Journal of Mathematical Research with Applications
Sept., 2017, Vol. 37, No. 5, pp. 520–526
DOI:10.3770/j.issn:2095-2651.2017.05.002
Http://jmre.dlut.edu.cn

A Relationship between the Walks and the Semi-Edge Walks of Graphs

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Abstract We establish a relation between the number of semi-edge walks of a connected graph and the number of walks of two auxiliary graphs. In addition, this relation gives upper bounds on the signless Laplacian spectral radius of connected graphs and planar graphs.

Keywords walks; semi-edge walks; signless Laplacian spectral radius; planar graphs

MR(2010) Subject Classification 05C50

1. Introduction

Throughout this paper, all graphs are finite, connected and simple unless stated otherwise. Let $G = (V(G), E(G))$ be a graph of order n with size m, where the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. We denote $N_G(v)$ (or $N(v)$ for short) as the set of neighbours of v in G, and $|N_G(v)|$ as the degree of v. For $i \in \{1, 2, \ldots, n\}$, let $d_i = d_G(v_i) = |N_G(v_i)|$. Moreover, the maximum and minimum degrees of G are denoted by Δ and δ, respectively. An edge with identical ends is called a loop. Two or more edges with the same pair of ends are said to be parallel edges. A linear k-forest is a graph whose components are paths of length at most k. The linear k-arborocity of G, denoted by $la_k(G)$, is the least integer p such that $E(G)$ can be decomposed into p linear k-forests. All notations undefined in this article are referred to the book [1].

Let $A(G)$ (or simply A) be the adjacency matrix of G and $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex degrees. Matrix $Q(G) = D(G) + A(G)$ (or simply Q) is called the signless Laplacian matrix of G. Obviously, all the eigenvalues of $A(G)$ and $Q(G)$ are real numbers since both of them are real symmetric matrices. Moreover, the largest eigenvalues of $A(G)$ and $Q(G)$, denoted by $\lambda_1(G)$ and $q_1(G)$ (or simply λ_1 and q_1), are called the spectral radius and the signless Laplacian spectral radius of G, respectively.

The following definitions facilitate the proof of our main results.

Definition 1.1 *A walk of length k in a graph G (not necessarily simple) is an alternating*

Received November 4, 2016; Accepted January 5, 2017
Supported by GRF of Hong Kong (Grant No. HKBU202413), FRG of Hong Kong Baptist University (Grant No. FRG2/14-15/012).
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sequence $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$ of vertices $v_1, v_2, \ldots, v_k, v_{k+1}$ and edges $e_1, e_2, \ldots, e_k$ such that for any $i \in \{1, 2, \ldots, k\}$ the vertices v_i and v_{i+1} are end-vertices (not necessarily distinct) of the edge e_i.

Definition 1.2 *Suppose $e = uv$ is an edge of a simple graph. A semi-edge, denoted by $u^v u$, is a 'walk' that starts from u toward v along e up to the midpoint of uv and then returns back to u.*

Note that, for an edge uv, there are two semi-edges $u^v u$ and $v^u v$.

Definition 1.3 *A semi-edge walk of length k in a graph G is an alternating sequence v_1, e_1, $v_2, e_2, \ldots, v_k, e_k, v_{k+1}$, where $v_1, v_2, \ldots, v_k, v_{k+1}$ are vertices and $e_1, e_2, \ldots, e_k$ are edges or semi-edges such that for any $i \in \{1, 2, \ldots, k\}$ the vertices v_i and v_{i+1} are end-vertices of e_i.*

It is well-known that the (i, j)-entry of the matrix A^k is the number of walks of length k from vertex i to vertex j. Based on this property, the number of walks has been widely used to study the spectral radius, the energy, the k-th spectral moment and other parameters (see [2–5] for details). Similarly, we have the following theorem.

Theorem 1.4 ([6]) *Let Q be the signless Laplacian matrix of a graph G. The (i, j)-entry of the matrix Q^k is equal to the number of semi-edge walks of length k starting at vertex i and terminating at vertex j.*

From this theorem, we can study some algebraic properties of a graph, such as signless Laplacian spectral radius, by counting its number of semi-edge walks. In this paper, we obtain a relation between the numbers of walks and semi-edge walks. Using this relation, we can study the number of semi-edge walks by the number of walks since it has been well studied.

2. Main results

Before stating our main result, we need to construct two new graphs.

Let G be a connected graph. For each edge uv, we add two loops $l_u(v)$ and $l_v(u)$ incident with u and v, respectively. The resulting graph is denoted by G_1.

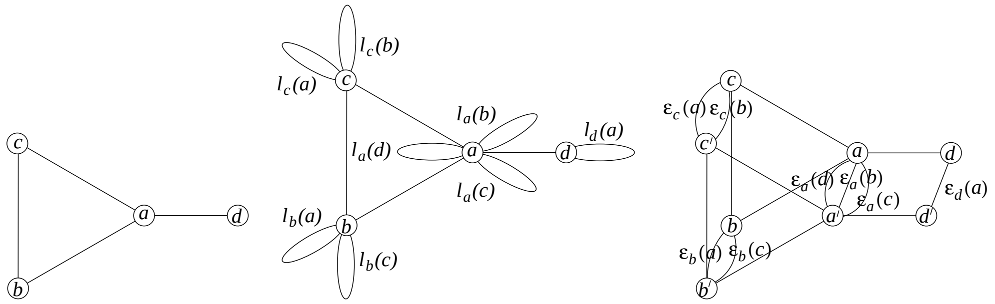
Remark 2.1 For each edge uv in G, the semi-edges $u^v u$ and $v^u v$ are corresponding to loops $l_u(v)$ and $l_v(u)$ in G_1, respectively. Clearly, this corresponding is a bijection between the set of all semi-edges of G and the set of all loops in G_1.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. We duplicate G to get G'. The vertex in G' corresponding to v_i is denoted by v_i'. Consider the disjoint union graph $G + G'$. For each $i \in \{1, 2, \ldots, n\}$, let $N_G(v_i) = \{u_1, \ldots, u_{d_i}\}$. We link v_i and v_i' with d_i parallel edges in $G + G'$ and denote these edges by $\varepsilon_{v_i}(u_j)$, for $1 \le j \le d_i$, respectively. The resulting graph is denoted by G_2.

Remark 2.2 Suppose $v_i u_j \in E(G)$. It associates a loop $l_{v_i}(u_j)$ incident to v_i in G_1. Then the corresponding $l_{v_i}(u_j) \mapsto \varepsilon_{v_i}(u_j)$ is a bijection from $E(G_1) \setminus E(G)$ onto $E(G_2) \setminus E(G + G')$.

Remark 2.3 There are no edges in G_2 incident with v_i and v_j' except $i = j$.

Note that G_1 and G_2 are not simple, where G_1 has loops and G_2 has parallel edges. The following figure shows the graphs that we have defined.



G G_1 G_2

Figure 1 Two constructions

The following theorem gives a relation between the semi-edge walks in G and the walks in G_1 and G_2.

Theorem 2.4 *Let G be a graph and let G_1 and G_2 be graphs constructed from G as above. If we ignore the direction of all loops, then the number of semi-edge walks in G equals to the number of walks in G_1 and equals to half of the number of walks in G_2.*

Proof The first part of the theorem follows from Remark 2.1.

Let S_i be the set of walks in G_i, $i = 1, 2$. Let $W_2 = x_1e_1x_2 \cdots e_kx_{k+1}$ be a walk of length k in G_2. According to the definition, x_j is either v_j or v'_j for some $v_j \in V(G)$, $1 \le j \le k+1$. Define $\phi(W_2) = v_1g_1v_2 \cdots g_kv_{k+1}$, where

$$g_j = \begin{cases} v_jv_{j+1} & \text{if } e_j = v_jv_{j+1} \text{ or } e_j = v'_jv'_{j+1} \text{ is a link;} \\ l_{v_j}(v) & \text{if } e_j = \varepsilon_{v_j}(v) \text{ for some } v \in V(G). \end{cases}$$

So $\phi(W_2) \in S_1$. Clearly, $\phi : S_2 \to S_1$ is a mapping. We will show by induction on the length of a walk that ϕ is a 2-to-1 surjection and hence we prove the second part of the theorem.

From the definition of ϕ, we have $\phi(WU) = \phi(W)\phi(U)$ if the end vertex of W is the initial vertex of U.

Suppose $W_1 = v_1g_1v_2 \in S_1$. Let $W_2 = x_1e_1x_2 \in \phi^{-1}[W_1]$ be the pre-image of W_1. According to the definition of ϕ, $x_i \in \{v_i, v'_i\}$ for $i = 1, 2$. Suppose $x_1 = v_1$. If g_1 is a link, then $v_1 \neq v_2$ and hence $g_1 = v_1v_2$ and $x_2 = v_2$ (by Remark 2.3). If g_1 is a loop, then $g_1 = l_{v_1}(v)$ for some $v \in N(v_1)$. Thus $e_1 = \varepsilon_{v_1}(v)$ and $x_2 = v'_1$. Suppose $x_1 = v'_1$. If g_1 is a link, then $v_1 \neq v_2$ and hence $g_1 = v_1v_2$ and $x_2 = v'_2$. If g_1 is a loop, then $g_1 = l_{v_1}(v)$ for some $v \in N(v_1)$. Thus $e_1 = \varepsilon_{v_1}(v)$ and $x_2 = v_1$ and hence $|\phi^{-1}[W_1]| = 2$.

Suppose $|\phi^{-1}[W]| = 2$ for any walk W of length $k-1$ in S_1, where $k \ge 2$.

Now suppose $W_1 = v_1g_1v_2 \cdots g_{k-1}v_kg_kv_{k+1} \in S_1$. Let $W_2 = x_1e_1x_2 \cdots e_{k-1}x_ke_kx_{k+1} \in \phi^{-1}[W_1]$. Let $U_1 = v_1g_1v_2 \cdots g_{k-1}v_k$ and $U_2 = x_1e_1x_2 \cdots e_{k-1}x_k$. Then $\phi(U_2) = U_1$. That means, $U_2 \in \phi^{-1}[U_1]$. Now $x_k \in \{v_k, v_k'\}$ and $x_{k+1} \in \{v_{k+1}, v_{k+1}'\}$. Similar to the case when the walk of length 1, x_{k+1} is determined uniquely. Therefore, according to the two choices of U_1 we obtain $|\phi^{-1}[W_1]| = 2$. $\Box$

3. Some upper bounds on q_1 of planar graphs

In this section, we provide some upper bounds on the signless Laplacian spectral radius of connected graphs and planar graphs. The spectral radius of a matrix can be represented as a limit of the matrix norms, which is presented as follows.

Lemma 3.1 ([7]) *Let $||\cdot||$ be a matrix norm on M_n, the set of all $n \times n$ complex matrices. For any $M \in M_n$, we have $\rho(M) = \lim_{k\to\infty} \|M^k\|^{1/k}$, where $\rho(M)$ is the spectral radius of M.*

This result holds for any matrix norm. Thus we can choose some particular norms to study the signless Laplacian matrix of a graph. Here we use the l_1-norm which is defined as

$$\|M\|_1 = \sum_{i,j=1}^{n} |m_{ij}|,$$

where m_{ij} is the (i,j)-entry of M.

Note that the signless Laplacian matrix of a graph is a nonnegative matrix. Hence, taking the matrix norm with l_1-norm in Lemma 3.1, we have

Corollary 3.2 *Let G be a graph of order n, Q be the signless Laplacian matrix of G and $\boldsymbol{u}$ be the n-vector with all entries 1. Then*

$$q_1 = \lim_{k\to\infty} \sqrt[k]{\boldsymbol{u}^T Q^k \boldsymbol{u}}.$$

Obviously, $\boldsymbol{u}^T Q^k \boldsymbol{u}$ stands for the sum of all entries of Q^k, which is the number of semi-edges walks of length k in G by Theorem 1.4. Therefore, we obtain some bounds on q_1 by estimating the number of semi-edges walks of length k in G. We can use Theorem 2.4 to estimate the number of semi-edges walks since the walks of a graph have been well studied.

Hayer gave an upper bound on the spectral radius by estimating the number of walks as follows.

Lemma 3.3 ([8]) *Let G be a graph with maximum degree Δ. If there is an orientation of G such that the maximum out-degree $\Delta^+ \le \Delta/2$, then $\lambda_1 \le 2\sqrt{\Delta^+(\Delta - \Delta^+)}$.*

In his proof, he used the following result on the number of walks.

Lemma 3.4 ([8]) *Let G be a graph of order n with maximum degree Δ. If there is an orientation of G such that the maximum out-degree $\Delta^+ \le \Delta/2$, then the number of walks of length k in G is at most $n2^k(\sqrt{\Delta^+(\Delta - \Delta^+)})^k$.*

Note that Lemma 3.4 holds for graphs with parallel edges but without loop. A similar result about semi-edge walks can be obtained by Theorem 2.4.

Lemma 3.5 *Let G be a (simple) graph of order n with maximum degree Δ. If there is an* *orientation of G such that the maximum out-degree $\Delta^+ \leq \Delta/2$, then the number of semi-edge walks of length k is at most $n2^k(\Delta^+ + \lceil\frac{\Delta}{2}\rceil)^{\frac{k}{2}}(\Delta + \lfloor\frac{\Delta}{2}\rfloor - \Delta^+)^{\frac{k}{2}}$.*

Proof Let G_2 be the graph constructed in Section 2 by adding some parallel edges to the disjoint union of two copies of G. Suppose there is an orientation of G such that $\Delta^+ \leq \Delta/2$. Define an orientation on G_2 from the orientation on G: For any edge $e = v_iv_j$ in G, if its direction is from v_i to v_j, then orient the edge v_iv_j from v_i to v_j and the edge $v'_iv'_j$ from v'_i to v'_j in G_2. For the $d_G(v_i)$ parallel edges between v_i and v'_i in G_2, we choose $\lceil d_G(v_i)/2\rceil$ parallel edges arbitrarily and orient them from v_i to v'_i while the other parallel edges from v'_i to v_i.

Under the above orientation of G_2, it is easy to see that its maximum out-degree and maximum degree are $\Delta^+ + \lceil\frac{\Delta}{2}\rceil$ and 2Δ, respectively. When Δ is even,

$$\Delta^+ + \lceil\tfrac{\Delta}{2}\rceil \leq \frac{\Delta}{2} + \frac{\Delta}{2} = \Delta = \frac{1}{2} \times 2\Delta.$$

And when Δ is odd, $\Delta^+ \leq \frac{\Delta-1}{2}$ as Δ^+ is an integer. Hence

$$\Delta^+ + \lceil\tfrac{\Delta}{2}\rceil \leq \frac{\Delta-1}{2} + \frac{\Delta+1}{2} = \Delta = \frac{1}{2} \times 2\Delta.$$

Therefore, by Lemma 3.4, the number of walks of length k in G_2 is at most

$$2n \cdot 2^k(\lceil\tfrac{\Delta}{2}\rceil + \Delta^+)^{\frac{k}{2}}(2\Delta - (\lceil\tfrac{\Delta}{2}\rceil + \Delta^+))^{\frac{k}{2}} = 2n \cdot 2^k(\lceil\tfrac{\Delta}{2}\rceil + \Delta^+)^{\frac{k}{2}}(\Delta + \lfloor\tfrac{\Delta}{2}\rfloor - \Delta^+)^{\frac{k}{2}}.$$

As a result, by Theorem 2.4, the number of semi-edge walks of length k in G is at most

$$n2^k(\lceil\tfrac{\Delta}{2}\rceil + \Delta^+)^{\frac{k}{2}}(\Delta + \lfloor\tfrac{\Delta}{2}\rfloor - \Delta^+)^{\frac{k}{2}}. \quad \Box$$

From Corollary 3.2, we know that $\boldsymbol{u}^TQ^k\boldsymbol{u}$ equals to the number of semi-edge walks of length k in G. Combined with Lemma 3.5, we have

Theorem 3.6 *Let G be a graph with maximum degree Δ. If there is an orientation such that the maximum out-degree $\Delta^+ \leq \frac{\Delta}{2}$, then*

$$q_1 \leq 2\sqrt{(\lceil\tfrac{\Delta}{2}\rceil + \Delta^+)(\Delta + \lfloor\tfrac{\Delta}{2}\rfloor - \Delta^+)}. \tag{3.1}$$

Remark 3.7 The bound in (3.1) is sharp. It is easy to show by induction that K_{2n+1} admits an orientation such that $\Delta^+ = n$. Then $q_1 = 2\sqrt{2n \cdot 2n} = 4n = 2\Delta$.

Remark 3.8 Not every graph can admit an orientation such that $\Delta^+ \leq \frac{\Delta}{2}$: Consider the complete graph K_{2n}, then

$$n(2n-1) = |E(K_{2n})| = \sum_{i=1}^{2n} d_i^+ \leq 2n\Delta^+.$$

Since Δ^+ is an integer, $\Delta^+ \geq \lceil\frac{2n-1}{2}\rceil = n > \Delta/2$.

If G is a tree, then it admits an orientation with maximum out-degree 1 by using the depth-first tree search and orienting each edge from child to parent. Hence we have

Corollary 3.9 *Let T be a tree with maximum degree $\Delta \geq 2$. Then*

$$q_1 \leq 2\sqrt{(\lceil\tfrac{\Delta}{2}\rceil + 1)(\Delta + \lfloor\tfrac{\Delta}{2}\rfloor - 1)}.$$

For planar graphs, Gonçalves proved the following decomposition result.

Lemma 3.10 ([9]) *If G is a planar graph, then $E(G) = E(T_1) \cup E(T_2) \cup E(T_3)$, where T_1, T_2 and T_3 are forests and $\Delta(T_3) \leq 4$.*

Lemma 3.11 *If G is a planar graph with the maximum degree $\Delta \geq 2$, then*

$$q_1 \leq 2\sqrt{(\lceil \tfrac{\Delta}{2} \rceil + 2)(\Delta + \lfloor \tfrac{\Delta}{2} \rfloor - 2)} + 2\sqrt{15}.$$

Proof If $\Delta \leq 3$, then

$$q_1 \leq 2\Delta \leq 6 < 2\sqrt{15} < 2\sqrt{(\lceil \tfrac{\Delta}{2} \rceil + 2)(\Delta + \lfloor \tfrac{\Delta}{2} \rfloor - 2)} + 2\sqrt{15}.$$

So we may assume $\Delta \geq 4$. Let $E(G) = E(T_1) \cup E(T_2) \cup E(T_3)$ be the edge decomposition of G by Lemma 3.10. Since T_1 and T_2 are forests, they admit an orientation with maximum out-degree 1, respectively. Hence $T_1 \cup T_2$ has an orientation with maximum out-degree at most 2. By Theorem 3.6, each component C of $T_1 \cup T_2$ has

$$q_1(C) \leq 2\sqrt{(\lceil \tfrac{\Delta}{2} \rceil + 2)(\Delta + \lfloor \tfrac{\Delta}{2} \rfloor - 2)}.$$

Therefore,

$$q_1(T_1 \cup T_2) \leq 2\sqrt{(\lceil \tfrac{\Delta}{2} \rceil + 2)(\Delta + \lfloor \tfrac{\Delta}{2} \rfloor - 2)}.$$

Note that each component T of T_3 is a tree with maximum degree at most 4. From Corollary 3.9, we have $q_1(T) \leq 2\sqrt{15}$ and thus $q_1(T_3) \leq 2\sqrt{15}$.

Obviously, $Q(G) = Q(T_1 \cup T_2) + Q(T_3)$. Hence, by Weyl inequality [7], we have

$$q_1 \leq 2\sqrt{(\lceil \tfrac{\Delta}{2} \rceil + 2)(\Delta + \lfloor \tfrac{\Delta}{2} \rfloor - 2)} + 2\sqrt{15}. \quad \square$$

We now use the linear 2-arboricity to determine the signless Laplacian spectral radius of a planar graph.

Lemma 3.12 *Let G be a graph and $la_2(G)$ be its linear 2-arboricity. Then*

$$q_1(G) \leq 3 \cdot la_2(G).$$

Proof Let $E(G) = E(T_1) \cup \cdots \cup E(T_{la_2(G)})$ be its edge decomposition, where T_i is a forest whose components are paths of length at most 2 for $i \in \{1, \ldots, la_2(G)\}$. And for each component P of T_i, we can easily obtain $q_1(P) \leq 3$ as P is a path of length at most 2. Hence $q_1(T_i) \leq 3$ for $i \in \{1, \ldots, la_2(G)\}$.

On the other hand, $Q(G) = Q(T_1) + \cdots + Q(T_{la_2(G)})$. Then by Weyl inequality, we have

$$q_1(G) \leq q_1(T_1) + \cdots + q_1(T_{la_2(G)}) \leq 3 \cdot la_2(G). \quad \square$$

Recently, Wang studied $la_2(G)$ for a planar graph and got the following upper bound on $la_2(G)$.

Lemma 3.13 ([10]) *If G is a planar graph with maximum degree Δ, then*

$$la_2(G) \leq \lceil \tfrac{\Delta+1}{2} \rceil + 6.$$

Combining these two lemmas, we have

Lemma 3.14 *If G is a planar graph with maximum degree Δ, then*

$$q_1 \le 3\lceil \tfrac{\Delta+1}{2} \rceil + 18.$$

Remark 3.15 We know that matrix $L(G) = D(G) - A(G)$ is the Laplacian matrix of G. Let I_n and O_n be the identity matrix and zero matrix of order n, respectively. Given a graph G of order n, we construct G_2 as in Section 2. Consider its adjacency matrix $A(G_2)$ and we have

$$\begin{aligned}
\det(\lambda I_{2n} - A(G_2)) &= \det \begin{pmatrix} \lambda I_n - A(G) & -D(G) \\ -D(G) & \lambda I_n - A(G) \end{pmatrix} \\
&= \det \begin{pmatrix} \lambda I_n - (A(G) + D(G)) & \lambda I_n - (A(G) + D(G)) \\ -D(G) & \lambda I_n - A(G) \end{pmatrix} \\
&= \det(\lambda I_n - Q(G)) \det \begin{pmatrix} I_n & I_n \\ -D(G) & \lambda I_n - A(G) \end{pmatrix} \\
&= \det(\lambda I_n - Q(G)) \det \begin{pmatrix} I_n & I_n \\ O_n & \lambda I_n - (A(G) - D(G) \end{pmatrix} \\
&= \det(\lambda I_n - Q(G)) \det(\lambda I_n - (-L(G))).
\end{aligned}$$

As a result, we have

Theorem 3.16 *Given a graph G, let G_2 be constructed from G as in Section 2. Then the spectrum of G_2 is exactly the union of the signless Laplacian spectrum and the minus of the Laplacian spectrum of G. In particular, $q_1(G) = \lambda_1(G_2)$ and $\mu_1(G) = -\lambda_{2n}(G_2)$, where $\mu_1(G)$ is the Laplacian spectral radius of G and $\lambda_{2n}(G_2)$ is the least eigenvalue of G_2.*

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