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Optimal Asset Control of the Dual Model with a Penalty at Ruin

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Abstract In this paper, we study the optimal financing problem in the dual model. We introduce a value function which considers both the expected present value of the dividends payout minus the equity issuance and a penalty at ruin. In order to get the optimal strategy, two categories of suboptimal models are constructed and studied. Based on these two suboptimal models, we identify the value function and the optimal strategy in the general optimal problem.

Keywords dual model; optimal dividend control; equity issuance; time value of ruin; proportional transaction costs

MR(2010) Subject Classification 49J20; 49J30; 60G51

1. Introduction

Optimal dividends have attracted a lot of interest since the early works of Borch [1,2] and Gerber [3]. For example, Avanzi et al. [4] considered the dividend problem in the dual model with the presence of barrier dividends strategy. Ng [5] considered the dual model of the compound Poisson model under a threshold dividend strategy. See, also, Jeanblanc-Picqué and Shiryaev [6], Asmussen et al. [7,8], Høgaard and Taksar [9, 10], Gerber and Shiu [11,12] and the references therein. Equity issuance is an important approach for the company to raise capital and reduce risk in financial market. This strategy has been extensively studied in various risk models. Sethi and Taksar [13] considered the model for the company that can control its risk exposure by equity issuance and dividends payout. Løkka and Zervos [14] studied the problem with the possibility of bankruptcy, and also considered the proportional transaction costs in their model. For more papers on this topic, we refer the readers to Avanzi et al. [15], He and Liang [16,17], Yao et al. [18] and the references therein. But we realize that just considering one of the two above strategies is not enough in the reality. Scholars have considered the effects of both these strategies. Dai et al. [19] and Yao et al. [20,21] applied these two strategies to study the optimal asset control of

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the dual model. On the other hand, the time of ruin, which is a classical topic in risk theory, is important for the company and performance evaluation. However, we observe that all these papers outlined above do not take it into account. During the past years, this disadvantage has gathered a lot of attentions. Thonhauser and Albrecher [22] considered this problem for the companies which can control their exposures only by means of their dividend payments.

In this paper, we focus on the control problem of maximizing the expected present value of dividends payout minus equity issuance before bankruptcy. Moreover, we add a component to the objective function that penalizes the early ruin of the controlled risk process.

The rest of this paper is organized as follows. In Section 2, the dual model is shortly discussed and some preliminaries in our model are introduced. In Section 3, we identify the solution to the control problem that allows for no issuance of new equity. In Section 4, we solve the control problem that arises when the admissible strategies are constrained to allow for no bankruptcy, while Section 5 is concerned with the solution to the general control problem that involves no constraints on the issuance of new equity and the reserves.

2. Mathematical model

In this paper, we model the uncontrolled reserves of a company by a dual risk model. Let $S(t) = \sum_{i=0}^{N(t)} Y_i$, a compound Poisson process with Poisson rate λ, be the positive gains or profits. We denote by $D(y)$ the distribution function of Y_i. We use the completed filtration $\{\mathscr{F}_t\}_{t\geq 0}$ with $\mathscr{F}_t = \sigma\{S(v), v \leq t\}$ satisfying the usual conditions. Let $B = \{B_t\}_{t\geq 0}$ be a standard Brownian motion adapted to that filtration. If there are no equity issuance and dividends payout to control the risk, the liquid reserve of the company evolves according to the following equation

$$R_t = x + \mu t + \sigma B_t + S(t),$$

where $0 \leq x < \infty$ is the initial reserve, $\mu < 0$ is the rate of expenses, and $\sigma > 0$. Furthermore, we assume that $\{B_t\}, \{N(t)\}$ and $\{Y_i\}$ are mutually independent, and $\mathbb{E}[Y_i] < \infty$. Moreover, we assume that the net profit condition is satisfied, i.e., $\mu + \lambda\mathbb{E}[Y_1] \geq 0$. As usual, we call such a model a dual model.

We can use such process to model companies (or financial institutions) that have occasional gains whose amount and frequency can be modeled by the process $\{S(t)\}$. For further discussions on the applications of this model, see e.g., Avanzi et al. [4,15], Bayraktar and Egami [23], Dong and Wang [24], Seal [25], Zhu and Yang [26] and the references therein.

To enrich the model, we assume that the company's manager can control the reserves by paying out dividends and by raising capital through issuing equity. We denote by L_t the cumulative amount of dividends paid from time zero up to time t, and by G_t the total amount of capital raised by issuing equity from time zero up to time t. We assume that both $L = \{L_t\}_{t\geq 0}$ and $G = \{G_t\}_{t\geq 0}$ are $(\mathscr{F}_t)$-adapted, increasing and right-continuous with left limits. A control policy π is described by the stochastic processes (L^π, G^π). Similar to Øksendal and Sulem [27], we only consider admissible policies. Let $\Pi = \{\pi\}$ denote the set of all admissible policies.

Given a control policy π, we model the liquid reserves of the company by

$$R_t^\pi = x + \mu t + \sigma B_t + S(t) - L_t^\pi + G_t^\pi, \tag{2.1}$$

and define the ruin time τ^π by $\tau^\pi = \inf\{t \geq 0 : R_t^\pi < 0\}$.

As usual, the proportional transaction costs are considered in our model. If the company pays l as dividends, then the shareholders can get $\beta_1 l$, $\beta_1 < 1$. On the other hand, in order to meet the cost of getting the amount of η by issuing new equity, the shareholders should pay out $\beta_2\eta$, $\beta_2 > 1$.

In this paper, we aim to identify the control strategy π that maximizes

$$J(x,\pi) = \mathbb{E}\Big[\int_0^{\tau^\pi} e^{-rt}\beta_1 \mathrm{d}L_t^\pi - \int_0^{\tau^\pi} e^{-rt}\beta_2 \mathrm{d}G_t^\pi + \int_0^{\tau^\pi} e^{-rt}\Lambda \mathrm{d}t\Big],$$

where $\Lambda > 0$, and r denotes the discount rate.

Here, we point out that compared to the classical value function, which maximizes the expected present value of the dividends payout minus the equity issuance before bankruptcy, there is an additional term depending on the lifetime of the controlled process. $e^{-rt}\Lambda$ can be interpreted as the present value of an amount which the insurer earns as long as the company is alive. In this way, the lifetime of the portfolio becomes part of the value function and is weighted according to the choice of Λ. Another interpretation is that in this way the Laplace transform of the ruin time is part of the value function.

At the end of this section, we introduce some preliminaries. The following ordinary differential equation plays an important role in our analysis below,

$$\frac{1}{2}\sigma^2 g''(x) + \mu g'(x) + \lambda\int_0^\infty g(x+y)\mathrm{d}D(y) - (\lambda+r)g(x) + \Lambda = 0. \tag{2.2}$$

The candidate solution to the equation (2.2) is given by

$$g(x) = c_1 e^{k_1 x} + c_2 e^{k_2 x} + \frac{\Lambda}{r}, \tag{2.3}$$

where $c_1, c_2 \in \mathbb{R}$ are constants, and the real numbers $k_1 > 0$ and $k_2 < 0$ are solutions to the following equation

$$\frac{1}{2}\sigma^2 k^2 + \mu k + \lambda\int_0^\infty e^{ky}\mathrm{d}D(y) - (\lambda + r) = 0.$$

It follows from Øksendal and Sulem [27] that k_1 and k_2 exist.

For simplicity, for each function $g(x) \in \mathcal{C}^2(\mathbb{R})$, define the integro-differential operator $\mathcal{A}$ by

$$\mathcal{A}g(x) = \frac{1}{2}\sigma^2 g''(x) + \mu g'(x) + \lambda\int_0^\infty g(x+y)\mathrm{d}D(y) - (\lambda+r)g(x).$$

3. The case without equity issuance

In this section, we consider the optimal dividend problem without equity issuance. We introduce the following notation. Given an initial reserve $x \geq 0$, let $\Pi_p = \big\{\pi_p = (L^{\pi_p}, G^{\pi_p}) \in$

$\Pi : G_t^{\pi_p} = 0 \text{ for all } t \geq 0\big\} \subset \Pi$. We define the associated value function $V_p(x)$ by

$$V_p(x) = \sup_{\pi_p \in \Pi_p} J(x, \pi_p). \tag{3.1}$$

Our aim is to find the value function $V_p(x)$ and an optimal strategy π^*. Similar to Dai et al. [19], here we try to construct a twice continuously differentiable concave solution to this problem, which is referred to as a classical solution to the optimal problem.

With reference to Øksendal and Sulem [27], the Hamilton-Jacobi-Bellman (HJB) equation corresponding to (3.1) is

$$\max\Big\{\frac{1}{2}\sigma^2 W''(x) + \mu W'(x) + \lambda \int_0^\infty W(x+y)\mathrm{d}D(y) - (\lambda + r)W(x) + \Lambda, \beta_1 - W'(x)\Big\} = 0 \tag{3.2}$$

with the boundary condition

$$W(0) = 0. \tag{3.3}$$

It follows from the Brownian motion, which takes us below zero in probability 1, and the negative drift of the process that the reserve $x = 0$ corresponds to bankruptcy. Then the boundary condition (3.3) naturally arises.

With regard to simple economic considerations, we conjecture that the value function $V_p(x)$ identifies with a solution $W(x)$ to the HJB equation (3.2) satisfying

$$\frac{1}{2}\sigma^2 W''(x) + \mu W'(x) + \lambda \int_0^\infty W(x+y)\mathrm{d}D(y) - (\lambda + r)W(x) + \Lambda = 0, \quad 0 < x \leq b^*, \tag{3.4}$$

and

$$\beta_1 - W'(x) = 0, \quad x \geq b^* \tag{3.5}$$

for some constant $b^* > 0$.

By (2.2), we would consider a solution to the equations (3.4) and (3.5) of the form

$$W(x) = c_1 e^{k_1 x} + c_2 e^{k_2 x} + \frac{\Lambda}{r}, \text{ if } 0 < x \leq b^*, \tag{3.6}$$

and

$$W(x) = \beta_1(x - b^*) + c_1 e^{k_1 b^*} + c_2 e^{k_2 b^*} + \frac{\Lambda}{r}, \text{ if } x \geq b^*. \tag{3.7}$$

Next, we specify the parameters c_1, c_2 and b^*. Our aim is to find a classical solution, so we need

$$c_1 k_1 e^{k_1 b^*} + c_2 k_2 e^{k_2 b^*} = \beta_1, \tag{3.8}$$

$$c_1 k_1^2 e^{k_1 b^*} + c_2 k_2^2 e^{k_2 b^*} = 0. \tag{3.9}$$

Using (3.8) and (3.9), we can express c_1 and c_2 in terms of b^*:

$$c_1(b^*) = -\frac{\beta_1 k_2}{k_1(k_1 - k_2)} e^{-k_1 b^*} > 0, \tag{3.10}$$

and

$$c_2(b^*) = \frac{\beta_1 k_1}{k_2(k_1 - k_2)} e^{-k_2 b^*} < 0. \tag{3.11}$$

By (3.3),

$$\frac{\Lambda}{r} + c_1(b^*) + c_2(b^*) = 0. \tag{3.12}$$

Lemma 3.1 *There exists a unique solution $b^* > 0$ to the equation (3.12). The function $W(x)$ given by (3.6) and (3.7) with b^* being the unique solution to (3.12) and with c_1, c_2 being given by (3.10) and (3.11) is concave in $[0, \infty)$ and satisfies the HJB equation (3.2) and the boundary condition (3.3).*

Proof For any $b \geq 0$, define $C(b) = \frac{\Lambda}{r} + c_1(b) + c_2(b)$. We have that

$$C'(b) = \frac{\beta_1 k_2}{k_1 - k_2} e^{-k_1 b} - \frac{\beta_1 k_1}{k_1 - k_2} e^{-k_2 b} < 0,$$
$$C(0) = \frac{\Lambda}{r} + \frac{\beta_1 k_1}{k_2(k_1 - k_2)} - \frac{\beta_1 k_2}{k_1(k_1 - k_2)} > 0.$$

Therefore, $C(b)$ is strictly decreasing. Since $C(0) > 0$ and $\lim_{b \to +\infty} C(b) = -\infty$, the function $C(b)$ has a unique positive root $b^* > 0$.

Since $W''(b^*) = 0$ and for $x < b^*$,

$$W'''(x) = c_1 k_1^3 e^{k_1 x} + c_2 k_2^3 e^{k_2 x} > 0,$$

we can get that $W''(x) < 0$ for all $x \in [0, b^*)$. Thus we can see that $W''(x) \leq 0$ for all $x \geq 0$. This shows that $W(x)$ is concave in $[0, \infty)$.

It follows from (3.12) that $W(x)$ satisfies the boundary condition (3.3).

The remaining problem is to prove that $W(x)$ satisfies the HJB equation (3.2). Noting (3.6) and (3.7), we only need to prove the following conditions:

$$W'(x) \geq \beta_1, \qquad x \in [0, b^*],$$
$$\mathcal{A}[W(x)] + \Lambda \leq 0, \quad x \geq b^*.$$

The proof is as follows. The concavity of $W(x)$ implies that $W'(x)$ is decreasing. Thus, we can get from (3.8) that for any $x \in [0, b^*]$, $W'(x) \geq W'(b^*) = \beta_1$. Moreover, for $x \geq b^*$,

$$\begin{aligned}\mathcal{A}[W(x)] + \Lambda &= \lambda\beta_1 \int_o^\infty y \mathrm{d}D(y) + \mu\beta_1 - r\beta_1(x - b^*) - rW(b^*) + \Lambda \\ &\leq \mu\beta_1 - rW(b^*) + \lambda\beta_1 \int_o^\infty y \mathrm{d}D(y) + \Lambda \\ &= \lim_{x \downarrow b^*} \mathcal{A}[W(x)] + \Lambda = \lim_{x \uparrow b^*} \mathcal{A}[W(x)] + \Lambda = 0.\end{aligned}$$

So $W(x)$ satisfies the HJB equation (3.2). The proof is completed. □

Theorem 3.2 *The value function V_p identifies with the concave solution $W(x)$ to the HJB equation (3.2). Moreover, define $\pi^* = (L^{\pi^*}, 0)$, where $(R_t^{\pi^*}, L_t^{\pi^*})$ is a solution to the following system of equation:*

$$R_t^{\pi^*} = x + \mu t + \sigma B(t) + \sum_{i=1}^{N(t)} Y_i - L_t^{\pi^*},$$
$$R_t^{\pi^*} \leq b^*, \quad t \geq 0,$$

$$\int_0^\infty \mathcal{I}\{R_t^{\pi^*} \le b^*\}\mathrm{d}L_t^{\pi^*} = 0,$$
$$G_t^{\pi^*} = 0.$$

π^* is an optimal strategy, i.e., $V_p(x) = J(x, \pi^*)$.

The proof of Theorem 3.2 can be developed by a straightforward modification of the proof of Theorem 5.2 below, so we omit it.

Remark 3.3 We can see that without equity issuance the optimal strategy is a barrier strategy, that is, any surplus above the level b^* would be paid as dividends to the shareholders of the company.

4. The case that never goes bankrupt

In this section we consider the optimal dividend problem with equity issuance. In this case, the bankruptcy is prohibited. Thereby, the reserve processes stay positive all the time. We aim at maximizing the discounted dividends payout minus the discounted costs of issuing new equity. Given an initial reserve $x \ge 0$, let $\Pi_s = \{\pi_s = (L^{\pi_s}, G^{\pi_s}) \in \Pi : R_t^{\pi_s} \ge 0 \text{ for all } \ge 0\} \subset \Pi$. We define the associated value function $V_s(x)$ by

$$V_s(x) = \sup_{\pi_s \in \Pi_s} J(x, \pi_s). \tag{4.1}$$

We aim at finding the value function $V_s(x)$ and an optimal strategy $\pi^{**} \in \Pi_s$.

Standard arguments, see Øksendal and Sulem [27], formally yield the associated HJB equation

$$\max\Big\{\frac{1}{2}\sigma^2 H''(x) + \mu H'(x) + \lambda\int_0^\infty H(x+y)\mathrm{d}D(y) - (\lambda + r)H(x) + \Lambda,\ \beta_1 - H'(x), H'(x) - \beta_2\Big\} = 0. \tag{4.2}$$

We now construct a classical solution $H(x)$ to the HJB equation (4.2). Considering the time value of money, we conjecture that an optimal strategy associated with a solution to the HJB equation (4.2) is characterized by

$$H'(0) = \lim_{x\downarrow 0} H(x) = \beta_2,$$
$$\frac{1}{2}\sigma^2 H''(x) + \mu H'(x) + \lambda\int_0^\infty H(x+y)\mathrm{d}D(y) - (\lambda + r)H(x) + \Lambda = 0, \quad 0 < x < b^{**},$$
$$H'(x) = \beta_1, \quad x \ge b^{**}.$$

By (2.2), we would conjecture that a function $H(x)$ satisfying (4.2) is given by

$$H(x) = d_1 e^{k_1 x} + d_2 e^{k_2 x} + \frac{\Lambda}{r}, \text{ if } 0 \le x < b^{**}, \tag{4.3}$$

and

$$H(x) = \beta_1(x - b^{**}) + d_1 e^{k_1 b^{**}} + d_2 e^{k_2 b^{**}} + \frac{\Lambda}{r}, \text{ if } x \ge b^{**}. \tag{4.4}$$

To find the solution, we must determine the parameters d_1, d_2 and b^{**}. Our aim is to find a $\mathcal{C}^2$

solution, so we need

$$d_1k_1e^{k_1b^{**}}+d_2k_2e^{k_2b^{**}}=\beta_1, \tag{4.5}$$

and

$$d_1k_1^2e^{k_1b^{**}}+d_2k_2^2e^{k_2b^{**}}=0. \tag{4.6}$$

Using equations (4.5) and (4.6), we can express d_1 and d_2 as functions of b^{**}:

$$d_1(b^{**})=-\frac{\beta_1k_2}{k_1(k_1-k_2)}e^{-k_1b^{**}}>0, \tag{4.7}$$

and

$$d_2(b^{**})=\frac{\beta_1k_1}{k_2(k_1-k_2)}e^{-k_2b^{**}}<0. \tag{4.8}$$

Moreover,

$$d_1(b^{**})k_1+d_2(b^{**})k_2=\beta_2. \tag{4.9}$$

Lemma 4.1 *The equation (4.9) has a unique solution $b^{**}>0$. The function $H(x)$ defined by (4.3) and (4.4) with b^{**} being the unique solution to (4.9) and d_1,d_2 being given by (4.7) and (4.8) is concave in $[0,\ \infty)$, and satisfies the HJB equation (4.2).*

Proof Define $D(b)=d_1(b)k_1+d_2(b)k_2$. We have that

$$\begin{aligned}
D'(b)&=\frac{\beta_1k_1k_2}{(k_1-k_2)}e^{-k_1b}-\frac{\beta_1k_2k_1}{(k_1-k_2)}e^{-k_1b},\\
D'(0)&=0,\\
D''(b)&=-\frac{\beta_1k_1^2k_2}{(k_1-k_2)}e^{-k_1b}+\frac{\beta_1k_1k_2^2}{(k_1-k_2)}e^{-k_1b}>0.
\end{aligned}$$

So we get that the function $D'(b)$ is strictly increasing. Thus, $D'(b)>0$ for all $b>0$. We easily see that $D(b)$ is increasing. Since $D(0)=\beta_1<\beta_2$ and $\lim_{b\to+\infty}D(b)=\infty$, we get that $D(b)=\beta_2$ has a unique positive root $b^{**}>0$.

Using the same method as the proof for Lemma 3.1, we can prove that $H(x)$ is concave and satisfies the HJB equation (4.2). The proof is completed. □

Theorem 4.2 *The value function $V_s(x)$ identifies with the concave solution $H(x)$ given by (4.3) and (4.4) to the HJB equation (4.2). Moreover, define $\pi^{**}=(L^{\pi^{**}},G^{\pi^{**}})$, where $(R_t^{\pi^{**}},L_t^{\pi^{**}},G_t^{\pi^{**}})$ is a solution to the following system of equation:*

$$\begin{cases}
R_t^{\pi^{**}}=x+\mu t+\sigma B(t)+\sum_{i=1}^{N(t)}Y_i-L_t^{\pi^{**}}+G_t^{\pi^{**}},\\
0\le R_t^{\pi^{**}}\le b^{**},\quad t\ge 0,\\
\int_0^\infty \mathcal{I}(R_t^{\pi^{**}}\le b^{**})\mathrm{d}L_t^{\pi^{**}}=0,\\
\int_0^\infty \mathcal{I}(R_t^{\pi^{**}}\ne 0)\mathrm{d}G_t^{\pi^{**}}=0.
\end{cases}$$

π^{**} is an optimal strategy, i.e., $V_s(x)=J(x,\pi^{**})=H(x)$.

Using the same method as the proof for Theorem 5.2 below, we can readily prove Theorem 4.2. Here we omit it.

5. The solution to the general problem

We now address the general problem of maximizing the expected discounted dividend flow minus the expected discounted costs of issuing new equity over all admissible strategies in the general case. Given an initial capital $x \geq 0$, we define the value function $V(x)$ by

$$V(x) = \sup_{\pi \in \Pi} J(x, \pi). \tag{5.1}$$

We mainly aim at finding a value function $V(x)$ and an optimal strategy.

Remark 5.1 Since $\pi_p, \pi_s \in \Pi$, $V(x) \geq \max\{V_p(x), V_s(x)\}$ for all $x \geq 0$.

The main result of this paper is the following.

Theorem 5.2 *Fix any initial capital $x \geq 0$, and consider the problem of maximizing the performance criterion $V(x)$ over all admissible strategies.*

(i) If $b^* \leq b^{**}$, then $V(x) = W(x) = V_p(x)$. An optimal strategy is $\pi^* = \{L^{\pi^*}, 0\}$ which is given by Theorem 3.2.

(ii) If $b^* \geq b^{**}$, then $V(x) = H(x) = V_s(x)$. An optimal strategy is $\pi^{**} = \{L^{\pi^{**}}, G^{\pi^{**}}\}$ which is given by Theorem 4.2.

In order to prove Theorem 5.2, we need the following Lemmas.

Lemma 5.3 *(I) If $b^* \geq b^{**}$, then $H(0) \geq 0$.*

(II) If $b^* \leq b^{**}$, then $W'(x) \leq \beta_2$.

Proof We first prove (I). It follows from (3.10), (3.11), (4.7) and (4.8) that $c_1(x) = d_1(x)$ and $c_2(x) = d_2(x)$. Thus,

$$H(0) = d_1(b^{**}) + d_2(b^{**}) + \frac{\Lambda}{r} = c_1(b^{**}) + c_2(b^{**}) + \frac{\Lambda}{r} = C(b^{**}).$$

Since $C(b) = c_1(b) + c_2(b) + \frac{\Lambda}{r}$ is strictly decreasing and $b^* \geq b^{**}$,

$$H(0) = C(b^{**}) = c_1(b^{**}) + c_2(b^{**}) + \frac{\Lambda}{r} \geq c_1(b^*) + c_2(b^*) + \frac{\Lambda}{r}.$$

By (3.12), $W(0) = c_1(b^*) + c_2(b^*) + \frac{\Lambda}{r} = C(b^*) = 0$. So $H(0) \geq 0$.

Next, we prove (II). From Lemma 3.1, we get that $W(x)$ is concave. The concavity of $W(x)$ implies that for all $x \geq 0$, $W'(x) \leq \beta_2$ if and only if $W'(0) \leq \beta_2$, i.e.,

$$W'(x) \leq \beta_2 \Leftrightarrow W'(0) = c_1(b^*)k_1 + c_2(b^*)k_2 \leq \beta_2.$$

Since $D(b)$ is increasing on $[0, \infty)$ and $b^* \leq b^{**}$,

$$\begin{aligned}\beta_2 = D(b^{**}) &= d_1(b^{**})k_1 + d_2(b^{**})k_2 = c_1(b^{**})k_1 + c_2(b^{**})k_2 \\ &\geq c_1(b^*)k_1 + c_2(b^*)k_2 = W'(0).\end{aligned}$$

So $W'(x) \leq \beta_2$. The proof has been done. $\square$

Lemma 5.4 *If $Q(x)$ satisfies the following HJB equation*

$$\max\Big\{\frac{1}{2}\sigma^2 Q''(x) + \mu Q'(x) + \lambda\int_0^\infty Q(x+y)\mathrm{d}D(y) - (r+\lambda)Q(x) + \Lambda,$$

$$\beta_1 - Q'(x), Q'(x) - \beta_2\Big\} = 0, \tag{5.2}$$

and the boundary condition

$$\max\{-Q(0), Q'(0) - \beta_2\} = 0, \tag{5.3}$$

then $Q(x) \geq J(x,\pi)$ *for any admissible strategy* π.

Proof For any fixed strategy π, put $\mathscr{D} = \{s : L^\pi_{s-} \neq L^\pi_s\}$ and $\mathscr{D}' = \{s : G^\pi_{s-} \neq G^\pi_s\}$. Moreover, let $\hat{L}^\pi_t$ be the discontinuous part of L^π_t and $\tilde{L}^\pi_t$ be the continuous part of L^π_t. Similarly, $\hat{G}^\pi_t$ and $\tilde{G}^\pi_t$ stand for discontinuous and continuous parts of G^π_t, respectively.

By the Dynkin formula [27, Theorem 1.23]

$$\begin{aligned}\mathbb{E}[e^{-r(t\wedge\tau^\pi)}Q(R^\pi_{t\wedge\tau^\pi})] =& Q(x) + \mathbb{E}\Big\{\int_0^{t\wedge\tau^\pi} e^{-rs}\mathcal{A}\big[Q(R^\pi_s)\big]\mathrm{d}s - \\ & \int_0^{t\wedge\tau^\pi} e^{-rs}Q'(R^\pi_s)\mathrm{d}\tilde{L}^\pi_s + \int_0^{t\wedge\tau^\pi} e^{-rs}Q'(R^\pi_s)\mathrm{d}\tilde{G}^\pi_s + \\ & \sum_{s\in\mathscr{D}\cup\mathscr{D}', s\leq t\wedge\tau^\pi} e^{-rs}[Q(R^\pi_s) - Q(R^\pi_{s-})]\Big\}.\end{aligned} \tag{5.4}$$

By (5.2),

$$\begin{aligned}\mathbb{E}[e^{-r(t\wedge\tau^\pi)}Q(R^\pi_{t\wedge\tau^\pi})] \leq& Q(x) - \mathbb{E}\Big[\int_0^{t\wedge\tau^\pi} e^{-rs}Q'(R^\pi_s)\mathrm{d}\tilde{L}^\pi_s + \int_0^{t\wedge\tau^\pi} e^{-rs}Q'(R^\pi_s)\mathrm{d}\tilde{G}^\pi_s\Big] + \\ & \mathbb{E}\Big[\sum_{s\in\mathscr{D}\cup\mathscr{D}', s\leq t\wedge\tau^\pi} e^{-rs}[Q(R^\pi_s) - Q(R^\pi_{s-})] - \int_0^{t\wedge\tau^\pi} e^{-rs}\Lambda\mathrm{d}s\Big].\end{aligned}$$

Since $\beta_1 \leq Q'(x) \leq \beta_2$, $Q(R^\pi_s) - Q(R^\pi_{s-}) \leq \beta_2(G^\pi_s - G^\pi_{s-}) - \beta_1(L^\pi_s - L^\pi_{s-})$. So

$$\begin{aligned}&\mathbb{E}\Big[e^{-r(t\wedge\tau^\pi)}Q(R^\pi_{t\wedge\tau^\pi}) + \int_0^{t\wedge\tau^\pi} e^{-rs}\beta_1\mathrm{d}L^\pi_s - \int_0^{t\wedge\tau^\pi} e^{-rs}\beta_2\mathrm{d}G^\pi_s + \\ &\int_0^{t\wedge\tau^\pi} e^{-rs}\Lambda\mathrm{d}s\Big] \leq Q(x).\end{aligned} \tag{5.5}$$

By (5.3), we take limits in (5.5) and then get

$$\mathbb{E}\Big[\int_0^{\tau^\pi} e^{-rs}\beta_1\mathrm{d}L^\pi_s - \int_0^{\tau^\pi} e^{-rs}\beta_2\mathrm{d}G^\pi_s + \int_0^{\tau^\pi} e^{-rs}\Lambda\mathrm{d}s\Big] \leq Q(x).$$

Therefore $J(x,\pi) \leq Q(x)$. The proof has been done. □

Next, we prove the main result of this paper.

Proof of Theorem 5.2 We first prove case (i) of the theorem. Since $b^* \leq b^{**}$, we deduce from Lemmas 3.1 and 5.3 that $W(x)$ satisfies the HJB equation (5.2) and the boundary condition (5.3). So $W(x) \geq V(x)$. On the other hand, we get from Remark 5.1 that $W(x) \leq V(x)$. Hence $W(x) = V(x)$.

In the sequel, we will show that $J(x,\pi^*) = W(x) = V(x)$, i.e., π^* is an optimal strategy. We deduce from Lemma 3.1 and Theorem 3.2 that for all $t \geq 0$,

$$\mathcal{A}[W(R^{\pi^*}_t)] + \Lambda = 0.$$

Applying the generalized Itô formula, we have that

$$\begin{aligned}e^{-r(t\wedge\tau^{\pi^*})}W(R^{\pi^*}_{t\wedge\tau^{\pi^*}}) =&W(x)+\int_0^{t\wedge\tau^{\pi^*}}e^{-rs}\mathcal{A}[W(R^{\pi^*}_s)]\mathrm{d}s-\int_0^{t\wedge\tau^{\pi^*}}e^{-rs}W'(R^{\pi^*}_s)\mathrm{d}\tilde{L}^{\pi^*}_s+\\&\sum_{s\in\mathscr{D},s\le t\wedge\tau^{\pi^*}}e^{-rs}[W(R^{\pi^*}_s)-W(R^{\pi^*}_{s-})]+M_1(t\wedge\tau^{\pi^*})+M_2(t\wedge\tau^{\pi^*})\\=&W(x)-\int_0^{t\wedge\tau^{\pi^*}}\beta_1e^{-rs}\mathrm{d}L^{\pi^*}_s+\int_0^{t\wedge\tau^{\pi^*}}\sigma e^{-rs}W'(R^{\pi^*}_s)\mathrm{d}B_s-\\&\int_0^{t\wedge\tau^{\pi^*}}e^{-rs}\Lambda\mathrm{d}s+M_1(t\wedge\tau^{\pi^*})+M_2(t\wedge\tau^{\pi^*}),\end{aligned}\tag{5.6}$$

where $M_1(t)=\int_0^t\sigma e^{-rs}W'(R^{\pi^*}_s)\mathrm{d}B_s$ and

$$M_2(t)=\int_0^t\int_0^{+\infty}e^{-rs}(W(R^{\pi^*}_{s-}+y)-W(R^{\pi^*}_{s-}))(N(\mathrm{d}s,\mathrm{d}y)-\lambda\mathrm{d}s\mathrm{d}D(y)).$$

By Theorem 3.2, $W(R^{\pi^*}_{t\wedge\tau^{\pi^*}})$ is bounded by $W(b^*)$. So M_1 and M_2 converge in $\mathcal{L}^1$ to two integrable random variables, respectively. Furthermore, M_1 and M_2 are uniformly integrable martingales. On the other hand, we have

$$\liminf_{t\to\infty}e^{-r(t\wedge\tau^{\pi^*})}W(R^{\pi}_{t\wedge\tau^{\pi^*}})=e^{-r\tau^{\pi^*}}W(0)=0.$$

From above arguments, we can take limits in (5.6) and then take the expectations at both sides of (5.6). Finally, we get

$$W(x)=\mathbb{E}\Big[\liminf_{t\to\infty}\Big\{\int_0^{t\wedge\tau^{\pi^*}}e^{-rs}\beta_1\mathrm{d}L^{\pi^*}_s+\int_0^{t\wedge\tau^{\pi^*}}e^{-rs}\Lambda\mathrm{d}s\Big\}\Big]=V(x,\pi^*).$$

So $W(x)=V(x)=V_p(x)$.

We now prove case (ii) of the theorem. Similar to the proof for case (i), since $b^*\ge b^{**}$, we deduce from Lemmas 4.1 and 5.3 that $H(x)$ defined by (4.3) and (4.4) satisfies the HJB equation (5.2) and the boundary condition (5.3). So $H(x)\ge V(x)$. On the other hand, we get from Remark 5.1 that $H(x)\le V(x)$. Hence $H(x)=V(x)$.

Next, we will prove that π^{**} is an optimal strategy, i.e., $V(x,\pi^{**})=H(x)$. The proof is as follows. We deduce from Lemma 4.1 and Theorem 4.2 that for all $t\ge0$,

$$\mathcal{A}[H(R^{\pi^{**}}_t)]+\Lambda=0.$$

Applying the generalized Itô formula gives

$$\begin{aligned}e^{-r(t\wedge\tau^{\pi^{**}})}H(R^{\pi^{**}}_{t\wedge\tau^{\pi^{**}}}) =&H(x)+\int_0^{t\wedge\tau^{\pi^{**}}}e^{-rs}\mathcal{A}[H(R^{\pi^{**}}_s)]\mathrm{d}s-\\&\int_0^{t\wedge\tau^{\pi^{**}}}e^{-rs}H'(R^{\pi^{**}}_s)\mathrm{d}\tilde{L}^{\pi^{**}}_s+\\&\int_0^{t\wedge\tau^{\pi^{**}}}e^{-rs}H'(R^{\pi^{**}}_s)\mathrm{d}\tilde{G}^{\pi^{**}}_s+\\&\sum_{s\in\mathscr{D},s\le t\wedge\tau^{\pi^{**}}}e^{-rs}[H(R^{\pi^{**}}_s)-H(R^{\pi^{**}}_{s-})]+M_2(t\wedge\tau^{\pi^*})+\end{aligned}$$

$$\sum_{s\in\mathscr{D}',s\leq t\wedge\tau^{\pi^{**}}} e^{-rs}[H(R_s^{\pi^{**}})-H(R_{s-}^{\pi^{**}})]+M_1(t\wedge\tau^{\pi^*})$$
$$=H(x)-\int_0^{t\wedge\tau^{\pi^{**}}} e^{-rs}\beta_1\mathrm{d}L_s^{\pi^{**}}+\int_0^{t\wedge\tau^{\pi^{**}}} e^{-rs}\beta_2\mathrm{d}G_s^{\pi^{**}}-$$
$$\int_0^{t\wedge\tau^{\pi^*}} e^{-rs}\wedge\mathrm{d}s+M_1(t\wedge\tau^{\pi^*})+M_2(t\wedge\tau^{\pi^*}). \tag{5.7}$$

By virtue of Theorem 4.2, $H(R^{\pi^{**}}_{t\wedge\tau^{\pi^{**}}})$ is bounded by $H(b^{**})$. Then,

$$\liminf_{t\to\infty} e^{-r(t\wedge\tau^{\pi^{**}})}H(R^{\pi}_{t\wedge\tau^{\pi^{**}}})=0.$$

Taking the expectations at both sides of (5.7) yields

$$H(x)=J(x,\pi^{**}).$$

So $V(x)=H(x)=V_q(x)$. $\square$

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