Journal of Mathematical Research with Applications
Jul., 2017, Vol. 37, No. 4, pp. 455–465
DOI:10.3770/j.issn:2095-2651.2017.04.007
Http://jmre.dlut.edu.cn

Morrey Spaces Associated to the Sections and Singular Integrals

Songbai WANG*, **Jinqiu WEN**
College of Mathematics and Statistics, Hubei Normal University, Hubei 435002*, P. R. China*

Abstract In this paper, we define the Morrey spaces $\mathcal{M}^{p,q}_{\mathcal{F}}(\mathbb{R}^n)$ and the Campanato spaces $\mathcal{E}^{p,q}_{\mathcal{F}}(\mathbb{R}^n)$ associated with a family $\mathcal{F}$ of sections and a doubling measure μ, where $\mathcal{F}$ is closely related to the Monge-Ampère equation. Furthermore, we obtain the boundedness of the Hardy-Littlewood maximal function associated to $\mathcal{F}$, Monge-Ampère singular integral operators and fractional integrals on $\mathcal{M}^{p,q}_{\mathcal{F}}(\mathbb{R}^n)$. We also prove that the Morrey spaces $\mathcal{M}^{p,q}_{\mathcal{F}}(\mathbb{R}^n)$ and the Campanato spaces $\mathcal{E}^{p,q}_{\mathcal{F}}(\mathbb{R}^n)$ are equivalent with $1\le q\le p<\infty$.

Keywords Morrey space; Campanato space; Monge-Ampère singular integral

MR(2010) Subject Classification 42B20;42B30

1. Introduction

To characterize the regularity of solutions to some partial differential equations, Morrey [1] first introduced the classical Morrey space as an extension of Lebesgue spaces. For $1\le q\le p\le\infty$, a function f is said to be in the Morrey space $\mathcal{M}^{p,q}(\mathbb{R}^n)$ if

$$\|f\|_{\mathcal{M}^{p,q}(\mathbb{R}^n)}:=\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{|B(x,r)|^{1/q-1/p}}\Big(\int_{B(x,r)}|f(y)|^q\mathrm{d}y\Big)^{1/q}<\infty. \tag{1.1}$$

A natural generalization of the $\mathrm{BMO}(\mathbb{R}^n)$ spaces is called Campanato space $\mathcal{L}^{p,q}(\mathbb{R}^n)$ (see [2]), via replacing the integrand $|f(y)|$ in (1.1) by $|f(y)-f_{B(x,r)}|$, where $f_{B(x,r)}:=\frac{1}{|B(x,r)|}\int_{B(x,r)}f(y)\mathrm{d}y$ denotes the average of f over the ball $B(x,r)$. Some tight relations between Morrey spaces and Campanato spaces were clarified by Peetre [3], for examples, $\mathcal{M}^{p,p}(\mathbb{R}^n)=L^p(\mathbb{R}^n)$ for all $p\in[1,\infty]$, $\mathcal{L}^{\infty,q}(\mathbb{R}^n)=\mathrm{BMO}(\mathbb{R}^n)$ for all $q\in[1,\infty)$ and $\mathcal{M}^{p,q}(\mathbb{R}^n)=\mathcal{L}^{p,q}(\mathbb{R}^n)$ whenever $1\le q<p<\infty$. Adams [4] seems to firstly start to research some properties of classical operators of harmonic analysis in Morrey spaces. The boundedness of Hardy-Littlewood maximal functions on classical Morrey spaces was obtained by Chiarenza and Frasca [5]. We could refer to [6–11] and the references therein for more information of these spaces and some recent developments.

The main purpose of this paper is to study some boundednesses of classical operators on Morrey spaces associated with a family $\mathcal{F}$ of sections which is closely related to the Monge-Ampère equation. We recall some notations and definitions associated with sections. We denote a family of sections by a collection of bounded convex sets $\mathcal{F}=\{S(x,t)\subset\mathbb{R}^n: x\in\mathbb{R}^n \text{ and } t>0\}$

Received August 7, 2016; Accepted February 27, 2017
Supported by Young Foundation of Education Department of Hubei Province (Grant No. Q20162504).
* Corresponding author
E-mail address: haiyansongbai@163.com (Songbai WANG)

satisfying certain axioms of affine invariance, which are based on the properties of the solutions of the real Monge-Ampère equations. In order to study real variable theory related to the Monge-Ampère equation, Caffarelli and Gutiérrez [12] introduced this concept, and meanwhile for a family of convex sets $\mathcal{F} = \{S(x,t) \subset \mathbb{R}^n : x \in \mathbb{R}^n \text{ and } t > 0\}$, they gave a Besicovitch type covering lemma to build a variant of the Calderón-Zygmund decomposition. As an application of the Calderón-Zygmund decomposition, the Hardy-Littlewood maximal operator M and $\mathrm{BMO}_{\mathcal{F}}$ space associated to the family $\mathcal{F}$ were well defined and the weak type $(1,1)$ boundedness of $M_{\mathcal{F}}$ and the John-Nirenberg inequality for $\mathrm{BMO}_{\mathcal{F}}$ were proved. In their another outstanding work [13], they studied the L^2 boundedness of the Monge-Ampère singular integral H and applied to the linear Monge-Ampère equation. We also refer the readers to [14]. Later, Incognito [15] used the theory of homogeneous space to prove the weak type $(1,1)$ of H. Ding and Lin [16] defined the Hardy space $H^1_{\mathcal{F}}$, which is just the dual of $\mathrm{BMO}_{\mathcal{F}}$, and showed that H is bounded from $H^1_{\mathcal{F}}$ to L^1. Tang [17] introduced the function space $\mathrm{BLO}_{\mathcal{F}}$ as a subset of $\mathrm{BMO}_{\mathcal{F}}$ and gave the boundedness of M from $\mathrm{BLO}_{\mathcal{F}}$ to $\mathrm{BMO}_{\mathcal{F}}$. Recently Lee [18] obtained the $H^1_{\mathcal{F}}$-boundedness of H by using the atom-molecule theory. In Lin's paper [19], Hardy spaces $H^p_{\mathcal{F}}$ with $1/2 < p \le 1$ and their dual spaces-Campanato spaces are studied, and the boundedness of H on these spaces are obtained.

For $x \in \mathbb{R}^n$ and $t > 0$, let $S(x,t)$ denote an open and bounded convex subset of $\mathbb{R}^n$ containing x. We call $S(x,t)$ a section if the family $\mathcal{F} := \{S(x,t) \subset \mathbb{R}^n : x \in \mathbb{R}^n, t > 0\}$ is monotone nondecreasing in t, i.e., $S(x,t) \subset S(x,t')$ for $t \le t'$, and satisfies the following three conditions:

(A) There are positive constants K_1, K_2, K_3 and ϵ_1, ϵ_2 such that for any two sections $S(x_0,t_0)$ and $S(x,t)$ with $t \le t_0$ satisfying

$$S(x_0,t_0) \cap S(x,t) \ne \emptyset$$

and an affine transformation T that "normalizes" $S(x_0,t_0)$, that is,

$$B(0,1/n) \subset T(S(x_0,t_0)) \subset B(0,1),$$

there exists $z \in B(0,K_3)$ depending on $S(x_0,t_0)$ and $S(x,t)$ which satisfies

$$B(z, K_2(t/t_0)^{\epsilon_2}) \subset T(S(x,t)) \subset B(z, K_1(t/t_0)^{\epsilon_1})$$

and

$$T(x) \in B(z, (1/2)K_2(t/t_0)^{\epsilon_2}).$$

Here and what follows $B(x,t)$ denotes the usual Euclidean ball centered at x with radius t.

(B) If T is an affine transformation that normalizes $S(x,t)$, then there is a constant $\delta > 0$ such that for any section $S(x,t)$, $y \in S(x,t)$ and $0 < \epsilon < 1$, $B(T(y), \epsilon^\delta) \cap T(S(x,(1-\epsilon)t)) = \emptyset$.

(C) $\bigcap_{t>0} S(x,t) = \{x\}$ and $\bigcup_{t>0} S(x,t) = \mathbb{R}^n$.

An important example of the family of sections comes from the Monge-Ampère equation [12]. Suppose $\phi : \mathbb{R}^n \to \mathbb{R}$ is a convex smooth function and let $\mathcal{L}(x)$ be a supporting hyperplane of ϕ at the point $(x, \phi(x))$ for any $x \in \mathbb{R}^n$. We define the set for any $t > 0$

$$S_\phi(x,t) = \{y \in \mathbb{R}^n : \phi(y) < \mathcal{L}(x) + t\}.$$

Then $\mathcal{F} = \{S_\phi(x,t) : x \in \mathbb{R}^n, t > 0\}$ is just the family of sections.

In addition, let the family $\mathcal{F}$ be equipped with a doubling Borel measure μ, which is finite on compact sets and $\mu(\mathbb{R}^n) = \infty$, where $\mathcal{F} = \{S(x,t) : x \in \mathbb{R}^n, t > 0\}$. There exists a constant A such that

$$\mu(S(x,2t)) \leq A\mu(S(x,t)) \text{ for any section } S(x,t) \in \mathcal{F}. \tag{1.2}$$

Aimar, Forzani and Toledano [20] used the properties (A) and (B) to get the following engulfing properties:

(D) There exists a constant $\theta > 1$, depending only on K_1, δ and ϵ_1, such that for $y \in S(x,r)$ we have

$$S(y,r) \subset S(x,\theta r) \text{ and } S(x,r) \subset S(y,\theta r). \tag{1.3}$$

By the family $\mathcal{F}$, we can define a quasi-metric $d(x,y)$ on $\mathbb{R}^n$ as

$$d(x,y) = \inf\{t : x \in S(y,t) \text{ and } y \in S(x,t)\}. \tag{1.4}$$

It is easy to check that for θ appearing in (1.3)

$$d(x,y) \leq \theta(d(x,z) + d(z,y)) \quad \text{for any } x,y,z \in \mathbb{R}^n. \tag{1.5}$$

Also,

$$S(x, \frac{t}{2\theta}) \subset B_d(x,t) \subset S(x,t) \text{ for any } x \in \mathbb{R}^n \text{ and } t > 0, \tag{1.6}$$

where $B_d(x,t) := \{y \in \mathbb{R}^n : d(x,y) < t\}$ is called a d-ball. From (1.2) and (1.6), it follows that $\mu(B_d(x,2t)) \leq A^{k_0}\mu(B_d(x,t))$ for any $x \in \mathbb{R}^n$ and $t > 0$ if we choose $k_0 \in N$ satisfying $2^{k_0-2} \geq \theta$. Hence, $(\mathbb{R}^n, d, \mu)$ is a homogeneous space of Coifman-Weiss type [21].

Let us define such a function ρ on $\mathbb{R}^n \times \mathbb{R}^n$ as $\rho(x,y) = \inf\{t > 0 : y \in S(x,t)\}$. The engulfing property of the sections (1.3) implies the following three properties of ρ (see [15]):

$$\rho(x,y) \leq \theta\rho(y,x), \quad \text{for all } x,y \in \mathbb{R}^n, \tag{1.7}$$

and

$$\rho(x,y) \leq \theta^2(\rho(x,z) + \rho(z,y)), \quad \text{for all } x,y,z \in \mathbb{R}^n, \tag{1.8}$$

and

(E) For a given section $S(x,t), y \in S(x,t)$ if and only if $\rho(x,y) < t$.

Next we introduce some definitions of operator associated to the family $\mathcal{F}$. At first, the maximal function Mf of f is defined by

$$Mf(x) := \sup_{x \in S \subset \mathcal{F}} \frac{1}{\mu(S)} \int_S |f(y)| \mathrm{d}\mu(y).$$

We consider the Monge-Ampère singular integral operator H as follows

$$H(f)(x) = \int_{\mathbb{R}^n} K(x,y) f(y) \mathrm{d}\mu(y),$$

where $K(x,y) = \sum_i^\infty k_i(x,y)$ satisfies the following conditions

(v1) $\mathrm{supp} k_i(\cdot,y) \subset S(y,2^i)$ for all $y \in \mathbb{R}^n$;

(v2) $\mathrm{supp} k_i(x,\cdot) \subset S(x,2^i)$ for all $x \in \mathbb{R}^n$;

(v3) $\int_{\mathbb{R}^n} k_i(x,y)\mathrm{d}\mu(y) = \int_{\mathbb{R}^n} k_i(x,y)\mathrm{d}\mu(x) = 0$ for all $x, y \in \mathbb{R}^n$;

(v4) $\sup_i \int_{\mathbb{R}^n} |k_i(x,y)|\mathrm{d}\mu(y) \le c_1$ for all $x \in \mathbb{R}^n$;

(v5) $\sup_i \int_{\mathbb{R}^n} |k_i(x,y)|\mathrm{d}\mu(x) \le c_2$ for all $y \in \mathbb{R}^n$;

(v6) If T is an affine transformation that normalizes the section $S(y, 2^i)$, then

$$|k_i(u,y) - k_i(v,y)| \le \frac{c_2}{\mu(S(y,2^i))}|T(u) - T(v)|^\alpha;$$

(v7) Finally, if T is an affine transformation that normalizes the section $S(x, 2^i)$, then

$$|k_i(x,v) - k_i(x,u)| \le \frac{c_2}{\mu(S(y,2^i))}|T(u) - T(v)|^\alpha,$$

where $0 < \alpha \le 1$ and $c_1, c_2 > 0$.

We emphasize here that Caffarelli and Gutiérrez [14] introduced and proved that H is bounded on $L^2(\mathbb{R}^n, \mathrm{d}\mu)$. Subsequently, the $L^p(\mathbb{R}^n, \mathrm{d}\mu)$, $1 < p < \infty$, and weak type $(1,1)$ estimate of H were obtained by Incognito [14].

For $\beta \in (0,1)$, the fractional integral operator I^β associated to the family $\mathcal{F}$ is defined by setting, for all real valued bounded functions f and $x \in \mathbb{R}^n$,

$$I^\beta(f)(x) := \int_{\mathbb{R}^n} \frac{f(y)}{\mu(S(x, d(x,y)))^{1-\beta}}\mathrm{d}\mu(y).$$

In this paper, we will study the boundedness of the above classical operators on Morrey space associated to the sections. It is pointed out here that many properties on our spaces could come down to the space of homogeneous type by the quasi-metric d.

2. Morrey spaces on sections

For $1 \le q \le p < \infty$, the Morrey space $\mathcal{M}_{\mathcal{F}}^{p,q}(\mathbb{R}^n)$ associated with the family $\mathcal{F}$ and the Borel measure μ satisfying the doubling condition (1.2) is defined to be the collection of all real-valued functions f on $\mathbb{R}^n$ such that

$$\|f\|_{\mathcal{M}_{\mathcal{F}}^{p,q}} := \sup_{S\in\mathcal{F}} \frac{1}{[\mu(S)]^{1/q-1/p}} \Big\{ \int_S |f(y)|^q \mathrm{d}\mu(y) \Big\}^{1/q} < \infty. \tag{2.1}$$

Remark 2.1 Because the family $\mathcal{F}$ is monotone increasing in t, the property of sections (C) implies that $L^p(\mathbb{R}^n, \mathrm{d}\mu) = \mathcal{M}_{\mathcal{F}}^{p,p}(\mathbb{R}^n)$ for all $p \in [1,\infty)$. And applying Hölder's inequality and the doubling condition (1.2) easily yields the following embedding relations: for all $1 \le q_1 \le q_2 \le p < \infty, \mathcal{M}_{\mathcal{F}}^{p,p}(\mathbb{R}^n) \subset \mathcal{M}_{\mathcal{F}}^{p,q_2}(\mathbb{R}^n) \subset \mathcal{M}_{\mathcal{F}}^{p,q_1}(\mathbb{R}^n)$ in the sense of continuous embedding.

Theorem 2.2 *If $1 < q \le p < \infty$, then the maximal operator M is bounded on $\mathcal{M}_{\mathcal{F}}^{p,q}(\mathbb{R}^n)$.*

Remark 2.3 This theorem was proved by Tang and Xu [22].

Proof By the definition of Morrey spaces, it suffices to show that for all $f \in \mathcal{M}_{\mathcal{F}}^{p,q}(\mathbb{R}^n)$ and $S \in \mathcal{F}$,

$$\frac{1}{\mu(S)^{1/q-1/p}} \Big\{ \int_S |M(f)(y)|\mathrm{d}\mu(y) \Big\}^{1/q} \le C\|f\|_{\mathcal{M}_{\mathcal{F}}^{p,q}}. \tag{2.2}$$

Fix a section $S = S(x_0, t_0)$ and set $\widetilde{S} = S(x_0, \theta^2 t_0)$. We split $f = f_1 + f_2$, where $f_1 = f\chi_{\widetilde{S}}$ and

$f_2 = f - f_1$. By the L^q-boundedness of M and (1.2), we obtain

$$\frac{1}{\mu(S)^{1/q-1/p}}\Big\{\int_S |M(f_1)(x)|\mathrm{d}\mu(x)\Big\}^{1/q} \le \frac{C}{[\mu(S)]^{1/q-1/p}}\Big\{\int_{\mathbb{R}^n}|f_1(y)|\mathrm{d}\mu(y)\Big\}^{1/q} \le C\|f\|_{\mathcal{M}^{p,q}_{\mathcal{F}}}.$$

Note that when $S(x',t)\cap S(x_0,t_0)\neq\emptyset$ and $S(x',t)\cap\widetilde{S}\neq\emptyset$, the property (D) implies that $t\ge t_0$, moreover, $S(x_0,t_0)\subset S(x_0,t)\subset S(y,\theta t)\subset S(x',\theta^2 t)$, where $y\in S(x',r)\cap S(x_0,t_0)$. Thus

$$\begin{aligned} M(f_2)(y) &= \sup_{S(x',t)\in\mathcal{F}}\frac{1}{\mu(S(x',t))}\int_{S(x',t)}|f(z)|\mathrm{d}\mu(z)\\ &\le \sup_{S(x',t):S\subset S(x',\theta^2 t)\in\mathcal{F}}\frac{1}{\mu(S(x',t))}\int_{S(x',t)}|f(z)|\mathrm{d}\mu(z). \end{aligned}$$

Using Hölder inequality, we can get for $S(x',r)$ with $S\subset S(x',\theta^2 r)$

$$\begin{aligned} \frac{1}{\mu(S(x',t))}\int_{S(x',t)}|f(z)|\mathrm{d}\mu(z) &\le \frac{1}{\mu(S(x',t))^{1/q}}\Big\{\int_{S(x',t)}|f(y)|^q\mathrm{d}\mu(z)\Big\}^{1/q}\\ &\le C\|f\|_{\mathcal{M}^{p,q}_{\mathcal{F}}}\mu(S(x',t))^{-1/p}\\ &\le C\|f\|_{\mathcal{M}^{p,q}_{\mathcal{F}}}\mu(S)^{-1/p}. \end{aligned}$$

Hence,

$$\frac{1}{\mu(S)^{1/q-1/p}}\Big\{\int_S |M(f_2)(y)|\mathrm{d}\mu(y)\Big\}^{1/q}\le C\|f\|_{\mathcal{M}^{p,q}_{\mathcal{F}}}.$$

We introduce another maximal function M_d as follows

$$M_d(f)(x) := \sup_{B_d\ni x}\frac{1}{\mu(B_d)}\int_{B_d}|f(y)|\mathrm{d}\mu(y),$$

where B_d is the d-ball. By (1.6), it is easy to see that $M(f)(x)$ is equivalent to $M_d(f)(x)$. As a consequence of Theorem 1.2 in [23], we also have the vector-valued inequality of M.

Lemma 2.4 *If $1<q\le\infty$ and $1<r\le\infty$, then there exists a constant C depending only on p,q,r,A,θ such that*

$$\Big\|\Big(\sum_{j\in\mathbb{N}}[M(f_j)]^r\Big)^{1/r}\Big\|_{L^{1,\infty}(\mathbb{R}^n,\mathrm{d}\mu)}\le C\Big\|\Big(\sum_{j\in\mathbb{N}}|f_j|^r\Big)^{1/r}\Big\|_{L^1(\mathbb{R}^n,\mathrm{d}\mu)}, \tag{2.3}$$

and

$$\Big\|\Big(\sum_{j\in\mathbb{N}}[M(f_j)]^r\Big)^{1/r}\Big\|_{L^q(\mathbb{R}^n,\mathrm{d}\mu)}\le C\Big\|\Big(\sum_{j\in\mathbb{N}}|f_j|^r\Big)^{1/r}\Big\|_{L^q(\mathbb{R}^n,\mathrm{d}\mu)}. \tag{2.4}$$

Theorem 2.5 *If $1<q\le p<\infty$ and $1<r\le\infty$, then there exists a constant C depending only on p,q,r,A,θ such that*

$$\Big\|\Big(\sum_{j\in\mathbb{N}}[M(f_j)]^r\Big)^{1/r}\Big\|_{\mathcal{M}^{p,q}_{\mathcal{F}}}\le C\Big\|\Big(\sum_{j\in\mathbb{N}}|f_j|^r\Big)^{1/r}\Big\|_{\mathcal{M}^{p,q}_{\mathcal{F}}}.$$

Proof Fix a section $S=S(x_0,t_0)\in\mathcal{F}$. For each $j\in\mathbb{N}$, set $f_j^1 := f_j\chi_{\widetilde{S}}$ and $f_j^2 := f - f_1$, where $\widetilde{S}=S(x_0,\theta^2 t_0)$. Since M is a sublinear operator, we have

$$M(f_j)(x)\le M(f_j^1)(x)+M(f_j^2)(x).$$

Then, applying Minkowski inequality, we see that

$$\begin{aligned}&\frac{1}{\mu(S)^{1/q-1/p}}\Big\{\int_S\Big(\sum_{j\in\mathbb{N}}[M(f_j)(x)]^r\Big)^{q/r}\mathrm{d}\mu(x)\Big\}^{1/q}\\&\le\frac{1}{\mu(S)^{1/q-1/p}}\Big\{\int_S\Big(\sum_{j\in\mathbb{N}}[M(f_j^1)(x)]^r\Big)^{q/r}\mathrm{d}\mu(x)\Big\}^{1/q}+\\&\quad\frac{1}{\mu(S)^{1/q-1/p}}\Big\{\int_S\Big(\sum_{j\in\mathbb{N}}[M(f_j^2)(x)]^r\Big)^{q/r}\mathrm{d}\mu(x)\Big\}^{1/q}=\mathrm{I}+\mathrm{II}.\end{aligned}$$

By (2.3), we can estimate

$$\begin{aligned}\mathrm{I}&\le\frac{C}{\mu(S)^{1/q-1/p}}\Big\{\int_{\widetilde{S}}\Big(\sum_{j\in\mathbb{N}}|f_j(x)|^r\Big)^{q/r}\Big\}^{1/q}\\&\le\frac{C\mu(\widetilde{S})^{1/q-1/p}}{\mu(S)^{1/q-1/p}}\Big\|\Big(\sum_{j\in\mathbb{N}}|f_j|^r\Big)^{1/r}\Big\|_{\mathcal{M}_{\mathcal{F}}^{p,q}}\le C\Big\|\Big(\sum_{j\in\mathbb{N}}|f_j|^r\Big)^{1/r}\Big\|_{\mathcal{M}_{\mathcal{F}}^{p,q}}.\end{aligned}$$

Indeed, we know that for any $j\in\mathbb{N}$ and $x\in S$,

$$M(f_j^2)(x)\le\sup_{S(x',t):S\subset S(x',\theta^2t)\in\mathcal{F}}\frac{1}{\mu(S(x',t))}\int_{S(x',t)}|f(z)|\mathrm{d}\mu(z).\tag{2.5}$$

For any $j\in\mathbb{N}$, there exists a sequence of sections, $\{S_{j,k}=S(x_{j,k},t_{j,k})\}_{k\in\mathbb{N}}$, satisfying that $S\subset S_{j,k}$ and that

$$\sup_{S(x',t):S\subset S(x',\theta^2t)\in\mathcal{F}}\frac{1}{|\mu(S(x',t))|}\int_{S(x',t)}|f(z)|\mathrm{d}\mu(z)=\lim_{k\to\infty}\frac{1}{\mu(S_{j,k})}\int_{S_{j,k}}|f_j(z)|\mathrm{d}\mu(z),$$

which, combined with (2.5) and Fatou's lemma, implies that

$$\begin{aligned}\mathrm{II}&\le\mu(S)^{1/p}\Big\{\sum_{j\in\mathbb{N}}\Big(\sum_{j\in\mathbb{N}}\sup_{S(x',t):S\subset S(x',\theta^2t)\in\mathcal{F}}\frac{1}{\mu(S(x',t))}\int_{S(x',t)}|f(z)|\mathrm{d}\mu(z)\Big)^r\Big\}^{1/r}\\&\le\liminf_{k\to\infty}\Big\{\mu(S)^{1/p}\Big(\sum_{j\in\mathbb{N}}\Big[\frac{1}{\mu(S_{j,k})}\int_{S_{j,k}}|f(z)|\mathrm{d}\mu(z)\Big]^r\Big)^{1/r}\Big\}.\end{aligned}$$

By duality, we need to prove that for all $k\in\mathbb{N}$ and all non-negative sequences $\{a_j\}_{j\in\mathbb{N}}$ satisfying that $[\sum_{j\in\mathbb{N}}a_j^{r'}]^{1/r'_j}=1$,

$$\mu(S)^{1/p}\sum_{j\in\mathbb{N}}\frac{a_j}{\mu(S_{j,k})}\int_{S_{j,k}}|f_j(z)|\mathrm{d}\mu(z)\le C\Big\|\Big(\sum_{j\in\mathbb{N}}|f_j|^r\Big)^{1/r}\Big\|_{\mathcal{M}_{\mathcal{F}}^{p,q}}.$$

To go on with our steps, we make the following geometric observation. For any $k\in\mathbb{N}$ and $i\in\mathbb{N}$, notice

$$J_{k,i}=\{j\in\mathbb{N}:2^i\mu(S)\le\mu(S(x_{j,k},\theta^2t_{j,k}))\le2^{i+1}\mu(S),S\subset S(x_{j,k},t_{j,k})\},$$

and $S^{k,i}=S(x_0,t^{k,i})$, where $t^{k,i}=\sup\{t_{j,k}:j\in J_{k,i}\}$. Then, we claim that for any $j\in J_{k,i}$, $S(x_{j,k},\theta^2t_{j,k})\subset S(x_0,\theta^3t^{k,i})$ and $\mu(S^{k,i})\sim2^i\mu(S)$ with implicit positive constants independent of k and i. In fact, for any $j\in J_{k,i}$, we have $S(x_{j,k},\theta^2t_{j,k})\subset S(x_0,\theta^3t_{j,k})\subset S(x_0,\theta^3t^{k,i})$ by the property (D). Using the doubling condition (1.2) and the previous inclusion relation, we get that for any $j\in J_{k,i}$,

$$2^i\mu(S)\le\mu(S(x_{j,k},\theta^2t_{j,k}))\le\mu(S(x_0,\theta^3t^{k,i}))\le C\mu(S^{k,i}).$$

On the other hand, notice that there exists a subsequence $\{j_v\}_{v\in\mathbb{N}}$ of $J_{k,i}$ such that $t_{j_v,k}$ increases to $t^{k,i}$ as $v\to\infty$. Using the continuity of μ, (1.2) and (1.3), we obtain that

$$\begin{aligned}\mu(S^{k,i}) &= \lim_{v\to\infty}\mu(S(x_0,t_{j_v,k}))\le\lim_{v\to\infty}\mu(S(x_0,\theta t_{j_v,k}))\\&\le\lim_{v\to\infty}\mu(S(x_{j_v,k},\theta^2 t_{j_v,k}))\le 2^{i+1}\mu(S).\end{aligned}$$

By the above claim and the Hölder inequality, we see that

$$\begin{aligned}&\mu(S)^{1/p}\sum_{j\in\mathbb{N}}\frac{a_j}{\mu(S_{j,k})}\int_{S_{j,k}}|f_j(z)|\mathrm{d}\mu(z)\\&=\mu(S)^{1/p}\sum_{i\in\mathbb{N}}\sum_{j\in J_{k,i}}\frac{a_j}{\mu(S_{j,k})}\int_{S_{j,k}}|f_j(z)|\mathrm{d}\mu(z)\\&\le C\mu(S)^{1/p-1}\sum_{i\in\mathbb{N}}2^{-i}\Big(\sum_{j\in J_{k,i}}a_j^{r'}\Big)^{1/r'}\int_{S(x_0,\theta^3t^{k,i})}\Big(\sum_{j\in J_{k,i}}|f_j(z)|^r\Big)^{1/r}\mathrm{d}\mu(z)\\&\le C\mu(S)^{1/p-1}\sum_{i\in\mathbb{N}}2^{-i}\mu(S(x_0,\theta^3t^{k,i}))^{1-1/p}\Big\|\Big(\sum_{j\in\mathbb{N}}|f_j|^r\Big)^{1/r}\Big\|_{\mathcal{M}^{p,q}_{\mathcal{F}}}\\&\le C\sum_{i\in\mathbb{N}}2^{-i/p}\Big\|\Big(\sum_{j\in\mathbb{N}}|f_j|^r\Big)^{1/r}\Big\|_{\mathcal{M}^{p,q}_{\mathcal{F}}}\\&\le C\Big\|\Big(\sum_{j\in\mathbb{N}}|f_j|^r\Big)^{1/r}\Big\|_{\mathcal{M}^{p,q}_{\mathcal{F}}}.\end{aligned}$$

Hence,

$$\mathrm{II}\le C\Big\|\Big(\sum_{j\in\mathbb{N}}|f_j|^r\Big)^{1/r}\Big\|_{\mathcal{M}^{p,q}_{\mathcal{F}}},$$

and we complete the proof. □

Here we also introduce the sharp maximal function $M^\sharp(f)$ of f associated to the family $\mathcal{F}$ as follows.

$$M^\sharp(f)(x):=\sup_{x\in S\subset\mathcal{F}}\frac{1}{\mu(S)}\int_S|f(y)-f_S|\mathrm{d}\mu(y)\sim\sup_{S\ni x}\inf_c\frac{1}{\mu(S)}\int_S|f(y)-c|\mathrm{d}\mu(y).$$

For $\delta>0$, we also define the following maximal function, $M^\delta(f)=M(|f|^\delta)^{1/\delta}$ and $M^{\sharp,\delta}(f)=M^\sharp(|f|^\delta)^{1/\delta}$.

Remark 2.6 *It is not difficult to see that these maximal functions are equivalent to the ones replaced by d-balls in the definitions.*

Lemma 2.7 ([24]) *Let $0<\delta<1$. Then there exists a constant $C>0$ such that*

$$M^\delta(Hf)(x)\le CM(f)(x),\tag{2.6}$$

for any smooth function f and every $x\in\mathbb{R}^n$.

By Remark 2.6 and Theorem 4.2 in [25], we have the following Fefferman-Stein inequality.

Lemma 2.8 *Let $0<q,\delta<\infty$. There exists a positive C such that*

$$\int_{\mathbb{R}^n}M^\delta(f)(x)^q\mathrm{d}\mu(x)\le C\int_{\mathbb{R}^n}M^{\sharp,\delta}(f)(x)^q\mathrm{d}\mu(x)$$

for any smooth function f for which the left-hand side is finite.

Similarly, we have the responding lemma on Morrey spaces as a consequent result.

Lemma 2.9 *Let $0 < \delta < \infty$ and $1 < q \le p \le \infty$. There exists a positive C such that*

$$\|M^\delta(f)\|_{\mathcal{M}_{\mathcal{F}}^{p,q}} \le C\|M^{\sharp,\delta}(f)\|_{\mathcal{M}_{\mathcal{F}}^{p,q}}$$

for any smooth function f for which the left-hand side is finite.

Proof For any section $S = S(x_0, t_0) \in \mathcal{F}$, set $\widetilde{S} = S(x_0, \theta^2 t_0)$. We can check that if $x \in S(x_0, \theta^{k+2}t_0)\backslash S(x_0, \theta^k t_0)$ for any $k \in \mathbb{N}$, then $M(\chi_S)(x) \le C\mu(S)/\mu(S(x_0, \theta^k t_0))$. So

$$\begin{aligned}
&\frac{1}{\mu(S)^{1/q-1/p}}\Big(\int_S M^\delta(f)(x)^q \mathrm{d}\mu(x)\Big)^{1/q}\\
&\le \frac{1}{\mu(S)^{1/q-1/p}}\Big(\int_{\mathbb{R}^n} M^\delta(f)(x)^q M(\chi_S)(x)\mathrm{d}\mu(x)\Big)^{1/q}\\
&\le \frac{C}{\mu(S)^{1/q-1/p}}\Big(\int_{\mathbb{R}^n} M^{\sharp,\delta}(f)(x)^q M(\chi_S)(x)\mathrm{d}\mu(x)\Big)^{1/q}\\
&\le \frac{C}{\mu(S)^{1/q-1/p}}\Big(\int_{\widetilde{S}} M^{\sharp,\delta}(f)(x)^q M(\chi_S)(x)\mathrm{d}\mu(x)+\\
&\quad \sum_{k=1}^\infty \int_{S(x_0,\theta^{2(k+1)}t_0)\backslash S(x_0,\theta^{2k}t_0))} M^{\sharp,\delta}(f)(x)^q M(\chi_S)(x)\mathrm{d}\mu(x)\Big)^{1/q}\\
&\le \frac{C}{\mu(S)^{1/q-1/p}}\Big\{\Big(\int_S M^{\sharp,\delta}(f)(x)^q M(\chi_S)(x)\mathrm{d}\mu(x)\Big)^{1/q}+\\
&\quad \Big(\sum_{k=1}^\infty \int_{S(x_0,\theta^{2(k+1)}t_0)\backslash S(x_0,\theta^{2k}t_0))} M^{\sharp,\delta}(f)(x)^q M(\chi_S)(x)\mathrm{d}\mu(x)\Big)^{1/q}\Big\}\\
&\le \frac{C}{\mu(S)^{1/q-1/p}}\Big\{\Big(\int_S M^{\sharp,\delta}(f)(x)^q M(\chi_S)(x)\mathrm{d}\mu(x)\Big)^{1/q}+\\
&\quad \Big(\sum_{k=1}^\infty \int_{S(x_0,\theta^{2(k+1)}t_0)\backslash S(x_0,\theta^{2k}t_0))} M^{\sharp,\delta}(f)(x)^q \frac{\mu(S)}{\mu(S(x_0,\theta^{2k}t_0))}\mathrm{d}\mu(x)\Big)^{1/q}\Big\}\\
&\le C\|M^{\sharp,\delta}(f)\|_{\mathcal{M}_{\mathcal{F}}^{p,q}}. \ \square
\end{aligned}$$

Theorem 2.10 *If $1 < q \le p < \infty$, then the Monge-Ampère singular integral operator H is bounded on $\mathcal{M}_{\mathcal{F}}^{p,q}(\mathbb{R}^n)$.*

Proof By Lemmas 2.9, 2.7 and Theorem 2.2, we have that for $0 < \delta < \infty$ and $1 < q \le p < \infty$,

$$\begin{aligned}
\|Hf\|_{\mathcal{M}_{\mathcal{F}}^{p,q}} &\le \|M(Hf)\|_{\mathcal{M}_{\mathcal{F}}^{p,q}} \le C\|M^{\sharp,\delta}(Hf)\|_{\mathcal{M}_{\mathcal{F}}^{p,q}}\\
&\le C\|M(f)\|_{\mathcal{M}_{\mathcal{F}}^{p,q}} \le C\|f\|_{\mathcal{M}_{\mathcal{F}}^{p,q}}. \ \square
\end{aligned}$$

Theorem 2.11 *Let $0 < \beta < 1$. If $1 < q \le p < \infty, 1 < t \le s < \infty$, and $q/t = p/s = 1 - \beta p$, then I^β is bounded from $\mathcal{M}_{\mathcal{F}}^{p,q}(\mathbb{R}^n)$ to $\mathcal{M}_{\mathcal{F}}^{s,t}(\mathbb{R}^n)$.*

Proof Fix $f \in \mathcal{M}_{\mathcal{F}}^{p,q}(\mathbb{R}^n)$. We claim that, for all $x \in \mathbb{R}^n$, the inequality of Hedberg type

$$|I^\beta(f)(x)| \le C\|f\|_{\mathcal{M}_{\mathcal{F}}^{p,q}}^{\beta p}[M(f)(x)]^{1-\beta p} \tag{2.7}$$

holds. Assume (2.7) holds for the moment and we prove Theorem 2.11. Indeed, by (2.7), $t(1-\beta p)=q$ and $\frac{q}{t}(\frac{1}{q}-\frac{1}{p})=\frac{1}{t}-\frac{1}{s}$ as well the fact that M is bounded on the space $\mathcal{M}_{\mathcal{F}}^{p,q}(\mathbb{R}^n)$, we conclude that for all $S\in\mathcal{F}$,

$$\begin{aligned}
&\frac{1}{\mu(S)^{1/t-1/s}}\Big\{\int_S |I^\beta(f)(y)|^t \mathrm{d}\mu(y)\Big\}^{1/t}\\
&\le C\|f\|^{\beta p}_{\mathcal{M}_{\mathcal{F}}^{p,q}}\frac{1}{\mu(S)^{1/t-1/s}}\Big\{\int_S |M(f)(y)|^{t(1-\beta p)}\mathrm{d}\mu(y)\Big\}^{1/t}\\
&\le C\|f\|^{\beta p}_{\mathcal{M}_{\mathcal{F}}^{p,q}}\frac{\mu(S)^{1/t-q/(pt)}}{\mu(S)^{1/t-1/s}}\Big\{\frac{1}{\mu(S)^{1/q-1/p}}\Big(\int_S |M(f)(y)|^q\mathrm{d}\mu(y)\Big)^{1/q}\Big\}^{q/t}\\
&\le C\|f\|^{\beta p}_{\mathcal{M}_{\mathcal{F}}^{p,q}}\|M(f)\|^{1-\beta p}_{\mathcal{M}_{\mathcal{F}}^{p,q}}\\
&\le C\|f\|_{\mathcal{M}_{\mathcal{F}}^{p,q}}.
\end{aligned}$$

Now we should show (2.7). Fix $x\in\mathbb{R}^n$ and $t>0$,

$$\begin{aligned}
|I^\beta(f)(x)| &\le \int_{S(x,t)}\frac{|f(y)|}{\mu(S(x,d(x,y)))^{1-\beta}}\mathrm{d}\mu(y)+\int_{S(x,t)^c}\frac{|f(y)|}{\mu(S(x,d(x,y)))^{1-\beta}}\mathrm{d}\mu(y)\\
&:= \mathrm{III}+\mathrm{IV}.
\end{aligned}$$

For III, by (1.2), we write

$$\begin{aligned}
\mathrm{III} &= \sum_{k=0}^{\infty}\int_{S(x,2^{-(k+1)}t)\subset S(x,d(x,y))\subset S(x,2^{-k}t)}\frac{|f(y)|}{\mu(S(x,d(x,y)))^{1-\beta}}\mathrm{d}\mu(y)\\
&\le \sum_{k=0}^{\infty}\frac{1}{\mu(S(x,2^{-(k+1)}t))^{1-\beta}}\int_{S(x,2^{-k}t)}|f(y)|\mathrm{d}\mu(y)\\
&\le C\sum_{k=0}^{\infty}A^{-k}\mu(S(x,t))^{\beta}M(f)(x)\le C\mu(S(x,t))^\beta M(f)(x).
\end{aligned}$$

As to IV, the Hölder inequality and (1.2) imply that

$$\begin{aligned}
\mathrm{IV} &= \sum_{k=0}^{\infty}\int_{S(x,2^{k}t)\subset S(x,d(x,y))\subset S(x,2^{k+1}t)}\frac{|f(y)|}{\mu(S(x,d(x,y)))^{1-\beta}}\mathrm{d}\mu(y)\\
&\le \sum_{k=0}^{\infty}\frac{\mu(S(x,2^{k+1}t))^{1/q'}}{\mu(S(x,2^{k}t))^{1-\beta}}\Big(\int_{S(x,2^{k+1}t)}|f(y)|^q\mathrm{d}\mu(y)\Big)^{1/q}\\
&\le \sum_{k=0}^{\infty}A^{k(\beta p-1)}\mu(S(x,t))^{(\beta p-1)/p}\|f\|_{\mathcal{M}_{\mathcal{F}}^{p,q}}\\
&\le C\mu(S(x,t))^{(\beta p-1)/p}\|f\|_{\mathcal{M}_{\mathcal{F}}^{p,q}}.
\end{aligned}$$

So,

$$|I^\beta(f)(x)|\le C(\mu(S(x,t))^\beta M(f)(x)+\mu(S(x,t))^{(\beta p-1)/p}\|f\|_{\mathcal{M}_{\mathcal{F}}^{p,q}}).$$

Now take $\mu(S(x,t))=\|f\|^p_{\mathcal{M}_{\mathcal{F}}^{p,q}}M(f)(x)^{-p}$, then

$$\mu(S(x,t))^\beta M(f)(x)=\mu(S(x,t))^{(\beta p-1)/p}\|f\|_{\mathcal{M}_{\mathcal{F}}^{p,q}}=\|f\|^{\beta p}_{\mathcal{M}_{\mathcal{F}}^{p,q}}M(f)(x)^{1-\beta p}.$$

Thus we complete the proof. □

Directly, we have the boundedness of the fractional integral I^β on Lebesgue space as follows.

Corollary 2.12 *Let $0<\beta<1$. If $1<q<\frac{1}{\beta}$ and $\frac{1}{t}=\frac{1}{q}-\beta$. Then I^β is bounded from $L^q(\mathbb{R}^n,d\mu)$ to $L^t(\mathbb{R}^n,\mathrm{d}\mu)$.*

Besides, by Minkowski's inequality, we have the following pointwise estimate

$$\Big[\sum_{j\in\mathbb{N}}|I^\beta(f_j)(x)|^r\Big]^{1/r}\le I^\beta\Big(\Big[\sum_{j\in\mathbb{N}}|f_j|^r\Big]^{1/r}\Big)(x),$$

then we obtain a vector-valued inequality of I^β.

Corollary 2.13 *Let $0<\beta<1$. If $1<q\le p<\infty$, $1<t\le s<\infty$, and $q/t=p/s=1-\beta p$. Then there exists a constant C such that*

$$\Big\|\Big[\sum_{j\in\mathbb{N}}|I^\beta(f_j)|^r\Big]^{1/r}\Big\|_{\mathcal{M}^{p,q}_{\mathcal{F}}}\le C\Big\|\Big(\sum_{j\in\mathbb{N}}|f_j|^r\Big)^{1/r}\Big\|_{\mathcal{M}^{p,q}_{\mathcal{F}}}.$$

3. Campanato spaces

For $q\in[1,\infty]$ and $p\in(1,\infty]$ a real-valued function f on $\mathbb{R}^n$ is said to be in the Campanato space $\mathcal{E}^{p,q}_{\mathcal{F}}(\mathbb{R}^n)$ provided that

$$\|f\|_{\mathcal{E}^{p,q}_{\mathcal{F}}}:=\sup_{S\in\mathcal{F}}\frac{1}{[\mu(S)]^{1/q-1/p}}\Big\{\int_S|f(y)-f_S|^q\mathrm{d}\mu(y)\Big\}^{1/q}<\infty,\tag{3.1}$$

where and in what follows, $f_S=\frac{1}{\mu(S)}\int_S f(y)\mathrm{d}\mu(y)$ denotes the mean of f over the section S.

Remark 3.1 This definition coincides with (1.2) in [19]. Clearly $\|\cdot\|_{\mathcal{E}^{p,q}_{\mathcal{F}}}$ is only a seminorm and $\|f\|_{\mathcal{E}^{p,q}_{\mathcal{F}}}=0$ if and only if f is constant μ-almost everywhere. We will assume the $\mathcal{E}^{p,q}_{\mathcal{F}}$ spaces to be quotient spaces from now on. As usual, when $p=\infty$, the space $\mathcal{E}^{\infty,1}_{\mathcal{F}}(\mathbb{R}^n)$ is reduced to $\mathrm{BMO}_{\mathcal{F}}$, which originated in [12]. From [16, Proposition 4.1], it is clear that $\mathcal{E}^{\infty,q}_{\mathcal{F}}(\mathbb{R}^n)=\mathrm{BMO}_{\mathcal{F}}(\mathbb{R}^n)$ for all $1\le q<\infty$. Lin [19] obtained that $\mathcal{E}^{p/(1-p),q'}_{\mathcal{F}}(\mathbb{R}^n)$ is the dual space of $H^{p,q}_{\mathcal{F}}(\mathbb{R}^n)$, for $p=1<q\le\infty$ or $1/2<p<1\le q<\infty$.

Theorem 3.2 *For $1\le q<p<\infty$, the spaces $\mathcal{M}^{p,q}_{\mathcal{F}}(\mathbb{R}^n)$ and $\mathcal{E}^{p,q}_{\mathcal{F}}(\mathbb{R}^n)$ coincide with equivalent norms.*

Proof For any $f\in\mathcal{M}^{p,q}_{\mathcal{F}}(\mathbb{R}^n)$, Minkowski's inequality implies that $\|f\|_{\mathcal{E}^{p,q}_{\mathcal{F}}}\le 2\|f\|_{\mathcal{M}^{p,q}_{\mathcal{F}}}$ naturally.

Conversely, we need to show that for all $f\in\mathcal{E}^{p,q}_{\mathcal{F}}(\mathbb{R}^n)$ and all $S\in\mathcal{F}$

$$\frac{1}{[\mu(S)]^{1/q-1/p}}\Big\{\int_S|f(y)|^q\mathrm{d}\mu(y)\Big\}^{1/q}\le C\|f\|_{\mathcal{E}^{p,q}_{\mathcal{F}}}.$$

It is not difficult to note that

$$\|f\|_{\mathcal{E}^{p,q}_{\mathcal{F}}}\le 2\sup_{S\in\mathcal{F}}\inf_c\frac{1}{\mu(S)^{1/q-1/p}}\Big(\int_S|f(y)-c|^q\mathrm{d}\mu(y)\Big)^{1/q}\le\|f\|_{\mathcal{E}^{p,q}_{\mathcal{F}}}.$$

Since $f\in\mathcal{M}^{p,q}_{\mathcal{F}}(\mathbb{R}^n)$, we have $|f_{S(x,t)}|\le\mu(S(x,t))^{-1/p}\|f\|_{\mathcal{M}^{p,q}_{\mathcal{F}}}\to 0$ as $t\to\infty$. Then, by Fatou's lemma, we can get for any $x\in\mathbb{R}^n$

$$\frac{1}{\mu(S)^{1/q-1/p}}\Big\{\int_S|f(y)|^q\mathrm{d}\mu(y)\Big\}^{1/q}=\frac{1}{\mu(S)^{1/q-1/p}}\Big\{\int_S|f(y)-\lim_{t\to\infty}f_{S(x,t)}|^q\mathrm{d}\mu(y)\Big\}^{1/q}$$

$$\leq \lim_{t\to\infty} \frac{1}{\mu(S)^{1/q-1/p}} \Big\{ \int_S |f(y) - f_{S(x,t)}|^q \mathrm{d}\mu(y) \Big\}^{1/q}$$
$$\leq C\|f\|_{\mathcal{E}^{p,q}_{\mathcal{F}}}.$$

Acknowledgements We thank the referees for their time and comments.

References

[1] C. B. MORREY. *On the solutions of quasi-linear elliptic partial differential equations*. Trans. Amer. Math. Soc., 1938, **43**(1): 126–166.

[2] S. CAMPANATO. *Proprietá di inclusione per spazi di Morrey*. Ric. Mat., 1963, **12**: 67–86. (in Italian)

[3] J. PEETRE. *On the theory of $\mathcal{L}^{p,\lambda}$ spaces*. J. Funct. Anal., 1969, **4**: 71–87.

[4] D. R. ADAMS. *A note on Riesz potentials*. Duke Math. J., 1975, **42**(4): 765–778.

[5] F. CHIARENZA, M. FRASCA. *Morrey spaces and Hardy-Littlewood maximal function*. Rend. Mat. Appl. (7), 1987, **7**(3-4): 273–279.

[6] D. R. ADAMS, Jie XIAO. *Morrey spaces in harmonic analysis*. Ark. Mat., 2012, **50**(2): 201–230.

[7] D. R. ADAMS, Jie XIAO. *Regularity of Morrey commutators*. Trans. Amer. Math. Soc., 2012, **364**(9): 4801–4818.

[8] X. T. DUONG, Jie XIAO, Lixin YAN. *Old and new Morrey spaces with heat kernel bounds*. J. Fourier Anal. Appl., 2007, **13**(1): 87–111.

[9] Liguang LIU, Y. SAWANO, Dachun YANG. *Morrey-type spaces on Gauss measure spaces and boundedness of singular integrals*. J. Geom. Anal., 2014, **24**(2): 1007–1051.

[10] H. RAFEIRO, N. SAMKO, S. SAMKO. *Morrey-Campanato spaces: an overview*. Oper. Theory Adv. Appl., 2013, **228**: 293–323.

[11] Wen YUAN, W. SICKEL, Dachun YANG. *Morrey and Campanato Meet Besov, Lizorkin and Triebel*. Springer-Verlag, Berlin, 2010.

[12] L. A. CAFFARELLI, C. E. GUTIÉRREZ. *Real analysis related to the Monge-Ampère equation*. Trans. Amer. Math. Soc., 1996, **348**(3): 1075–1092.

[13] L. A. CAFFARELLI, C. E. GUTIÉRREZ. *Properties of the solutions of the linearized Monge-Ampère equation*. Amer. J. Math., 1997, **119**(2): 423–465.

[14] L. A. CAFFARELLI, C. E. GUTIÉRREZ. *Singular integrals related to the Monge-Ampère equation, Wavelet theory and harmonic analysis in applied sciences (Buenos Aires, 1995)*. Birkhäuser Boston, Boston, MA, 1997.

[15] A. INCOGNITO. *Weak-type $(1-1)$ inequality for the Monge-Ampère SIO's*. J. Fourier Anal. Appl., 2001, **7**(1): 41–48.

[16] Yong DING, Chincheng LIN. *Hardy spaces associated to the sections*. Tohoku Math. J., 2005, **57**(2): 147–170.

[17] Lin TANG. *BLO spaces associated with the sections*. Proc. Amer. Math. Soc., 2007, **135**(8): 2423–2432.

[18] Mingyi LEE. *The boundedness of Monge-Ampére singular integral operators*. J. Fourier Anal. Appl., 2012, **18**(2): 211–222.

[19] Chincheng LIN. *Boundedness of Monge-Ampère singular-integral operators action on Hardy spaces and their duals*. Trans. Amer. Math. Soc., 2016, **368**: 3075–3104.

[20] H. AIMAR, L. FORZANI, R. TOLEDANO. *Balls and quasi-metrics: a space of homogeneous type modeling the real analysis related to the Monge-Ampère equation*. J. Fourier Anal. Appl., 1998, **4**: 377–381.

[21] R. COIFMAN, G. WEISS. *Extensions of hardy spaces and their use in analysis*. Bull. Am. Math. Soc., 1977, **83**(4): 569–645.

[22] Lin TANG, Jingshi XU. *Boundedness of maximal function related to the Monge-Ampère equation*. J. Math. Anal. Appl., 2007, **333**(2): 943–949.

[23] L. GRADAKOA, Liguang LIU, Dachun YANG. *Vector-valued singular integrals and maximal functions on spaces of homogeneous type*. Math. Scand., 2009, **104**(2): 296–310.

[24] Lin TANG. *Weighted estimates for singular integral operators and commutators associated with the sections*. J. Math. Anal. Appl., 2007, **333**(2): 577–590.

[25] J. MARTELL. *Sharp maximal functions associated with approximations of the identity in spaces of homogeneous type and applications*. Studia Math., 2004, **161**(2): 113–145.