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Weighted Estimates for the Iterated Commutators of Multilinear Operators with Mild Regularity Kernels

Jie SUN[1,*], **Xianliang SHI**[2]
1. *Department of Mathematics, Mudanjiang Normal University, Heilongjiang* 157011*, P. R. China;*
2. *College of Mathematics and Computer Science, Hunan Normal University, Hunan* 410081*, P. R. China*

Abstract In this paper, we study the iterated commutators of BMO functions and multilinear operators kernel of which satisfies some mild regularity condition. Some new properties for the commutator on weighted Lebesgue spaces are established.

Keywords multilinear operator; weight function; sharp maximal function; commutator; BMO function

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1. Introduction and results

In 1975, Coifman and Meyer [1] studied the bilinear singular integral operators. Then many researchers were interested in the bilinear or multilinear sigular integrals [2–10]. The multilinear operators T are initially defined on the m-fold product of Schwartz space $\mathscr{S}(\mathbb{R}^n)$, and take their values into the space of tempered distributions on $\mathscr{S}'(\mathbb{R}^n)$,

$$T:\mathscr{S}(\mathbb{R}^n)\times\cdots\times\mathscr{S}(\mathbb{R}^n)\to\mathscr{S}'(\mathbb{R}^n).$$

We will assume that the distributional kernel of the operator K on $(\mathbb{R}^n)^{m+1}$ is defined away from the diagonal $\{(x,y_1,\dots,y_m): x=y_1=\cdots=y_m\}$ in $(\mathbb{R}^n)^{m+1}$ so that

$$T(f_1,f_2,\dots,f_m)(x)=\int_{(\mathbb{R}^n)^m}K(x,y_1,\dots,y_m)f_1(y_1)\cdots f_m(y_m)\mathrm{d}y_1\cdots\mathrm{d}y_m,$$

whenever $f_1,\dots,f_m\in C_0^\infty(\mathbb{R}^n)$ and $x\notin\cap_{j=1}^m\mathrm{supp}\,f_j$.

Grafakos and Torres [7] discussed the multilinear operator T with kernel K satisfying the s-tandard estimates and obtained the weak endpoint estimate of the multilinear Calderón-Zygmund operator. Grafakos and Torres [8] established the weighted estimates with A_p weights for T and the corresponding maximal operator. Pérez and Torres [10] used a variant of the sharp maximal

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* Corresponding author
E-mail address: sj800816@163.com (Jie SUN); xianliangshi@yahoo.com (Xianliang SHI)

operator $M^\sharp$ of Fefferman and Stein [11] to get the weighted estimates of the operator T. One can see [12]–[19] for more estimates about multilinear operators.

In this paper, we consider the multilinear operator which was discussed in [16]. Suppose that T is a multilinear operator and there exist $p_0 \geq 1$ and a constant $C > 0$ so that the following conditions hold

(H1) For all $p_0 \leq q_1, \dots, q_m < \infty$ and $0 < q < \infty$ satisfying

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q},$$

T is bounded from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$.

(H2) There exists $\delta > \frac{n}{p_0}$, so that for the conjugate exponent p_0' of p_0, one has

$$\begin{aligned}&\Big(\int_{S_{j_m}(Q)} \cdots \int_{S_{j_1}(Q)} |K(x, y_1, \dots, y_m) - K(x', y_1, \dots, y_m)|^{p_0{}'} \mathrm{d}y_1 \cdots \mathrm{d}y_m\Big)^{1/p_0{}'}\\ &\leq C \frac{(|x-x'|)^{m(\delta - \frac{n}{p_0})}}{|Q|^{\frac{m\delta}{n}}} 2^{-m\delta j_0},\end{aligned}$$

for all balls Q, all $x, x' \in \frac{Q}{2}$, and $(y_1, \dots, y_m) \neq (0, \dots, 0)$, where $j_0 = \max\{j_k : k = 1, \dots, m\}$, and $S_j(Q) = 2^j Q \setminus 2^{j-1} Q$, if $j \geq 1$, otherwise $S_j(Q) = Q$.

Bui and Duong [16] proved the weighted boundedness of commutator $T_{\sum \mathbf{b}}(\vec{f})$ which is defined as follows:

$$T_{\sum \mathbf{b}}(\vec{f})(x) = \sum_{j=1}^{m} T_{b_j}^{j}(\vec{f})(x),$$

where $\vec{f} = (f_1, \dots, f_m)$, $\mathbf{b} = (b_1, \dots, b_m)$, $b_j \in \mathrm{BMO}(\mathbb{R}^n)$, $1 \leq j \leq m$, and

$$\begin{aligned}&[b_j, T]_j(\vec{f})(x) = T_{b_j}^{j}(\vec{f})(x)\\ &\quad = b_j(x) T(f_1, \dots, f_j, \dots, f_m)(x) - T(f_1, \dots, b_j f_j, \dots, f_m)(x), \quad j = 1, \dots, m.\end{aligned}$$

For the multilinear operator T, $\mathbf{b} = (b_1, \dots, b_m)$ is a family of locally integrable functions. In 2014, Pérez, Pradolini et al. [20] considered the iterated commutator of multilinear operator

$$T_{\Pi \mathbf{b}}(f_1, \dots, f_m) = [b_1, [b_2, \cdots [b_{m-1}, [b_m, T]_m]_{m-1}, \dots]_2]_1(f_1, \dots, f_m),$$

that is

$$T_{\Pi \mathbf{b}}(\vec{f})(x) = \int_{(\mathbf{R}^n)^m} \prod_{j=1}^{m} (b_j(x) - b_j(y_j)) K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m) \mathrm{d}y_1 \cdots \mathrm{d}y_m.$$

When $m = 1$, $T_{\sum \mathbf{b}}(\vec{f}) = T_{\Pi \mathbf{b}}(\vec{f}) = [b, T]f = bT(f) - T(bf)$, which is the well-known classical commutator with BMO function in [21]. They got that $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$. In this paper, we mainly discuss the weighted boundedness of the iterated commutator $T_{\Pi \mathbf{b}}(\vec{f})$, when the multilinear operator T satisfies (H1), (H2) and $b_j \in \mathrm{BMO}(\mathbb{R}^n), 1 \leq j \leq m$.

Our main results are the following theorems.

Theorem 1.1 *Assume that T satisfies (H1) and (H2). For all $p_0 < p_1, \dots, p_m < \infty$ and*

$0 < p < \infty$ *with*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p},$$

and $b_j \in \mathrm{BMO}(\mathbb{R}^n), 1 \leq j \leq m$, $w \in A_{\tilde{p}/p_0}$, $\tilde{p} = \min\{p_1, \ldots, p_m\}$. *Then, there exists a constant* $C > 0$ *such that*

$$\|T_{\Pi\mathbf{b}}(\vec{f})\|_{L^p(w)} \leq C \prod_{j=1}^{m} \|b_j\|_{\mathrm{BMO}} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w)}.$$

Theorem 1.2 *Assume that* T *satisfies (H1) and (H2). For all* $p_0 < p_1, \ldots, p_m < \infty$ *and* $0 < p < \infty$ *with*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p},$$

and $b_j \in \mathrm{BMO}(\mathbb{R}^n)$, $1 \leq j \leq m$, $\vec{w} \in A_{\vec{P}/p_0}$. *Then, there exists a constant* $C > 0$ *such that*

$$\|T_{\Pi\mathbf{b}}(\vec{f})\|_{L^p(v_{\vec{w}})} \leq C \prod_{j=1}^{m} \|b_j\|_{\mathrm{BMO}} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}.$$

The rest of this paper is organized as follows. After recalling some preliminary notations and lemmas in Section 2, we will prove our results in Section 3. We would like to remark that the main methods employed in this paper is a combination of ideas and arguments from [16] and [19].

Throughout this paper, we let p' satisfy $1/p + 1/p' = 1$. The letter C, sometimes with additional parameters, will stand for positive constants, not necessarily the same one at each occurrence but is independent of the essential variables.

2. Preliminaries and lemmas

In order to prove the theorems, we will formulate some lemmas and preliminaries. For a function $f \in L_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal and Sharp maximal functions are defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \big|f(y)\big| \mathrm{d}y,$$

and

$$M^\sharp f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \big|f(y) - f_B\big| \mathrm{d}y \approx \sup_{B \ni x} \inf_{C} \frac{1}{|Q|} \int_Q \big|f(y) - C\big| \mathrm{d}y$$

respectively, where f_Q denotes the average of f over ball Q.

For $\delta > 0$, we denote $M_\delta(f)$ and $M^\sharp_\delta(f)$ by

$$M_\delta(f) = M(|f|^\delta)^{\frac{1}{\delta}}, \quad M^\sharp_\delta(f) = [M^\sharp(|f|^\delta)]^{1/\delta}.$$

Let $0 < p$, $\delta < \infty$, $\omega \in A_\infty$. Then there exists a constant C, such that

$$\int_{\mathbb{R}^n} M_\delta f(x)^p \omega(x) \mathrm{d}x \leq C \int_{\mathbb{R}^n} M^\sharp_\delta f(x)^p \omega(x) \mathrm{d}x,$$

for any function f for which the left hand side is finite.

Following the notation in [22], for m exponents $p_1, \dots, p_m$, we denote by p the number given by $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\vec{P}$ for the vector $\vec{P} = (p_1, \dots, p_m)$. Let $1 \le p_1, \dots, p_m < \infty$. A multiple weight $\vec{w} = (w_1, \dots, w_m)$ is said to satisfy the multiple weight $A_{\vec{P}}$ condition if for

$$v_{\vec{w}} = \Pi_{j=1}^m {w_j}^{p/p_j},$$

it holds that

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q v_{\vec{w}}\Big)^{1/p_1} \Pi_{j=1}^m \Big(\frac{1}{|Q|}\int_Q {w_j}^{1-p_j'}\Big)^{1/p_j'} < \infty.$$

When $p_j = 1$, $(\frac{1}{|Q|}\int_Q {w_j}^{1-p_j'})^{1/p_j'}$ is understood as $(\sup_Q w_j)^{-1}$. One can check that $A_{(1,\dots,1)}$ is contained in $A_{\vec{P}}$ for each $\vec{P}$. In fact, one has

$$\Pi_{j=1}^m A_{p_j} \subset A_{\vec{P}}$$

with strict containment. Moreover,

$$\vec{w} \in A_{\vec{P}} \Longleftrightarrow \begin{cases} w_j^{1-p_j'} \in A_{mp_j'}, \ j = 1, \dots, m \\ v_{\vec{w}} \in A_{mp} \end{cases}$$

where the condition $w_j^{1-p_j'} \in A_{mp_j'}$ in the case $p_j = 1$ is understood as $w_j^{1/m} \in A_1$. Observe that in the linear case ($m = 1$) the above condition represents the same A_p condition.

We will use the following Kolmogorov inequality

$$\|f\|_{L^p(Q, \frac{\mathrm{d}x}{|Q|})} \le C\|f\|_{L^{q,\infty}(Q, \frac{\mathrm{d}x}{|Q|})},$$

where $0 < p < q < \infty$. See ([22,23]).

The following lemma is our main ingredient in the proof of our main results.

Lemma 2.1 ([22]) *If $w \in A_{\vec{P}}$, then $v_{\vec{w}} \in A_{mp}$ and there exists $\min\{p_1, \dots, p_m\} > r > 1$ such that $\vec{w} \in A_{\vec{P}/r}$, where $\vec{P}/r = (p_1/r, \dots, p_m/r)$.*

For $\vec{f} = (f_1, \dots, f_m)$ and $p \ge 1$, we define the operator

$$\mathcal{M}_p(\vec{f})(x) = \sup_{Q \ni x} \Pi_{j=1}^m \Big(\frac{1}{|Q|}\int_Q |f_j(y_j)|^p \mathrm{d}y_j\Big)^{1/p}.$$

As a result of [16, Proposition 2.3], we have

Lemma 2.2 *Assume that T satisfies (H1) and (H2). And let $0 < \delta < \epsilon < p_0/m$. Then for any $q_0 > p_0$, we have*

$$M_\delta^\sharp(T_{b_j}^j(\vec{f}))(x) \le C\|b_j\|_{\mathrm{BMO}}\mathcal{M}_{q_0}(\vec{f})(x).$$

Lemma 2.3 ([16]) *Assume that $p_0 \ge 1$ and $p_j > p_0$ ($\forall j = 1, \dots, m$) with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. We have*

$$\|\mathcal{M}_p(\vec{f})\|_{L^p(v_{\vec{w}})} \le C\Pi_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}$$

if and only if $\vec{w} \in A_{\vec{P}/p_0}$, where $\vec{P}/p_0 = (p_1/p_0, \dots, p_m/p_0)$.

Lemma 2.4 ([16]) *Suppose T satisfies (H1) and (H2). For any $p_0 < p_1, \dots, p_m < \infty$ and p so* *that* $1/p = 1/p_1 + \cdots + 1/p_m$ *and* $\vec{w} \in A_{\vec{P}/p_0}$, *we have*

$$\|T(\vec{f})\|_{L^p(v_{\vec{w}})} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}.$$

3. Proofs of main theorems

For the sake of simplicity, we only consider the case $m = 2$ and the proof of Theorems is based on the following estimate of sharp maximal function.

Lemma 3.1 *Suppose that* T *satisfies (H1) and (H2). Let* $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$, $0 < \delta < 1/3 < \varepsilon < 1/2$. *For* $q_0 > p_0$, *then we have*

$$\begin{aligned} M_\delta^\sharp(T_{\Pi\mathbf{b}}(f_1, f_2))(x) \leq & C\|b_1\|_{\mathrm{BMO}} M_\varepsilon(T_{b_2}^2(f_1, f_2))(x) + C\|b_2\|_{\mathrm{BMO}} M_\varepsilon(T_{b_1}^1(f_1, f_2))(x) + \\ & C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}} M_\varepsilon(T(f_1, f_2))(x) + C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}} M_{q_0}(\vec{f})(x). \end{aligned}$$

Proof For any constants λ_1 and λ_2, write

$$\begin{aligned} T_{\Pi\mathbf{b}}(\vec{f})(x) = & (b_1(x) - \lambda_1)(b_2(x) - \lambda_2)T(f_1, f_2)(x) - (b_1(x) - \lambda_1)T(f_1, (b_2 - \lambda_2)f_2)(x) - \\ & (b_2(x) - \lambda_2)T((b_1 - \lambda_1)f_1, f_2)(x) + T((b_1 - \lambda_1)f_1, (b_2 - \lambda_2)f_2)(x) \\ = & -(b_1(x) - \lambda_1)(b_2(x) - \lambda_2)T(f_1, f_2)(x) + (b_1(x) - \lambda_1)T^2_{(b_2 - \lambda_2)}(f_1, f_2)(x) + \\ & (b_2(x) - \lambda_2)T^1_{(b_1 - \lambda_1)}(f_1, f_2)(x) + T((b_1 - \lambda_1)f_1, (b_2 - \lambda_2)f_2)(x). \end{aligned}$$

For any fixed $x \in R^n$, a ball Q centered at x and a constant c, since $0 < \delta < \frac{1}{3}$, we can estimate

$$\begin{aligned} & \Big(\frac{1}{|Q|}\int_Q \big||T_{\Pi\mathbf{b}}(f_1, f_2)(z)|^\delta - |c|^\delta\big| \mathrm{d}z\Big)^{\frac{1}{\delta}} \\ & \leq \Big(\frac{1}{|Q|}\int_Q |T_{\Pi\mathbf{b}}(f_1, f_2)(z) - c|^\delta \mathrm{d}z\Big)^{\frac{1}{\delta}} \\ & \leq C\Big(\frac{1}{|Q|}\int_Q |(b_1(z) - \lambda_1)(b_2(z) - \lambda_2)T(f_1, f_2)(z)|^\delta \mathrm{d}z\Big)^{\frac{1}{\delta}} + \\ & \quad C\Big(\frac{1}{|Q|}\int_Q |(b_1(z) - \lambda_1)T^2_{(b_2 - \lambda_2)}(f_1, f_2)(z)|^\delta \mathrm{d}z\Big)^{\frac{1}{\delta}} + \\ & \quad C\Big(\frac{1}{|Q|}\int_Q |(b_2(z) - \lambda_2)T^1_{(b_1 - \lambda_1)}(f_1, f_2)(z)|^\delta \mathrm{d}z\Big)^{\frac{1}{\delta}} + \\ & \quad C\Big(\frac{1}{|Q|}\int_Q |T((b_1 - \lambda_1)f_1, (b_2 - \lambda_2)f_2)(z) - c|^\delta \mathrm{d}z\Big)^{\frac{1}{\delta}} \\ & = I + II + III + IV. \end{aligned}$$

To deal with I. Set $\lambda_j = (b_j)_{Q^*}$, $Q^* = 8Q$, there exist $s_1, s_2, s_3 > 1$, $\delta s_3 < \varepsilon < \frac{p_0}{2}$ and $\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} = 1$. Using Hölder's inequality and Jensen's inequality, we have

$$\begin{aligned} I \leq & C\Big(\frac{1}{|Q|}\int_Q |b_1(z) - \lambda_1|^{s_1\delta} \mathrm{d}z\Big)^{\frac{1}{s_1\delta}} \Big(\frac{1}{|Q|}\int_Q |b_2(z) - \lambda_2|^{s_2\delta} \mathrm{d}z\Big)^{\frac{1}{s_1\delta}} \times \\ & \Big(\frac{1}{|Q|}\int_Q |T(f_1, f_2)(z)|^{s_3\delta} \mathrm{d}z\Big)^{\frac{1}{s_3\delta}} \end{aligned}$$

$$\leq C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}}M_\varepsilon(T(f_1,f_2))(x).$$

Since II is similarly as III. We only consider II. For $1<t_1,t_2<\infty, t_2<\varepsilon/\delta<\frac{p_0}{2\delta}$ satisfying $\frac{1}{t_1}+\frac{1}{t_2}=1$, we get

$$\begin{aligned}II \leq & C\Big(\frac{1}{|Q|}\int_Q|b_1(z)-\lambda_1|^{t_1\delta}\mathrm{d}z\Big)^{\frac{1}{t_1\delta}}\Big(\frac{1}{|Q|}\int_Q|T^2_{b_2-\lambda}(f_1,f_2)(z)|^{t_2\delta}\mathrm{d}z\Big)^{\frac{1}{t_2\delta}}\\ \leq & C\|b_1\|_{\mathrm{BMO}}M_\varepsilon(T^2_{b_2-\lambda}(f_1,f_2))(x)=C\|b_1\|_{\mathrm{BMO}}M_\varepsilon(T^2_{b_2}(f_1,f_2))(x).\end{aligned}$$

Similarly, we have

$$III\leq C\|b_2\|_{\mathrm{BMO}}M_\varepsilon(T^1_{b_1}(f_1,f_2))(x).$$

Next we consider IV, decompose $f_i=f_i\chi_{Q^*}+f_i\chi_{(Q^*)^c}$, $c=\sum_{j=1}^3 c_j$, where

$$c_1=T(f_1\chi_{Q^*},(b_2-\lambda_2)f_2\chi_{(Q^*)^c})(x),$$

$$c_2=T(f_1\chi_{(Q^*)^c},(b_2-\lambda_2)f_2\chi_{Q^*})(x),$$

$$c_3=T(f_1\chi_{(Q^*)^c},(b_2-\lambda_2)f_2\chi_{(Q^*)^c})(x).$$

Thus

$$\begin{aligned}IV \leq & \Big(\frac{C}{|Q|}\int_Q|T((b_1-\lambda_1)f_1\chi_{Q^*},(b_2-\lambda_2)f_2\chi_{Q^*})(z)|^\delta\mathrm{d}z\Big)^{\frac{1}{\delta}}+\\ & \Big(\frac{C}{|Q|}\int_Q|T((b_1-\lambda_1)f_1\chi_{Q^*},(b_2-\lambda_2)f_2\chi_{(Q^*)^c})(z)-c_1|^\delta\mathrm{d}z\Big)^{\frac{1}{\delta}}+\\ & \Big(\frac{C}{|Q|}\int_Q|T((b_1-\lambda_1)f_1\chi_{(Q^*)^c},(b_2-\lambda_2)f_2\chi_{Q^*})(z)-c_2|^\delta\mathrm{d}z\Big)^{\frac{1}{\delta}}+\\ & \Big(\frac{C}{|Q|}\int_Q|T((b_1-\lambda_1)f_1\chi_{(Q^*)^c},(b_2-\lambda_2)f_2\chi_{(Q^*)^c})(z)-c_3|^\delta\mathrm{d}z\Big)^{\frac{1}{\delta}}\\ :=& IV_1+IV_2+IV_3+IV_4.\end{aligned}$$

There exists $1<s<\frac{1}{3\delta}$, such that $s\delta<\frac{1}{3}<\frac{p_0}{2}$, and there exists $q_0>p_0$, we get

$$\begin{aligned}IV_1 \leq & \Big(\frac{C}{|Q|}\int_Q|T((b_1-\lambda_1)f_1\chi_{Q^*},(b_2-\lambda_2)f_2\chi_{Q^*})(z)|^{s\delta}\mathrm{d}z\Big)^{\frac{1}{s\delta}}\\ \leq & C\|T((b_1-\lambda_1)f_1\chi_{Q^*},(b_2-\lambda_2)f_2\chi_{Q^*})\|_{L^{\frac{p_0}{2},\infty}(Q,\frac{dx}{|Q|})}\\ \leq & C\Big(\frac{1}{|Q|}\int_{Q^*}|(b_1(z)-\lambda_1)f_1(z)|^{p_0}\mathrm{d}z\Big)^{\frac{1}{p_0}}\times\\ & \Big(\frac{1}{|Q|}\int_{Q^*}|(b_2(z)-\lambda_2)f_2(z)|^{p_0}\mathrm{d}z\Big)^{\frac{1}{p_0}}\\ \leq & C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}}M_{q_0}(\vec{f})(x).\end{aligned}$$

The proofs of IV_2, IV_4 are similarly as IV_3, we only consider IV_2,

$$\begin{aligned}&|T((b_1-\lambda_1)f_1\chi_{Q^*},(b_2-\lambda_2)f_2\chi_{(Q^*)^c})(z)-T(f_1\chi_{Q^*},(b_2-\lambda_2)f_2\chi_{(Q^*)^c})(x)|\\ &\leq\int_{Q^*}|(b_1(y_1)-\lambda_1)f_1(y_1)|\int_{R^n\setminus Q^*}|K(z,y_1,y_2)-K(x,y_1,y_2)||(b_2(y_2)-\lambda_2)f_2(y_2)|\mathrm{d}y_2\mathrm{d}y_1\end{aligned}$$

$$\begin{aligned}
&\le C\Big(\int_{Q^*}|(b_1(y_1)-\lambda_1)f_1(y_1)|^{p_0}\mathrm{d}y_1\Big)^{\frac{1}{p_0}}\times\\
&\quad\sum_{j_1=1}^{m}\Big(\int_{Q^*}\int_{S_{j_1}(Q^*)}|K(z,y_1,y_2)-K(x,y_1,y_2)|^{p_0'}\mathrm{d}y_2\mathrm{d}y_1\Big)^{\frac{1}{p_0'}}\times\\
&\quad\Big(\int_{S_{j_1}(Q^*)}|(b_2(y_2)-\lambda_2)f_2(y_2)|^{p_0}\mathrm{d}y_2\Big)^{\frac{1}{p_0}}\\
&\le C\Big(\int_{Q^*}|(b_1(y_1)-\lambda_1)f_1(y_1)|^{p_0}\mathrm{d}y_1\Big)^{\frac{1}{p_0}}\times\\
&\quad\sum_{j_1=1}^{\infty}\frac{|x-z|^{2(\delta-\frac{n}{p_0})}}{|Q^*|^{\frac{2\delta}{n}}}2^{-2\delta j_1}\Big(\int_{S_{j_1}(Q^*)}|(b_2(y_2)-\lambda_2)f_2(y_2)|^{p_0}\mathrm{d}y_2\Big)^{\frac{1}{p_0}}\\
&\le C\sum_{j_1=1}^{\infty}2^{2j_1(-\delta+\frac{n}{p_0})}\Big(\frac{1}{|2^{j_1}Q^*|}\int_{2^{j_1}Q^*}|(b_1(y_1)-\lambda_1)f_1(y_1)|^{p_0}\mathrm{d}y_1\Big)^{\frac{1}{p_0}}\times\\
&\quad\Big(\frac{1}{|2^{j_1}Q^*|}\int_{2^{j_1}Q^*}|(b_2(y_2)-\lambda_2)f_2(y_2)|^{p_0}\mathrm{d}z\Big)^{\frac{1}{p_0}}\\
&\le C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}}M_{q_0}(\vec{f})(x).
\end{aligned}$$

Thus

$$IV_2\le C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}}M_{q_0}(\vec{f})(x),$$

$$IV_3\le C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}}M_{q_0}(\vec{f})(x),$$

$$IV_4\le C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}}M_{q_0}(\vec{f})(x).$$

Thus we complete the proof of Lemma 3.1. □

When Theorem 1.2 is proved, Theorem 1.1 is obviously true. We only prove Theorem 1.2.

Proof of Theorem 1.2 For $0<\delta<1/3<\varepsilon<1/2$, and by Lemma 3.1, we have

$$\begin{aligned}
&\|T_{\Pi\mathbf{b}}(f_1,f_2)\|_{L^p(v_{\vec{w}})}\le\|M_\delta(T_{\Pi\mathbf{b}}(f_1,f_2))\|_{L^p(v_{\vec{w}})}\le C\|M_\delta^\sharp(T_{\Pi\mathbf{b}}(f_1,f_2))\|_{L^p(v_{\vec{w}})}\\
&\quad\le C\|b_1\|_{\mathrm{BMO}}\|M_\varepsilon(T_{b_2}^2(f_1,f_2))\|_{L^p(v_{\vec{w}})}+C\|b_2\|_{\mathrm{BMO}}\|M_\varepsilon(T_{b_1}^1(f_1,f_2))\|_{L^p(v_{\vec{w}})}+\\
&\quad\quad C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}}\|M_\varepsilon(T(f_1,f_2))\|_{L^p(v_{\vec{w}})}+C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}}\|M_{q_0}(\vec{f})\|_{L^p(v_{\vec{w}})}.
\end{aligned}$$

When $\varepsilon<\frac{1}{2}$. Since $\vec{w}\in A_{\vec{P}/p_0}$, there exists $r>1$ such that $\vec{w}\in A_{\vec{P}/rp_0}$. Take $q_0=rp_0$, by Lemma 2.2

$$M_\varepsilon(T_{b_j}^j(f_1,f_2))(x)\le C\|b_j\|_{\mathrm{BMO}}M_{q_0}(f_1,f_2)(x).$$

Thus

$$\|T_{\Pi\mathbf{b}}(f_1,f_2)\|_{L^p(v_{\vec{w}})}\le C\|b_1\|_{\mathrm{BMO}}\|b_2\|_{\mathrm{BMO}}\|M_{q_0}(f_1,f_2)+M_\varepsilon(T(f_1,f_2))\|_{L^p(v_{\vec{w}})}.$$

Since $\vec{w}\in A_{\vec{P}/p_0}$, $\vec{w}\in A_{\vec{P}/q_0}$, by Lemma 2.3, we have

$$\|M_{q_0}(f_1,f_2)\|_{L^p(v_{\vec{w}})}\le C\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(w_j)}.$$

Since $v_{\vec{w}} \in A_{2p/p_0}$, by Lemma 2.4, we get

$$\|M_\varepsilon(T(f_1,f_2))\|_{L^p(v_{\vec{w}})} \le C\|T(f_1,f_2)\|_{L^p(v_{\vec{w}})} \le C\prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

Thus we complete the proof of Theorem 1.2. $\square$

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