Journal of Mathematical Research with Applications
Sept., 2017, Vol. 37, No. 5, pp. 505–519
DOI:10.3770/j.issn:2095-2651.2017.05.001
Http://jmre.dlut.edu.cn

Height Estimates for Spacelike Hypersurfaces with Constant k-Mean Curvature in GRW Spacetimes

Ximin LIU*, **Ning ZHANG**
Department of Mathematics, Dalian University of Technology, Liaoning 116024*, P. R. China*

Abstract In this paper, using the generalized Omori-Yau maximum principle, we obtain height estimates for spacelike hypersurface in a generalized Robertson-Walker (GRW) spacetime with constant higher order mean curvature and whose boundary is contained in a slice. Furthermore, we apply these results to draw some topological conclusions. Finally, considering the Omori-Yau maximum principle for the Laplacian and for more general elliptic trace type differential operators, we have some further non-existence results.

Keywords height estimates; generalized Robertson-Walker spacetimes; spacelike hypersurface; higher order mean curvature

MR(2010) Subject Classification 53C42; 53B30; 53C50

1. Introduction

In the seventies, constant mean curvature spacelike hypersurfaces in Lorentzian manifolds have been studied not only from their mathematical interest, but also from their important in general relativity.

Recently, the height estimates for constant (higher order) mean curvature compact spacelike hypersurfaces with boundary have been studied by more and more authors. This is because the height estimates are very useful tool to research existence and uniqueness problems of spacelike hypersurfaces with constant mean curvature, more generally, constant higher order mean curvature, as well as to obtain some relevant topological property of such hypersurfaces.

There are some articles studying the height estimates of spacelike hypersurfaces in a generalized Robertson-Walker spacetimes. López [1] obtained the the height estimates of constant mean curvature compact spacelike hypersurfaces in the Lorentz-Minkowski spacetime $\mathbb{L}^{n+1}$ and with boundary on a spacelike hyperplan. Later, the height estimates for compact spacelike graphs in the steady state spacetime were obtained by Montiel in [2] and he used them to prove some existence and uniqueness theorems for complete spacelike hypersurfaces in the de Sitter spacetime with constant mean curvature $H > 1$ and prescribed asymptotic future boundary. More generally, Lima [3] extended the height estimates proved by López to any n and he also got sharp height estimates for compact spacelike hypersurfaces with some positive constant higher order

Received September 5, 2016; Accepted December 9, 2016
Supported by the National Natural Science Foundation of China (Grant No. 11371076).
* Corresponding author
E-mail address: ximinliu@dlut.edu.cn (Ximin LIU); znlt@mail.dlut.edu.cn (Ning ZHANG)

mean curvature in the Lorentz-Minkowski spacetime with boundary on a spacelike hyperplan. More recently, the authors [4] obtained a sharp height estimate concerning compact hypersurfaces immersed into warped product spaces with some constant higher order mean curvature and whose boundary is contained in a slice, and they applied these results to study of topological properties of spacelike hypersurfaces. Furthermore, García-Martínez and Impera [5] obtained height estimates and half-space theorems for spacelike hypersurfaces of constant k-mean curvature, $1 \leq k \leq n$ in a generalized Robertson-Walker spacetimes and whose boundary is contained in a slice.

In this paper we study compact spacelike hypersurfaces in a wider family of spacetimes, that is, the generalized Robertson-Walker (GRW) spacetimes. i.e., a spacetime considering a warped product of a negative definite interval as a base and a Riemannian manifold as a fiber, moreover, a positive smooth function as a warped function (see Section 2). Notice that the family of GRW spacetimes is very wide, for instance, the Lorentz-Minkowski spacetime, the Einstein-de Sitter spacetime, the static Einstein spacetime, and the Roberston-Walker (RW) spacetimes (fiber of constant sectional curvature). In any GRW spacetime there is a distinguished family of spacelike hypersurfaces, that is so-called slices, which are defined as level hypersurfaces of the time coordinate of the spacetime. Furthermore, any slice is totally umbilical and has constant higher order mean curvature.

In this paper, first we study the height estimates of compact spacelike hypersurface with constant k-th mean curvature H_k for $1 \leq k \leq n$ in a GRW spacetime and with boundary contained in a slice. By controlling the specific value $\frac{H_{k+1}}{H_k}$ for $1 \leq k \leq n-1$ and imposing suitable conditions on the geometry of the ambient spacetime, see Theorems 3.1, 3.5 in Section 3 and Theorems 4.1, 4.3 in Section 4.

Following the same spirit as in [6] and using these results in previous sections, we obtain several non-existence results about the topological properties of complete spacelike hypersurfaces properly immersed into spatially closed GRW spacetime, see Theorems 5.2 and 5.6 in Section 5.

Finally, in Section 6, we prove a number of further non-existence results in the form of half-space theorem, extending to the complete spacelike hypersurface in a GRW spacetime, see Theorems 6.5 and 6.8. Notice that, since the height estimates do not apply in this more general case. So, in this case, our approach is based on applying a generalized version of the Omori-Yau maximum principle for trace type differential operators associated to the Newton transformations.

2. Preliminaries

Consider M^n an n-dimensional Riemannian manifold, and let I be an open interval in $\mathbb{R}$ endowed with the metric $-dt^2$. We let $f : I \to \mathbb{R}^+$ be a positive smooth function. Denote $\overline{M}^{n+1} := -I \times_f M^n$ to be the warped product endowed with the Lorentzian metric

$$\langle , \rangle = -\pi_I^*(\mathrm{d}t^2) + f(\pi_I)^2 \pi_M^*(\langle , \rangle_M) \tag{2.1}$$

where π_I and π_M denote the projections onto I and M, respectively. This spacetime is a warped product in the sense of ([7], Chap.7), with fiber $(M, \langle , \rangle)$, base $(I, -dt^2)$ and warping function f. Following the terminology used in [8] we will refer to $-I \times_f M^n$ as a generalized Robertson-Walker (GRW) spacetime. In particular, if the fiber M^n has constant section curvature, it is called a Robertson-Walker (RW) spacetime. Notice that $f(t)\frac{\partial}{\partial t}$ is closed conformal timelike vector field on $\overline{M}^{n+1}$ which determines a foliation $t \to M_t := t \times M$ of $\overline{M}^{n+1}$ by complete totally umbilical spacelike hypersurfaces with constant mean curvature.

Now consider a spacelike hypersurface $\varphi : \Sigma^n \to \overline{M}^{n+1}$. In this case, since vector field $\partial_t := \partial/\partial t$ is a (unitary) timelike globally defined on $\overline{M}$, there exists a unique unitary timelike normal field N globally defined on Σ with the same time-orientation as ∂_t, i.e., such that $\langle N, \partial_t \rangle < 0$. From the wrong-way Cauchy-Schwarz inequality ([7, Proposition 5.30], for instance), we have $\langle N, \partial_t \rangle \leq -1$, and the equality holds at a point $p \in M$ if and only if $N = \partial_t$ at p. Moreover, we will denote the function $\Theta : \Sigma \to (-\infty, -1]$, $\Theta := -\langle N, \partial_t \rangle$, as the angle function. In this case, we will refer to that normal field N as the future-pointing Gauss map of the hypersurface.

Let $A : T\Sigma \to T\Sigma$ stand for the shape operator of Σ with respect to the future-pointing Gauss map N. It is well known that A restricts to a self-adjoint linear operator $A_p : T_p\Sigma \to T_p\Sigma$, and its eigenvalues $k_1(p), \ldots, k_n(p)$ are the principal curvatures of the hypersurface Σ. Associated to the shape operator there are n algebraic invariants, which are the elementary symmetric function S_k of its eigenvalues, given by

$$S_k(p) = S_k(k_1(p), \ldots, k_n(p)) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}, \quad k = 1, \ldots, n.$$

Observe that the characteristic polynomial of A satisfies

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by construction. The kth-mean curvature H_k of the hypersurface is then defined by

$$\binom{n}{k} H_k = (-1)^k S_k.$$

Thus $H_1 = -\frac{1}{n}Tr(A)$ is the mean curvature and

$$n(n-1)H_2 = \overline{S} - S + 2\overline{\mathrm{Ric}}(N, N),$$

where S is the scalar curvature of Σ, while $\overline{S}$ and $\overline{\mathrm{Ric}}$ are, respectively, the scalar curvature and the Ricci tensor of the GRW spacetime $\overline{M}^{n+1}$. Furthermore, if k is even, it follows from the Gauss equation that H_k is a geometric quantity which is related to the intrinsic curvature of Σ^n.

In the following, we introduce the corresponding Newton transformations $P_k : T\Sigma \to T\Sigma$ which are inductively defined by

$$P_0 = I, \quad P_k = \binom{n}{k} H_k I + AP_{k-1}, \quad k = 1, \ldots, n.$$

It is not difficult to see that the Newton transformations P_k are all self-adjoint operators which commute with the shape operator A. Even more, if $\{e_k\}$ is an orthonormal frame on $T_p\Sigma$

which is diagonalizable with A_p, $A_p(e_i) = k_i(p)e_i$, then

$$(P_k)_p(e_i) = (-1)^k \sum_{i_1<\cdots<i_k, i_j\neq i} k_{i_1}\cdots k_{i_k}(p)e_i,$$

It can be easily seen that the Newton transformations satisfy the following properties:

(a) $\mathrm{Tr}(P_k) = c_k H_k$;

(b) $\mathrm{Tr}(A\circ P_k) = -c_k H_{k+1}$;

(c) $\mathrm{Tr}(A^2\circ P_k) = \binom{n}{k+1}(nH_1H_{k+1} - (n-k-1)H_{k+2})$

where $c_k = (n-k)\binom{n}{k} = (k+1)\binom{n}{k+1}$, and $H_k = 0$ if $k > n$. We refer the reader to [9] for further details.

Let ∇ stand for the Levi-Civita connection of Σ and let $g \in \mathcal{C}^\infty(\Sigma)$. Associated to each Newton transformations P_k, we define the second order linear differential operator $L_k : \mathcal{C}^\infty(\Sigma) \to \mathcal{C}^\infty(\Sigma)$, given by

$$L_k(g) = \mathrm{Tr}(P_k \circ \nabla^2 g). \tag{2.2}$$

Here $\nabla^2 g : T\Sigma \to T\Sigma$ denotes the self-adjoint linear operator metrically equivalent to the hessian of g, and it is given by

$$\langle \nabla^2 g(X), Y\rangle = \langle \text{hess } g(X), Y\rangle = \langle \nabla_X(\nabla g), Y\rangle, \quad X, Y \in T\Sigma.$$

It follows from (2.2) that the operator L_k is elliptic if and only if P_k is positive definite. Clearly, $L_0 = \Delta$ is always elliptic. For our applications, it is useful to state two lemmas in which geometric conditions are given in order to guarantee the ellipticity of L_k when $k \geq 1$.

Lemma 2.1 ([10]) *Let Σ be a spacelike hypersurface in a GRW spcetime. If $H_2 > 0$ on Σ, then L_1 is an elliptic operator (for an appropriate choice of the Gauss map N).*

For a proof of Lemma 2.1 [10, Lemma 3.2], where Alías and Colares proved it as a consequence of [11, Lemma 3.10]. The next lemma is a consequence of [12, Proposition 3.2].

Lemma 2.2 ([10]) *Let Σ be a spacelike hypersurface in a GRW spcetime. If there exists an elliptic point of Σ, with respect to an appropriate choice of the Gauss map N, and $H_k > 0$ on Σ, $3 \leq k \leq n$, then the operator L_j is elliptic for any $1 \leq j \leq k-1$.*

If $\varphi : \Sigma^n \to \overline{M}^{n+1}$ is a Riemannian immersion, with Σ oriented by unit vector field N. We will refer to the normal vector field N as future-pointing Gauss map of the hypersurface. In what follows, we will consider two particular functions naturally attached to Σ, namely, the angle function Θ and the (vertical) height function $h = (\pi_I)\,|_\Sigma$.

Let $\overline{\nabla}$ and ∇ stand for gradients with respect to the metrics of $\overline{M}^{n+1}$ and Σ^n, respectively. A simple computation shows that the gradient of π_I on $\overline{M}^{n+1}$ is given by

$$\overline{\nabla}\pi_I = -\langle \overline{\nabla}\pi_I, \partial_t\rangle \partial_t = -\partial_t.$$

So, the gradient of h on Σ^n is

$$\nabla h = (\overline{\nabla}\pi_I)^\top = -\partial_t^\top = -\partial_t - \Theta N.$$

Particularly, we have

$$|\nabla h|^2 = -1 + \Theta^2,$$

where $|\ |$ denotes the norm of a vector field on Σ^n.

In the following, we give the technical proposition that will be essential for the proofs of our main results (for further details of the proof see Sections 4 and 8 in [10]).

Proposition 2.3 ([5]) *Let $\varphi : \Sigma^n \to -I \times_f M^n$ be a spacelike hypersurface with angle function Θ and height function $h = \pi_I \circ \varphi$. Then the following formulas hold:*

(a) Let $\sigma(t)$ be a primitive of $f(t)$. Then

$$L_{k-1}h = -(\log f)'(h)(c_{k-1}H_{k-1} + \langle P_{k-1}\nabla h, \nabla h\rangle) - \Theta c_{k-1}H_k, \tag{2.3}$$

$$L_{k-1}\sigma(h) = -c_{k-1}(f'(h)H_{k-1} + \Theta f(h)H_k). \tag{2.4}$$

(b) Let $\widehat{\Theta} = f(h)\Theta$. Then

$$\begin{aligned} L_{k-1}\widehat{\Theta} = &\binom{n}{k} f(h)\langle \nabla h, \nabla H_k\rangle + f'(h)c_{k-1}H_k + \\ &\widehat{\Theta}\binom{n}{k}(nH_1H_k - (n-k)H_{k+1}) + \\ &\frac{\widehat{\Theta}}{f^2(h)}\sum_{i=1}^{n} \mu_{k-1,i}K_M(N^* \wedge E_i^*)|N^* \wedge E_i^*|^2 - \\ &\widehat{\Theta}(\log f)''(h)(|\nabla h|^2 c_{k-1}H_{k-1} - \langle P_{k-1}\nabla h, \nabla h\rangle), \end{aligned} \tag{2.5}$$

where $\{E_i\}_{i=1}^n$ is an orthonormal frame on Σ and, for any vector field X in $-I \times_f M^n$, we let $X^* = \pi_{M*}X$.

3. Height estimate for constant mean curvature hypersurfaces

In this section, we will use the results that we have discussed in the previous section to state and prove our main result about the height estimate of spacelike hypersurfaces in GRW spacetime $-I \times_f M^n$. We point out that, to prove this result we are not assuming that the mean curvature of the spacelike hypersurface is constant.

Theorem 3.1 *Let Σ^n be a compact spacelike hypersurface in a GRW spacetime $-I \times_f M^n$ with $H_2 > 0$. Assume that $\partial\Sigma \subset \{s\} \times M$ for some $s \in I$. If either*

(i) $f'(h) \le 0$ and $\frac{H_2}{H_1} \le \inf_I(\log f)'$ or

(ii) $f'(h) \ge 0$ and $\frac{H_2}{H_1} \ge \sup_I(\log f)'$

then $h \le s$.

Proof First, we prove part (i). Note that, in this case, H_1 is a negative function and $\frac{H_2}{H_1} \le \inf_I(\log f)' \le (\log f)'(h)$. Applying (2.4) with $k = 2$, we then obtain

$$L_1\sigma(h) = -c_1(f'(h)H_1 + \Theta f(h)H_2) = -c_1 f(h)H_1((\log f)'(h) + \Theta\frac{H_2}{H_1}) \ge 0.$$

and we conclude by Lemma 2.1 that L_1 is an elliptic operator. It follows then by the classical maximum principle for the elliptic operator L_1 that $\sigma(h)$ must attain its maximum on $\partial\Sigma$, that is $\sigma(h) \leq \sigma(s)$. Since σ is an increasing function, this implies that $h \leq s$.

For part (ii), observe that, in this case, H_1 is a positive function and $\frac{H_2}{H_1} \geq \sup_I (\log f)' \geq (\log f)'(h)$. Hence

$$L_1\sigma(h) = -c_1(f'(h)H_1 + \Theta f(h)H_2) = -c_1 f(h)H_1((\log f)'(h) + \Theta \frac{H_2}{H_1}) \geq 0.$$

and we conclude again by the classical maximum principle for elliptic operator L_1 that h must attain its maximum on $\partial\Sigma$, that is $h \leq s$. □

Now, let $\mathbb{R}_1^{n+2}$ be the (n+2)-dimensional Lorentz-Minkowski space and consider the hyperquadric

$$\mathbb{S}_1^{n+1} = \{p \in \mathbb{R}_1^{n+2} : \langle p, p\rangle = 1\}.$$

As known to all, the hyperquadric $\mathbb{S}_1^{n+1}$ is the de Sitter space and its sectional curvature is 1. If we choose a unit timelike vector $a \in \mathbb{R}_1^{n+2}$, then we consider on $\mathbb{S}_1^{n+1}$ the closed conformal vector field

$$\mathcal{T}_a(p) = a - \langle a, p\rangle p, \quad p \in \mathbb{S}_1^{n+1}, \quad \langle a, a\rangle = -1.$$

It is easy to see that $\mathcal{T}_a$ is a timelike vector field. So, we can denote the de Sitter space $\mathbb{S}_1^{n+1}$ as the warped product $-I \times_{\cosh t} \mathbb{S}^n$, where $\mathbb{S}^n$ means the Riemannian unit sphere [13].

Particularly, if a is a null vector, the vector field $\mathcal{T}_a$ is timelike on the open set

$$\{p \in \mathbb{S}_1^{n+1} : \langle p, a\rangle \neq 0\}.$$

Let $\mathcal{H}^{n+1}$ be the connected component of this set characterized by $\langle p, a\rangle > 0$, which is called the (n+1)-dimensional steady state spacetime. Moreover, as well-explained in [14], the steady state spacetime is isometric to the GRW spacetime $-I \times_{e^t} M^n$.

In the following, we consider spacelike hypersurfaces immersed in the de Sitter space $\mathbb{S}_1^{n+1}$ and the steady state spacetime $\mathcal{H}^{n+1}$, then from Theorem 3.1, we have:

Corollary 3.2 *Let $\varphi : \Sigma^n \to \mathbb{S}_1^{n+1}$ be a compact spacelike hypersurface in the de Sitter space $\mathbb{S}_1^{n+1}$ with $H_2 > 0$. Suppose that the boundary of Σ^n satisfies $\varphi(\partial\Sigma) \subset \mathbb{S}_\tau$ for some $\tau \in I$. If either*

(i) $h \leq 0$ and $\frac{H_2}{H_1} \leq \inf_I(\tanh)(h)$;

(ii) $h \geq 0$ and $\frac{H_2}{H_1} \geq \sup_I(\tanh)(h)$

then $h \leq \tau$.

Corollary 3.3 *Let $\varphi : \Sigma^n \to \mathcal{H}^{n+1}$ be a compact spacelike hypersurface in the steady state spacetime $\mathcal{H}^{n+1}$ with $H_2 > 0$. Assume that the boundary $\varphi(\partial\Sigma) \subset M_\tau$ for some $\tau \in I$. If $\frac{H_2}{H_1} \geq 1$, then $h \leq \tau$.*

Recall that a spacetime obeys the null convergence condition (NCC) if its Ricci tensor is non-negative on null (or lightlike) directions. It is not difficult to see that a GRW spacetime $-I \times_f M^n$ obeys the NCC if and only if

$$\mathrm{Ric}_M \geq (n-1)\sup_I (ff'' - f'^2)\langle , \rangle_M \tag{3.1}$$

where Ric_M and $\langle , \rangle_M$ are respectively the Ricci and metric tensors of Riemannian manifold M^n.

In the following, we state a lemma which will be essential for the proof of our main results (see also [5] Proposition 7 for more detailed proof).

Lemma 3.4 ([5]) *Let $-I \times_f M^n$ be a GRW spacetime which obeys the NCC. Assume that Σ^n is a spacelike hypersurface of constant mean curvature in $-I \times_f M^n$. Then the function $\phi = H_1\sigma(h) + \widehat{\Theta}$ is superharmonic.*

By Theorem 3.1, Lemma 3.4 and the classical maximum principle, we have one of the main theorem of this section.

Theorem 3.5 *Let $-I \times_f M^n$ be a GRW spacetime obeying the NCC and Σ^n be a compact spacelike hypersurface in $-I \times_f M^n$ with non-vanishing constant mean curvature satisfying $H_2 > 0$. Assume that $\partial\Sigma \subset \{s\} \times M^n$ for some $s \in I$ and that f is a monotone function on I.*

(i) If $f' \geq 0$ and $\frac{H_2}{H_1} \geq \sup_I(\log f)'$, then $\Sigma^n \subset [s-\alpha, s] \times M^n$ where

$$\alpha = \frac{\frac{f(s)}{f(\min_\Sigma h)} \max_{\partial\Sigma}(-\Theta) - 1}{H_1} \geq 0.$$

(ii) If $f' \leq 0$ and $\frac{H_2}{H_1} \leq \inf_I(\log f)'$ then $\Sigma^n \subset [s-\beta, s] \times M^n$ where

$$\beta = \frac{\max_{\partial\Sigma}(-\Theta) - 1}{H_1} \geq 0.$$

Proof First, we prove (i). Applying Theorem 3.1, we have $h \leq s$. Furthermore, by Lemma 3.4, the function $-\phi = -H_1\sigma(h) - \widehat{\Theta}$ is subharmonic. So it can attain its maximum on the $\partial\Sigma$, then

$$-H_1\sigma(h) + f(h) \leq -H_1\sigma(h) - \widehat{\Theta} \leq -H_1\sigma(s) + f(s)\max_{\partial\Sigma}(-\Theta). \tag{3.2}$$

Note that, for any $t \leq s$,

$$\sigma(s) - \sigma(t) = \int_s^t f(v)\mathrm{d}v \geq \inf_{v\in(t,s)}(f(v))(s-t).$$

Therefore, when $f' \geq 0$, for any $x \in \Sigma$, we have

$$H_1 f(h)(h-s) \geq H_1(\sigma(h) - \sigma(s)) \geq f(h) - f(s)\max_{\partial\Sigma}(-\Theta).$$

Particularly,

$$H_1(h-s) \geq 1 - \frac{f(s)}{f(h)}\max_{\partial\Sigma}(-\Theta) \geq 1 - \frac{f(s)}{f(\min_\Sigma h)}\max_{\partial\Sigma}(-\Theta). \tag{3.3}$$

So, we prove the part (i) of the theorem.

As for part (ii), if $f' \leq 0$, the discussion is similar to the above, we also have

$$H_1 f(s)(h-s) \geq H_1(\sigma(h) - \sigma(s)) \geq f(h) - f(s)\max_{\partial\Sigma}(-\Theta) \geq f(s)(1 - \max_{\partial\Sigma}(-\Theta)).$$

Hence, we prove (ii). □

As a consequence of Theorem 3.5, we have the following results.

Corollary 3.6 *Let $\varphi : \Sigma^n \to -\mathbb{R}^+ \times_{\cosh t} M^n$ be a compact spacelike hypersurface of constant* *mean curvature in de Sitter spacetime* $-\mathbb{R}^+ \times_{\cosh t} M^n$ *with* $H_2 > 0$ *and* $\mathrm{Ric}_M \geq n-1$. *Assume that* $\varphi(\partial\Sigma) \subset M_\tau$ *for some* $\tau \in I$. *If* $\frac{H_2}{H_1} \geq \sup_I(\tanh)$, *then* $\Sigma^n \subset [\tau-\alpha, \tau] \times M^n$ *where*

$$\alpha = \frac{\frac{\cosh(\tau)}{\cosh(\min_\Sigma h)} \max_{\partial\Sigma}(-\Theta) - 1}{H_1} \geq 0.$$

Corollary 3.7 *Let* $\varphi : \Sigma^n \to -\mathbb{R}^- \times_{\cosh t} M^n$ *be a compact spacelike hypersurface of constant mean curvature in de Sitter spacetime* $-\mathbb{R}^- \times_{\cosh t} M^n$ *with* $H_2 > 0$ *and* $\mathrm{Ric}_M \geq n-1$. *Assume that* $\varphi(\partial\Sigma) \subset M_\tau$ *for some* $\tau \in I$. *If* $\frac{H_2}{H_1} \leq \inf_I(\tanh)$, *then* $\Sigma^n \subset [\tau-\beta, \tau] \times M^n$ *where*

$$\beta = \frac{\max_{\partial\Sigma}(-\Theta) - 1}{H_1} \geq 0.$$

Corollary 3.8 *Let* $\varphi : \Sigma^n \to -I \times_{e^t} M^n$ *be a compact spacelike hypersurface of constant mean curvature in steady state spacetime* $-I \times_{e^t} M^n$ *with* $H_2 > 0$ *and* $\mathrm{Ric}_M \geq 0$. *Assume that* $\varphi(\partial\Sigma) \subset M_\tau$ *for some* $\tau \in I$. *If* $\frac{H_2}{H_1} \geq 1$, *then* $\Sigma^n \subset [\tau-\alpha, \tau] \times M^n$ *where*

$$\alpha = \frac{e^{\tau - \min_\Sigma h} \max_{\partial\Sigma}(-\Theta) - 1}{H_1} \geq 0.$$

Corollary 3.9 *Let* $\varphi : \Sigma^n \to -I \times M^n$ *be a compact spacelike hypersurface of constant mean curvature* $H_1 \neq 0$ *in the static spacetime* $-I \times M^n$ *with* $H_2 > 0$ *and* $\mathrm{Ric}_M \geq 0$. *Assume that* $\varphi(\partial\Sigma) \subset M_\tau$ *for some* $\tau \in I$. *Then* $\Sigma^n \subset [\tau-\alpha, \tau] \times M^n$ *where*

$$\alpha = \frac{\max_{\partial\Sigma}(-\Theta) - 1}{H_1} \geq 0.$$

4. Height estimates of constant k-th mean curvatures

In this section, we generalize the estimates of previous section spacelike hypersurface of constant k-th mean curvatures, $2 \leq k \leq n$. In the following we give the version of Theorem 3.1 in the higher order mean curvature case.

Theorem 4.1 *Let* Σ^n *be a compact spacelike hypersurface in a GRW spacetime* $-I \times_f M^n$ *with positive k-th mean curvature for* $2 \leq k \leq n$. *Assume that* $\partial\Sigma \subset \{s\} \times M$ *for some* $s \in I$ *and that, if* $k \geq 3$, *there exists an elliptic point* $p \in \Sigma$. *If either*

(i) $f'(h) \leq 0$ *or*

(ii) $f'(h) \geq 0$ *and* $\frac{H_{k+1}}{H_k} \geq \sup_I(\log f)'(h)$,

then $h \leq s$.

Proof Firstly, if $f'(h) \leq 0$, since $H_k > 0$ for $2 \leq k \leq n$, using (2.4), we have

$$L_k\sigma(h) = -c_k(f'(h)H_k + \Theta f(h)H_{k+1}) \geq 0.$$

Furthermore, by Lemmas 2.1 and 2.2 that L_{k-1} is an elliptic operator, then we can obtain the conclusion by applying the classical maximum principle for elliptic operator L_k.

Finally, for part (ii), $\frac{H_{k+1}}{H_k} \geq \sup_I(\log f)' \geq (\log f)'(h)$. Hence we have

$$L_k\sigma(h) = -c_k(f'(h)H_k + \Theta f(h)H_{k+1}) = -c_k f(h)H_k((\log f)'(h) + \Theta\frac{H_{k+1}}{H_k}) \geq 0,$$

so the conclusion follows again by the maximum principle for the elliptic operator L_k. □

Now, we are ready to extend the Theorem 3.5 to the case of constant higher order mean curvature spacelike hypersurfaces. Here, in order to do that we need to impose on $-I \times_f M^n$ the following strong condition instead of the null convergence condition, that is

$$K_M \geq \sup_I (ff'' - f'^2) \tag{4.1}$$

where K_M is the sectional curvature of M^n. We will refer to (4.1) as the strong null convergence condition (strong NCC).

Firstly, we state the following lemma which is very useful to prove our main results, and we refer to [10] for the proof of the Lemma.

Lemma 4.2 ([10]) *Let $-I \times_f M^n$ be a GRW spacetime obeying the strong NCC and let Σ^n be a spacelike hypersurface in $-I \times_f M^n$ with positive constant k-th mean curvature H_k for some $2 \leq k \leq n$. Moreover, if $k \geq 3$, assume that there exists an elliptic point on Σ. Let $\phi = H_k^{1/k}\sigma(h) + \widehat{\Theta}$. If $f'(h) \geq 0$, then*

$$L_{k-1}\phi \leq 0$$

on Σ.

Using Theorem 4.1 and Lemma 4.2, we can prove the following

Theorem 4.3 *Let $-I \times_f M^n$ be a GRW spacetime obeying the strong NCC and with f is non-decreasing function. Let Σ^n be a spacelike hypersurface in $-I \times_f M^n$ with positive constant k-th mean curvature satisfying $\frac{H_{k+1}}{H_k} \geq \sup_I (\log f)'(h)$, for some $2 \leq k \leq n$. Suppose that $\partial\Sigma \subset \{s\} \times M^n$ for some $s \in I$ and, if $k \geq 3$, there exists an elliptic point on Σ. Then*

$$\Sigma^n \subset [s - \alpha, s] \times M^n$$

where

$$\alpha = \frac{\frac{f(s)}{f(\min_\Sigma h)} \max_{\partial\Sigma}(-\Theta) - 1}{H_k^{1/k}} \geq 0.$$

Proof Applying Theorem 4.1, we have $h \leq s$. Moreover, by Lemma 4.2, the function ϕ is subharmonic and Σ is compact, we can obtain the following by the classical maximum principle for the elliptic operator L_{k-1}

$$-H_k^{1/k}\sigma(h) - \widehat{\Theta} \leq -H_k^{1/k}\sigma(s) + f(s)\max_{\partial\Sigma}(-\Theta).$$

So

$$H_k^{1/k}(\sigma(h) - \sigma(s)) \geq f(h) - f(s)\max_{\partial\Sigma}(-\Theta).$$

Therefore, we can obtain the conclusion reasoning as in Theorem 3.5. □

Considering the de Sitter space $-\mathbb{R}^+ \times_{\cosh t} M^n$, as a consequence of Theorem 4.3, we have the following corollary.

Corollary 4.4 *Let Σ^n be a compact spacelike hypersurface in de Sitter space $-\mathbb{R}^+ \times_{\cosh t} M^n$ satisfying $K_M \geq 1$. Suppose that Σ has constant k-th mean curvature H_k, $2 \leq k \leq n$, obeying*

$\frac{H_{k+1}}{H_k} \geq \sup_I(\tanh)'$ *and, if* $k \geq 3$*, assume that there exists an elliptic point on* Σ*. Suppose that* $\partial\Sigma \subset \{\tau\} \times M^n$ *for some* $\tau \in I$*. Then* $\Sigma^n \subset [\tau-\alpha, \tau] \times M^n$ *where*

$$\alpha = \frac{\frac{\cosh(\tau)}{\cosh(\min_\Sigma h)} \max_{\partial\Sigma}(-\Theta) - 1}{H_k^{1/k}} \geq 0.$$

Considering the steady state space $-I \times_{e^t} M^n$ and using Theorem 4.3, we have the following.

Corollary 4.5 *Let* Σ^n *be a compact spacelike hypersurface in steady state spacetime* $-I \times_{e^t} M^n$ *with* $K_M \geq 0$*. Assume that* Σ *has constant k-th mean curvature* H_k*,* $2 \leq k \leq n$*, satisfying* $\frac{H_{k+1}}{H_k} \geq 1$ *and, if* $k \geq 3$*, assume that there exists an elliptic point on* Σ*. Suppose that* $\partial\Sigma \subset \{\tau\} \times M^n$ *for some* $\tau \in I$*. Then* $\Sigma^n \subset [\tau - \alpha, \tau] \times M^n$ *where*

$$\alpha = \frac{e^{\tau - \min_\Sigma h} \max_{\partial\Sigma}(-\Theta) - 1}{H_k^{1/k}} \geq 0.$$

5. Half-space theorems of spacelike hypersurfaces

In this section, we will consider complete spacelike hypersurfaces in spatially closed GRW spacetimes $-I \times_f M^n$, that is Riemannian fiber M^n is compact. First, we introduce the following definition.

Definition 5.1 ([5]) *Let* Σ *be a spacelike hypersurface in a GRW spacetime* $-I \times_f M^n$*. We say that* Σ *lies in an upper or lower half-space if it is respectively contained in a region of* $-I \times_f M^n$ *of the form*

$$[\tau, +\infty) \times M^n \quad \text{or} \quad (-\infty, \tau] \times M^n,$$

for some real number τ*.*

By the height estimates which are obtained in the above section, we have the following

Theorem 5.2 *Let* Σ^n *be a complete spacelike hypersurface properly immersed in a spatially closed GRW spacetime* $-I \times_f M^n$ *obeying the NCC and with monotone warping function. Suppose that* $H_2 > 0$ *and* $H_1 \neq 0$*. If either*

(i) $f'(h) \leq 0$ *and* $\frac{H_2}{H_1} \leq \inf_I(\log f)'$ *or*

(ii) $f'(h) \geq 0$ *and* $\frac{H_2}{H_1} \geq \sup_I(\log f)'$

then Σ *cannot lie in a lower half-space. Particularly,* Σ *must have at least one top end.*

Proof Suppose that Σ lies in a lower half-space $(-\infty, \tau] \times M^n$, $\tau \in I$. For any $s \in I$, $s < \tau$, denote by Σ_s^+ the hypersurface

$$\Sigma_s^+ = \{(t, x) \in \Sigma | t \geq s\}.$$

Furthermore, since M^n is compact and the immersion is proper, then Σ_s^+ is a compact spacelike hypersurface with boundary contained in M_s. By Theorem 3.1, we have the height function of Σ_s^+ obeys $h \leq s$, leading to a contradiction since s is arbitrary. So, we obtain the conclusion. □

Combining Theorem 5.2 and the conclusion in the previous section we can get the following

Corollary 5.3 *Let Σ^n be a complete spacelike hypersurface properly immersed in a spatially closed de Sitter spacetime $-\mathbb{R}^+ \times_{\cosh t} M^n$ with $H_2 > 0$ and $\mathrm{Ric}_M \geq n-1$. If $\frac{H_2}{H_1} \geq \sup_{\mathbb{R}^+}(\tanh)$, then Σ cannot lie in a lower half-space. Particularly, Σ must have at least one top end.*

Corollary 5.4 *Let Σ^n be a complete spacelike hypersurface properly immersed in a spatially closed de Sitter spacetime $-\mathbb{R}^- \times_{\cosh t} M^n$ with $H_2 > 0$ and $\mathrm{Ric}_M \geq n-1$. If $\frac{H_2}{H_1} \leq \inf_{\mathbb{R}^-}(\tanh)$, then Σ cannot lie in a lower half-space. Particularly, Σ must have at least one top end.*

Corollary 5.5 *Let Σ^n be a complete spacelike hypersurface properly immersed in a spatially closed steady state spacetime $-I \times_{e^t} M^n$ with $H_2 > 0$ and $\mathrm{Ric}_M \geq 0$. If $\frac{H_2}{H_1} \geq 1$, then Σ cannot lie in a lower half-space. Particularly, Σ must have at least one top end.*

By Theorem 3.5, we can extend Theorem 5.2 to the hypersurfaces of positive constant k-th mean curvature.

Theorem 5.6 *Let Σ^n be a complete spacelike hypersurface properly immersed in a spatially closed GRW spacetime $-I \times_f M^n$ obeying the strong NCC and with non-decreasing warping function. Suppose that Σ^n has positive constant k-th mean curvature satisfying $\frac{H_{k+1}}{H_k} \geq \sup_I(\log f)'$ for some $2 \leq k \leq n$ and that, if $k \geq 3$, there exists an elliptic point on Σ. Then Σ cannot lie in a lower half-space. In particular, Σ must have at least one top end.*

Last, as an application of Theorem 5.6, we can have the following corollaries.

Corollary 5.7 *Let Σ^n be a complete spacelike hypersurface properly immersed in a spatially closed de Sitter space $-\mathbb{R}^+ \times_{\cosh t} M^n$ satisfying $K_M \geq 1$. Suppose that Σ has constant k-th mean curvature H_k, $2 \leq k \leq n$, obeying $\frac{H_{k+1}}{H_k} \geq \sup_{\mathbb{R}^+}(\tanh)'$ and, if $k \geq 3$, there exists an elliptic point on Σ. Then Σ cannot lie in a lower half-space. In particular, Σ must have at least one top end.*

Corollary 5.8 *Let Σ^n be a complete spacelike hypersurface properly immersed in a spatially closed steady state spacetime $-I \times_{e^t} M^n$ with $K_M \geq 0$. Assume that Σ has constant k-th mean curvature H_k, $2 \leq k \leq n$, satisfying $\frac{H_{k+1}}{H_k} \geq 1$ and, if $k \geq 3$, there exists an elliptic point on Σ. Then Σ cannot lie in a lower half-space. In particular, Σ must have at least one top end.*

6. Further half-space theorems for spacelike hypersurfaces

In this section, we will extend the previous theorems to the complete noncompact situation. In order to do that, the main tool is a generalization of the Omori-Yau maximum principle for trace type differential operators that we are introducing the next.

Let Σ be a Riemannian manifold and let $L = Tr(P \circ \mathrm{hess})$ be a semi-elliptic operator, where $P : T\Sigma \to T\Sigma$ is a positive semi-definite symmetric tensor. Following the terminology introduced in [15], we say that the Omori-Yau maximum principle holds on Σ for the operator L if, for any function $u \in C^2(\Sigma)$ with $u^* = \sup_\Sigma u < +\infty$, there exists a sequence $\{p_j\}_{j\in\mathbb{N}} \subset \Sigma$ with the properties

$$\text{(i) } u(p_j) > u^* - \frac{1}{j}, \quad \text{(ii) } \parallel \nabla u(p_j) \parallel < \frac{1}{j}, \quad \text{(iii) } Lu(p_j) < \frac{1}{j}$$

for every $j \in \mathbb{N}$. Equivalently, for any function $u \in C^2(\Sigma)$ with $u_* = \inf_\Sigma u > -\infty$, there exists a sequence $\{p_j\}_{j\in\mathbb{N}} \subset \Sigma$ with the properties

$$\text{(i) } u(p_j) < u_* + \frac{1}{j}, \quad \text{(ii) } \parallel \nabla u(p_j) \parallel < \frac{1}{j}, \quad \text{(iii) } Lu(p_j) > -\frac{1}{j}$$

for every $j \in \mathbb{N}$.

In the following, we state some useful facts obtained in [16], [17], and [5] where the validity of the Omori-Yau maximum principle has been proved for the trace type differential operators.

Lemma 6.1 ([16]) *Let $(\Sigma, \langle,\rangle)$ be a Riemannian manifold, and let $L = Tr(P \circ \text{hess})$ be a semi-elliptic operator, where $P : T\Sigma \to T\Sigma$ is a positive semi-definite symmetric tensor satisfying $\sup_\Sigma TrP < +\infty$. Assume the existence of a non-negative C^2 function γ with the properties*

(1) $\gamma(p) \to +\infty$ as $p \to \infty$,

(2) $\exists A > 0$ such that $\parallel \nabla\gamma \parallel \leq A\sqrt{\gamma}$ outside a compact set,

(3) $\exists B > 0$ such that $L\gamma \leq B\sqrt{\gamma G(\sqrt{\gamma})}$ outside a compact set,

where G is a smooth function on $[0, +\infty)$ such that:

(i) $G(0) > 0$,

(ii) $G'(t) \geq 0$ on $[0, +\infty)$,

(iii) $1/\sqrt{G(t)} \notin L^1(+\infty)$,

(iv) $\limsup_{t\to\infty} \frac{tG(\sqrt{t})}{G(t)} < +\infty$,

then, the Omori-Yau maximum principle holds on Σ for the operator L.

Lemma 6.2 ([17]) *Let $(\Sigma, \langle,\rangle)$ be a complete, noncompact Riemannian manifold with sectional curvature bounded from below. Then, the Omori-Yau maximum principle holds on Σ for any semi-elliptic operator $L = Tr(P \circ \text{hess})$ with $\sup_\Sigma TrP < +\infty$.*

Lemma 6.3 ([5]) *Let $-I \times_f M^n$ be a GRW spacetime satisfying the strong NCC. Let $\varphi : \Sigma^n \to -I \times_f M^n$ be a complete spacelike hypersurface contained in a slab with $\sup_\Sigma \parallel A \parallel^2 < +\infty$ and $\inf_\Sigma \frac{f''(h)}{f(h)} > -\infty$. Then the sectional curvature of Σ is bounded from below and the Omori-Yau maximum principle holds on Σ for any semi-elliptic operator $L = Tr(P \circ \text{hess})$ with $\sup_\Sigma TrP < +\infty$.*

Remark 6.4 ([16]) From the equality

$$\parallel A \parallel^2 = n^2 H_1^2 - n(n-1)H_2$$

it follows that under the condition $\inf_\Sigma H_2 > -\infty$, the assumption $\sup_\Sigma \parallel A \parallel^2 < +\infty$ is equivalent to $\sup_\Sigma |H_1| < +\infty$.

With previous preparation we are ready to prove the main results.

Theorem 6.5 *Let Σ^n be a complete spacelike hypersurface of constant mean curvature in a GRW spacetime $-I \times_f M^n$ which satisfies the strong NCC. Suppose that $H_2 > 0$ and $\inf_\Sigma \frac{f''(h)}{f(h)} > -\infty$.*

(i) If $\frac{H_2}{H_1} > \sup_I(\log f)'$, *then* Σ *cannot lie in a lower half-space.*

(ii) If $\frac{H_2}{H_1} < \inf_I(\log f)'$, *then* Σ *cannot lie in an upper half-space.*

Proof First we notice that the basic inequality $H_1^2 \geq H_2 > 0$, it follows that we can orient the hypersurface so that $H_1 > 0$ on Σ. Now we define the operator $\widehat{L}_1 = Tr(\widehat{P}_1 \circ \text{hess})$ with $\widehat{P}_1 = (\frac{1}{H_1})P_1$. Note that $Tr(\widehat{P}_1) = c_1$. So, by the Lemma 6.3 and Remark 6.4, we have the Omori-Yau maximum principle holds on Σ for the operator $\widehat{L}_1$.

In the following, let us prove part (i). Applying contradiction, we assume that Σ lies in a lower half-space, that is, $\sup_\Sigma(h) := h^* < +\infty$. By the definition of the operator $\widehat{L}_1$, there exists a sequence $\{p_j\}_{j\in\mathbb{N}}$ such that

(i) $\lim_{j\to\infty} \sigma(h(p_j)) = \sup_\Sigma \sigma(h)$,

(ii) $\| \nabla\sigma(h)(p_j) \|= f(h(p_j)) \| \nabla h(p_j) \|< \frac{1}{j}$,

(iii) $\widehat{L}_1\sigma(h)(p_j) < \frac{1}{j}$.

Observe that condition (i) implies that $\lim_{j\to+\infty} h(p_j) = h^*$, since $\sigma(t)$ is strictly increasing. Therefore by condition (ii) we obtain $\lim_{j\to+\infty} \| \nabla h(p_j) \|= 0$, and $\lim_{j\to+\infty} \Theta(p_j) = -1$. Thus, using

$$\widehat{L}_1\sigma(h) = -c_1(f'(h) + \Theta f(h)\frac{H_2}{H_1}),$$

we have

$$\frac{1}{j} > \widehat{L}_1\sigma(h)(p_j) = -c_1 f(h(p_j))((\log f)'(h(p_j)) + \Theta(p_j)\frac{H_2}{H_1}).$$

Letting $j \to +\infty$, we get

$$0 \geq (\frac{H_2}{H_1} - (\log f)'(h^*)).$$

So, we complete the proof of (i), since $\frac{H_2}{H_1} \leq (\log f)'(h^*) \leq \sup_I(\log f)'$, contradicting the initial assumption on $\frac{H_2}{H_1}$.

For the part (ii), we assume by contradiction that Σ lies in an upper half-space, that is, $\inf_\Sigma(h) := h_* > -\infty$. Again, we can find a sequence $\{q_j\}_{j\in\mathbb{N}}$ satisfying the following conditions

(i) $\lim_{j\to\infty} \sigma(h(q_j)) = \inf_\Sigma \sigma(h)$,

(ii) $\| \nabla\sigma(h)(q_j) \|= f(h(q_j)) \| \nabla h(q_j) \|< \frac{1}{j}$,

(iii) $\widehat{L}_1\sigma(h)(q_j) > -\frac{1}{j}$.

Hence

$$-\frac{1}{j} < \widehat{L}_1\sigma(h)(q_j) = -c_1 f(h(q_j))((\log f)'(h(q_j)) + \Theta(q_j)\frac{H_2}{H_1}).$$

If $j \to +\infty$, adopting the similar arguments leads to a contradiction with the initial condition. So, part (ii) is proved. □

By Theorem 6.5, we can have the following corollaries immediately.

Corollary 6.6 *Let* Σ^n *be a complete spacelike hypersurface of constant mean curvature in a de Sitter spacetime* $-I \times_{\cosh t} M^n$ *with* $H_2 > 0$ *and* $\text{Ric}_M \geq n-1$.

(i) If $\frac{H_2}{H_1} > \sup_I(\tanh)$, *then* Σ *cannot lie in a lower half-space.*

(ii) If $\frac{H_2}{H_1} < \inf_I(\tanh)$, *then* Σ *cannot lie in an upper half-space.*

Corollary 6.7 *Let Σ^n be a complete spacelike hypersurface of constant mean curvature in a steady state spacetime $-I \times_{e^t} M^n$ with $H_2 > 0$ and $\mathrm{Ric}_M \geq 0$.*

(i) If $\frac{H_2}{H_1} > 1$, then Σ cannot lie in a lower half-space.

(ii) If $\frac{H_2}{H_1} < 1$, then Σ cannot lie in an upper half-space.

The next results extend to the complete spacelike hypersurface of constant k-th mean curvature in case $2 \leq k \leq n$.

Theorem 6.8 *Let Σ^n be a complete spacelike hypersurface of positive constant k-th mean curvature for some $2 \leq k \leq n$ in a GRW spacetime $-I \times_f M^n$ which satisfies the strong NCC and whose warping function is non-decreasing. Suppose that $\sup_\Sigma |H_1| < +\infty$ and $\inf_\Sigma \frac{f''(h)}{f(h)} > -\infty$. Moreover, if $k \geq 3$, there exists an elliptic point on Σ.*

(i) If $\frac{H_{k+1}}{H_k} > \sup_I (\log f)'$, then Σ cannot lie in a lower half-space.

(ii) If $\frac{H_{k+1}}{H_k} < \inf_I (\log f)'$, then Σ cannot lie in an upper half-space.

Proof First, we observe that the existence of an elliptic point and $H_k > 0$ implies that $H_i > 0$ and the operators P_i are positive definite for all $1 \leq i \leq k-1$. Since $H_k > 0$, we consider the operator $\widehat{L}_k = Tr(\widehat{P}_k \circ \mathrm{hess})$ with $\widehat{P}_k = (\frac{1}{H_k})P_k$. Note that $Tr(\widehat{P}_k) = c_k$. Therefore, using Lemma 6.3 and Remark 6.4, the Omori-Yau maximum principle holds on Σ for the operator $\widehat{L}_k$. We conclude then as in Theorem 6.5 with the aid of the equation

$$\widehat{L}_k \sigma(h) = -c_k(f'(h) + \Theta f(h)\frac{H_{k+1}}{H_k}) = -c_k f(h)((\log f)' + \Theta \frac{H_{k+1}}{H_k}). \quad \Box$$

Finally, as an application of Theorem 6.8, we can straightforwardly have the following corollaries about hypersurface of constant higher mean curvature in spacial spacetime.

Corollary 6.9 *Let Σ^n be a complete spacelike hypersurface of positive constant k-th mean curvature for $2 \leq k \leq n$ in a de Sitter space $-\mathbb{R}^+ \times_{\cosh t} M^n$ satisfying $K_M \geq 1$. Suppose that $\sup_\Sigma |H_1| < +\infty$ and, if $k \geq 3$, there exists an elliptic point on Σ.*

(i) If $\frac{H_{k+1}}{H_k} > \sup_{\mathbb{R}^+} (\tanh)'$, then Σ cannot lie in a lower half-space.

(ii) If $\frac{H_{k+1}}{H_k} < \inf_{\mathbb{R}^+} (\tanh)'$, then Σ cannot lie in an upper half-space.

Corollary 6.10 *Let Σ^n be a complete spacelike hypersurface of positive constant k-th mean curvature for $2 \leq k \leq n$ in steady state spacetime $-I \times_{e^t} M^n$ with $K_M \geq 0$. Suppose that $\sup_\Sigma |H_1| < +\infty$ and, if $k \geq 3$, there exists an elliptic point on Σ.*

(i) If $\frac{H_{k+1}}{H_k} > 1$, then Σ cannot lie in a lower half-space.

(ii) If $\frac{H_{k+1}}{H_k} < 1$, then Σ cannot lie in an upper half-space.

Acknowledgements We thank the referees for their time and comments.

References

[1] R. LÓPEZ. *Area monotonicity for spacelike surfaces with constant mean curvature.* J. Geom. Phys., 2004, **52**(3): 353–363.

[2] S. MONTIEL. *Complete non-compact spacelike hypersurfaces of constant mean curvature in de Sitter spaces.* J. Math. Soc. Japan., 2003, **55**(4): 915–938.

[3] H. F. DE LIMA. *A sharp height estimate for compact spacelike hypersurfaces with constant r-mean curvature in the Lorentz-Minkowski space and application*. Differential Geom. Appl., 2008, **26**(4): 445–455.
[4] S. C. GARCÍA-MARTÍNEZ, D. IMPERA, M. RIGOLI. *A sharp height estimate for compact hypersurfaces with constant k-mean curvature in warped product spaces*. Proc. Edinb. Math. Soc. (2), 2015, **58**(2): 403–419.
[5] S. C. GARCÍA-MARTÍNEZ, D. IMPERA. *Height estimates and half-space theorems for spacelike hypersurfaces in generalized Robertson-Walker spacetimes*. Differential Geom. Appl., 2014, **32**: 46–67.
[6] A. G. COLARES, H. F. DE LIMA. *Space-like hypersurfaces with positive constant r-mean curvature in Lorentzian product spaces*. Gen. Relativity Gravitation, 2008, **40**(10): 2131–2147.
[7] B. O'NELL. *Semi-Riemannian Geometry with Applications to Relativity*. Academic Press, Inc., Harcourt Brace Jovanovich, Publishers, New York, 1983.
[8] L. J. ALÍAS, A. ROMERO, M. SÁNCHEZ. *Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes*. Gen. Relativity Gravitation, 1995, **27**: 71–84.
[9] L. J. ALÍAS, A. J. BRASIL, A. G. COLARES. *Integral formulae for spacelike hypersurfaces in conformally stationary spacetimes and applications*. Proc. Edinb. Math. Soc. (2), 2003, **46**(2): 465–488.
[10] L. J. ALÍAS, A. G. COLARES. *Uniqueness of spacelike hypersurfaces with constant higher order mean curvature in generalized Robertson-Walker spacetimes*. Math. Proc. Cambridge Philos. Soc., 2007, **143**(3): 703–729.
[11] M. F. ELBERT. *Constant positive 2-mean curvature hypersurfaces*. Illinois J. Math., 2002, **46**(1): 247–267.
[12] J. L. BARBOSA, A. G. COLARES. *Stability of hypersurfaces with constant r-mean curvature*. Ann. Global Anal. Geom., 1997, **15**(3): 277–297.
[13] S. MONTIEL. *Uniqueness of spacelike hypersurfaces of constant mean curvature in foliated spacetimes*. Math. Ann., 1999, **314**(3): 529–553.
[14] A. L. ALBUJER, L. J. ALÍAS. *Spacelike hypersurfaces with constant mean curvature in the steady state space*. Proc. Amer. Math. Soc., 2009, **137**(2): 711–721.
[15] S. PIGOLA, M. RIGOLI, A. G. SETTI. *Maximum principles on Riemannian manifolds and applications*. Mem. Amer. Math. Soc., 2005, **174**(822): x+99.
[16] L. J. ALÍAS, D. IMPERA, M. RIGOLI. *Hypersurfaces of constant higher order mean curvature in warped products*. Trans. Amer. Math. Soc., 2013, **365**(2): 591–621.
[17] L. J. ALÍAS, D. IMPERA, M. RIGOLI. *Spacelike hypersurfaces of constant higher order mean curvature in generalized Robertson-Walker spacetimes*. Math. Proc. Cambridge Philos. Soc., 2012, **152**(2): 365–383.
[18] L. J. ALÍAS, D. IMPERA, M. RIGOLI. *Erratum to "Spacelike hyypersurfaces of constant higher order mean curvature in generalized Robertson-Walker spacetimes". Math. Proc. Camb. Phil. Soc. (2012), 152, 365-383*. Math. Proc. Cambridge Philos. Soc., 2013, **155**(2): 375–377.