Let τ be a regular inductive subgroup functor. Suppose that E is a normal subgroup of G and P is a Sylow p-subgroup of E such that $(|E|, p-1) = 1$. If every cyclic subgroup of P of prime order or order 4 (when P is a non-abelian 2-group) is Φ-τ-supplement in G, then E is p-nilpotent.*

Proof Suppose that it is false and let (G, E) be a counterexample for which $|G|+|E|$ is minimal. We prove theorem via the following steps.

(1) $O_{p'}(E) = 1$

See step (1) in the proof of Theorem 3.2.

(2) $E/O_p(E)$ is a chief factor of G and $O_p(E) \leq Z_\infty(E)$.

Let N be a normal subgroup of G such that $N < E$. It is easy to see that (G, N) satisfies the hypothesis of the theorem, hence by the choice of (G, E), N is p-nilpotent. It follows from

(1) that N is a p-group and so $N \leq O_p(E)$. It shows that $E/O_p(E)$ is a chief factor of G.

Since $(|E|, p-1) = 1$, $Z_{\mathfrak{U}}(E) = Z_{\infty}(E)$. It follows from Proposition 3.4 that $O_p(E) \leq Z_{\mathfrak{U}}(G) \cap E \leq Z_{\mathfrak{U}}(E) = Z_{\infty}(E)$.

(3) $p = 2$ and $E/O_2(E)$ is a non-abelian chief factor of G.

If $p \nmid |E/O_p(E)|$, then $E/O_p(E)$ is p-nilpotent, and so by (2), E is p-nilpotent, a contradiction. Hence $p | E/O_p(E)|$. Since $E/O_p(E)$ is a chief factor of G, $E/O_p(E)$ is non-abelian, and thereby E is not soluble. Since $(|E|, p-1) = 1$, by Feit-Thompson Theorem, we have $p = 2$.

(4) Final contradiction.

By [16, Chap. IV, Satz 5.4], E has a 2-closed minimal non 2-nilpotent subgroup A. Let A_2 be a Sylow 2-subgroup of A contained in P. Then by [1, Chap. VII, Theorem 6.18], $A_2/\Phi(A_2)$ is a chief factor of A; $\Phi(A) = Z_{\infty}(A)$; $\Phi(A_2) = A_2 \cap \Phi(A)$ and the exponent of A_2 is p or 4 (when P is a non-abelian 2-group). By (2), $A_2 \cap O_2(E) \leq A_2 \cap Z_{\infty}(E) \leq A_2 \cap Z_{\infty}(A) = A_2 \cap \Phi(A) = \Phi(A_2)$. Hence there exists an element $x \in A_2 \backslash O_2(E)$. Let $H = \langle x \rangle$. Then $|H| = p$ or 4 (when P is a non-abelian 2-group). By hypothesis, G/H_G has a subnormal subgroup T/H_G and a τ-subgroup S/H_G contained in H/H_G such that $G = HT = A_2T$ and $(H/H_G) \cap (T/H_G) \leq (S/H_G)\Phi(H/H_G)$. If $H/H_G = S/H_G$, then $HO_2(E)/O_2(E)$ is a τ-subgroup of $G/O_2(E)$ because τ is inductive. Since τ is regular and $E/O_2(E)$ is a minimal normal subgroup of $G/O_2(E)$, $|G/O_2(E) : N_{G/O_2(E)}(E/O_2(E) \cap HO_2(E)/O_2(E))| = |G : N_G(HO_2(E))|$ is a power of 2. Hence $(HO_2(E))^G$ is a 2-group, and so $H \leq O_2(E)$, a contradiction. Therefore, we assume that $S/H_G < H/H_G$. Then $H \cap T \leq \Phi(H)$, and so $A \nleq T$. Since $A = A_2(A \cap T)$, $A \cap T \neq 1$. It implies that $A \cap T$ is a 2-nilpotent group because that A is a minimal non 2-nilpotent group. Let $A_{2'}$ be a normal 2-complement of $A \cap T$. Since $A_{2'}$ is a subnormal Hall subgroup of A, $A_{2'} \trianglelefteq A$. It implies that $A_{2'}$ is a normal 2-complement of A, which is impossible. The proof of the theorem is completed. □

Proposition 3.6 *Let E be a normal subgroup of G and τ a regular inductive subgroup functor. Suppose that every cyclic subgroup of P of prime order or order 4 (when P is a non-abelian 2-group) is Φ-τ-supplement in G. Then $E \leq Z_{\mathfrak{U}}(G)$.*

Proof See the proof of Proposition 3.3 and use Proposition 3.4 and Theorem 3.5 instead of Proposition 3.1 and Theorem 3.2. □

Theorem 3.7 *Let $\mathfrak{F}$ be a formation containing all supersoluble groups, τ a regular inductive subgroup functor and E a normal subgroup of G such that $G/E \in \mathfrak{F}$. Suppose that $X = E$ or $X = F^*(E)$. If one of the following holds:*

(i) Every τ-subgroup of G contained in E is subnormally embedded in G and every maximal subgroup of every noncyclic Sylow subgroup of X is Φ-τ-supplement in G;

(ii) For every noncyclic Sylow subgroup P of X, every cyclic subgroup of P of prime order or order 4 (when P is a non-abelian 2-group) is Φ-τ-supplement in G.

Then $G \in \mathfrak{F}$.

Proof By Propositions 3.3 and 3.6, we have that $X \leq Z_{\mathfrak{U}}(G) \leq Z_{\mathfrak{F}}(G)$. Therefore, by Lemma 2.6, $E \leq Z_{\mathfrak{F}}(G)$. Consequently, $G \in \mathfrak{F}$. □

4. Further applications

In view of [11, Examples 1.5, 1.7 and 1.9] and [12, Examples 4.6 and 4.9], many results in former literatures can be generalized by our theorem. For example, [17, Theorems 3.2 and 3.7]; [18, Theorems 3.1 and 3.3]; [19, Theorem 3.3]; [20, Theorems 3.4, 3.7 and Corollary 3.5]; [21, Theorems 3.2, 3.5 and Corollary 3.2]; [10, Lemmas 2.2, 2.3 2.4 and Theorem 3.1]; [22, Theorems 3.1 and 3.2]; [23, Theorem]; [24, Theorems 3.2, 3.3, 3.6 and 3.7] and so on.

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