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D_C-Projective Dimension of Complexes

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Abstract Let R be a commutative ring and C a semidualizing R-module. We introduce the notion of D_C-projective dimension for homologically bounded below complexes and give some characterizations of this dimension.

Keywords complexes; semidualizing modules; D_C-projective dimension

MR(2010) Subject Classification 18G25; 18G35; 55U15

1. Introduction

Over a commutative Noetherian ring, Foxby [1], Golod [2] and Vasconcelos [3] independently initiated the study of semidualizing modules under different names, which provided a common generalization of dualizing modules and free modules of rank one. By using these modules, Golod [2] defined the G_C-dimension, a refinement of projective dimension, for finitely generated modules. When $C = R$, this recovers the G-dimension introduced by Auslander and Bridger [4]. Motivated by Enochs and Jenda's extensions in [5] of G-dimension, Holm and Jøgensen [6] extended the G_C-dimension to arbitrary modules over a commutative Noetherian ring (where they used the name of C-Gorenstein projective dimension). Later, White [7] further extended this concept to the non-Noetherian setting, named G_C-projective dimension of modules, and she showed that it shares many common properties with the Gorenstein projective dimension. As a special case of Gorenstein projective modules, strongly Gorenstein flat modules were studied in [8], and later in [9] under different name-the Ding projective modules. The relative versions of Ding projective modules and Ding projective dimension of modules with respect to a semidualizing module were investigated in [10–12]. In a different direction, homological dimensions have been extended to complexes. Avramov and Foxby [13] defined projective, injective, and flat dimensions for arbitrary complexes of left modules over associative rings. Over commutative local rings, Yassemi [14] and Christensen [15] introduced a Gorenstein projective dimension for complexes with bounded below homology. Christensen, Frankild and Holm [16] gave a nice functorial descriptions for the Gorenstein projective dimensions to homologically bounded below complexes. The Gorenstein projective dimension of complexes with respect to a semidualizing module over commutative rings were investigated in [17].

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Motivated by these works, in this paper, we introduce a concept of Ding projective dimension with respect to a semidulizing module for homologically bounded below complexes, and give some characterizations of this dimension. Our result extends [10, Theorem 2.4] and [11, Proposition 2.11] to the context of complexes.

Next we shall recall some notation and definitions which we need in the sequel. In order to make things less technical, throughout this article, by a ring R, we always mean a commutative ring with identity, all modules are unitary R-modules. We use $\mathcal{C}(R)$ to denote the category of complexes of R-modules. To every complex

$$X : \cdots \longrightarrow X_{n+1} \xrightarrow{\delta_{n+1}^X} X_n \xrightarrow{\delta_n^X} X_{n-1} \longrightarrow \cdots$$

in $\mathcal{C}(R)$, the nth cycle (resp., boundary, homology) of X is denoted by $\mathrm{Z}_n(X)$ (resp., $\mathrm{B}_n(X)$, $\mathrm{H}_n(X)$), and we set $\mathrm{C}_n(X) = \mathrm{Coker}\delta_{n+1}^X$. Given an R-module M, we identify it with the complex that all entries 0 except M in degree 0. Given an $X \in \mathcal{C}(R)$ and an integer m, then $\Sigma^m X$ denotes the complex X shifted m degrees (to the left); it is given by $(\Sigma^m X)_n = X_{n-m}$ and whose boundary operators are $(-1)^m \delta_{n-m}^X$. The supremum and infimum of X capture its homological position; they are defined as follows

$$\sup X = \sup\{s \in \mathbb{Z} \mid \mathrm{H}_s(X) \neq 0\}, \quad \text{and} \quad \inf X = \inf\{s \in \mathbb{Z} \mid \mathrm{H}_s(X) \neq 0\}.$$

For every R-complex X, the underlying graded module $X^\natural$ is an R-complex with zero-differential, so one has

$$\sup X^\natural = \sup\{s \in \mathbb{Z} \mid X_s \neq 0\}, \quad \text{and} \quad \inf X^\natural = \inf\{s \in \mathbb{Z} \mid X_s \neq 0\}.$$

Let $\alpha : X \longrightarrow Y$ be a morphism of R-complexes. The mapping cone $\mathrm{Cone}(\alpha)$ of α is the complex with $\mathrm{Cone}(\alpha)_n = Y_n \oplus X_{n-1}$ and differential

$$\delta_n^{\mathrm{Cone}(\alpha)}(y_n, x_{n-1}) = (\delta_n^Y(y_n) + \alpha_{n-1}(x_{n-1}), -\delta_{n-1}^X(x_{n-1})).$$

Given $X, Y \in \mathcal{C}(R)$, $\mathrm{Hom}_R(X,Y)$ denotes the complex with $\mathrm{Hom}_R(X,Y)_n = \prod_{t\in\mathbb{Z}} \mathrm{Hom}_R(X_t, Y_{n+t})$, and with differential given by

$$\delta_n\left((f_t)_{t\in\mathbb{Z}}\right) = \left(\delta_{n+t}^Y f_t - (-1)^n f_{t-1}\delta_t^X\right)_{t\in\mathbb{Z}}.$$

A quasi-isomorphism $\phi : X \longrightarrow Y$, denoted by $\phi : X \xrightarrow{\simeq} Y$ is a morphism such that the induced map $\mathrm{H}_n(\phi) : \mathrm{H}_n(X) \longrightarrow \mathrm{H}_n(Y)$ is an isomorphism for all $n \in \mathbb{Z}$. The complexes X and Y are equivalent and denoted by $X \simeq Y$ [15, A.1.11], if they can be linked by a sequence of quasi-isomorphisms with arrows in the alternating directions. Let $X \in \mathcal{C}(R)$, and let $s, t \in \mathbb{Z}$. The hard truncation above, $X_{\leqslant s}$, of X at s, and the hard truncation below, $X_{\geqslant t}$, of X at t are given by:

$$X_{\leqslant s} : 0 \longrightarrow X_s \xrightarrow{\delta_s^X} X_{s-1} \xrightarrow{\delta_{s-1}^X} X_{s-2} \longrightarrow \cdots$$

and

$$X_{\geqslant t} : \cdots \longrightarrow X_{t+2} \xrightarrow{\delta_{t+2}^X} X_{t+1} \xrightarrow{\delta_{t+1}^X} X_t \longrightarrow 0.$$

The soft truncation above, $X_{\subset s}$, of X at s and the soft truncation below, $X_{\supset t}$, of X at t are given by

$$X_{\subset s} : 0 \longrightarrow \mathrm{C}_s(X) \xrightarrow{\delta_s^X} X_{s-1} \xrightarrow{\delta_{s-1}^X} X_{s-2} \longrightarrow \cdots$$

and

$$X_{\supset t} : \cdots \longrightarrow X_{t+2} \xrightarrow{\delta_{t+2}^X} X_{t+1} \xrightarrow{\delta_{t+1}^X} \mathrm{Z}_t(X) \longrightarrow 0.$$

We use subscripts $\sqsubset, \sqsupset, \square$ to denote boundedness conditions and $(\sqsubset), (\sqsupset), (\square)$ to denote homological boundedness conditions. For example, $\mathcal{C}_{\sqsupset}(R)$ is the full subcategory of $\mathcal{C}(R)$ of bounded below complexes, and $\mathcal{C}_{(\sqsupset)}(R)$ is the full subcategory of $\mathcal{C}(R)$ of homologically bounded below complexes. For a class $\mathcal{L}$ of R-modules, $\mathcal{C}^{\mathrm{L}}(R)$ denotes the full subcategories of $\mathcal{C}(R)$ with modules in $\mathcal{L}$.

We will use C to denote an arbitrary but fixed semidualizing R-module [7, 1.8], i.e., the follow three conditions are satisfied:

(1) C admits a degreewise finite projective resolution.

(2) the natural homothety map $\chi_C^R : R \longrightarrow \mathrm{Hom}_R(C, C)$ is an isomorphism.

(3) $\mathrm{Ext}_R^{\geq 1}(C, C)$=0.

Recall from [7,18] that the R-modules in the following classes

$$\mathcal{P}_C = \{C \otimes P | P \text{ is a projective } R\text{-module}\},$$
$$\mathcal{F}_C = \{C \otimes F | F \text{ is a flat } R\text{-module}\}$$

are called C-projective and C-flat, respectively. When $C = R$, we omit the subscript and recover the classes of projective and flat R-modules.

Let $\mathcal{L}$ be a class of R-modules. Recall that a complex X of R-modules is $\mathrm{Hom}_R(-, \mathcal{L})$-exact if the complex $\mathrm{Hom}_R(X, L)$ is exact for any $L \in \mathcal{L}$.

Definition 1.1 ([10,11]) *An R-module M is called D_C-projective if there exists a* $\mathrm{Hom}_R(-, \mathcal{F}_C)$-*exact exact complex X of R-modules with $X_i \in \mathcal{P}$ for all $i \geq 0$ and $X_i \in \mathcal{P}_C$ for all $i < 0$ such that $M \cong \mathrm{Im}\delta_0^X$.*

The class of D_C-projective R-modules denoted by $\mathcal{D}_C\mathcal{P}$. Putting $C = R$, then D_C-projective modules are just Ding projective modules [8,9], and we denote it by $\mathcal{DP}$.

2. Main results

According to [15, A.3.1], a projective resolution of a complex $X \in \mathcal{C}_{(\sqsupset)}(R)$ is a quasi-isomorphism $P \xrightarrow{\simeq} X$ where $P \in \mathcal{C}_{\sqsupset}^{\mathrm{P}}(R)$. By [15, A.3.2], every complex $X \in \mathcal{C}_{(\sqsupset)}(R)$ has a projective resolution $P \xrightarrow{\simeq} X$ with $P_l = 0$ for $l < \inf X$. Thus, for every $X \in \mathcal{C}_{(\sqsupset)}(R)$, there exists a quasi-isomorphism $D \xrightarrow{\simeq} X$ with $D \in \mathcal{C}_{\sqsupset}^{\mathrm{D_C P}}(R)$ by [10, Proposition 1.8]. Hence we have

Definition 2.1 *The D_C-projective dimension of $X \in \mathcal{C}_{(\sqsupset)}(R)$, denoted by D_C-$\mathrm{pd}_R X$, is defined* as

$$D_C\text{-pd}_R X = \inf\{\sup\{i \in \mathbb{Z} \mid D_i \neq 0\} \mid X \simeq D \in \mathcal{C}_{\sqsupset}^{\mathrm{D_CP}}(R)\}.$$

In order to characterize the D_C-projective dimension of complexes, we need the following preparations.

Lemma 2.2 *If $D \in \mathcal{C}_{\sqsupset}^{\mathrm{D_CP}}(R)$ is exact and $F \in \mathcal{C}_{\square}^{\mathrm{F}_C}(R)$, then the complex $\mathrm{Hom}_R(D,F)$ is exact.*

Proof We can assume that F is nonzero and that $\sup F^{\natural} = n$. We proceed by induction on n. Without loss of generality, we may assume that $D_i = 0$ and $F_i = 0$ for $i < 0$.

If $n = 0$, then F is a C-flat module, and so $\mathrm{Ext}_R^i(D_j, F) = 0$ for all $i > 0$ and $j \in \mathbb{Z}$ by [10, Proposition 1.4]. Since D is exact and $\mathrm{C}_i(D) = 0$ for all $i \leq 0$, it follows by [15, Lemma 4.1.1(c)] that $\mathrm{Ext}_R^1(\mathrm{C}_i(D), F) = \mathrm{Ext}_R^{i+1}(\mathrm{C}_0(D), F) = 0$ for all $i > 0$. Thus $\mathrm{Hom}_R(D,F)$ is exact again by [15, Lemma 4.1.1(c)].

Let $n > 0$ and assume that $\mathrm{Hom}_R(D, \widetilde{F})$ is exact for all $\widetilde{F} \in \mathcal{C}_{\square}^{\mathrm{F}_C}(R)$ concentrated in at most $n-1$ degrees. Consider the degreewise split exact sequence

$$0 \longrightarrow F_{\leqslant n-1} \longrightarrow F \longrightarrow \textstyle\sum^n F_n \longrightarrow 0.$$

It remains exact after the application of $\mathrm{Hom}_R(D,-)$, so $\mathrm{Hom}_R(D,F)$ is exact since $\mathrm{Hom}_R(D,F_n)$ and $\mathrm{Hom}_R(D, F_{\leqslant n-1})$ are exact by the induction base and hypothesis, respectively. □

Lemma 2.3 *If $X \simeq D \in \mathcal{C}_{\sqsupset}^{\mathrm{D_CP}}(R)$ and $U \simeq F \in \mathcal{C}_{\square}^{\mathrm{F}_C}(R)$, then $\mathbf{R}\mathrm{Hom}_R(X,U)$ can be represented by $\mathrm{Hom}_R(D,F)$.*

Proof Let $P \in \mathcal{C}_{\sqsupset}^{\mathrm{P}}(R)$ be a projective resolution of X, then $\mathbf{R}\mathrm{Hom}_R(X,U)$ is represented by $\mathrm{Hom}_R(P,F)$. Since $P \simeq X \simeq D$, there exists a quasi-isomorphism $\alpha : P \xrightarrow{\simeq} D$ by [15, A.3.6], and hence we have a morphism $\mathrm{Hom}_R(\alpha, F) : \mathrm{Hom}_R(D,F) \longrightarrow \mathrm{Hom}_R(P,F)$. Since $\mathrm{Cone}(\alpha)$ is exact by [15, A.1.19] and it belongs to $\mathcal{C}_{\sqsupset}^{\mathrm{D_CP}}(R)$ by [10, Proposition 1.8], we conclude from the isomomorphism $\mathrm{Cone}(\mathrm{Hom}_R(\alpha,F)) \cong \Sigma^1\mathrm{Hom}_R(\mathrm{Cone}(\alpha), F)$ that $\mathrm{Cone}(\mathrm{Hom}_R(\alpha,F))$ is exact by Lemma 2.2 and, hence $\mathrm{Hom}_R(\alpha,F)$ is a quasi-isomorphism by [15, A.1.19]. Thus, $\mathrm{Hom}_R(P,F) \simeq \mathrm{Hom}_R(D,F)$. This implies that $\mathbf{R}\mathrm{Hom}_R(X,U)$ is represented by $\mathrm{Hom}_R(D,F)$. □

Lemma 2.4 *Let W be a C-flat R-module and $X \in \mathcal{C}_{(\square)}(R)$. If $X \simeq D \in \mathcal{C}_{\sqsupset}^{\mathrm{D_CP}}(R)$ and $\sup X \leq n$, then $\mathrm{Ext}_R^m(\mathrm{C}_n(D), W) = \mathrm{H}_{-(m+n)}(\mathbf{R}\mathrm{Hom}_R(X,W))$ for any $m > 0$.*

Proof Since $\sup D = \sup X \leq n$, $D_{\geqslant n} \simeq \Sigma^n \mathrm{C}_n(D)$ by [15, A.1.14.3], so $\mathbf{R}\mathrm{Hom}_R(\mathrm{C}_n(D), W)$ is represented by $\mathrm{Hom}_R(\Sigma^{-n}D_{\geqslant n}, W)$ by Lemma 2.3. Thus for any $m > 0$, by [15, A.2.1.3, A.1.3.1 and A.1.20.2], we have

$$\begin{aligned}\mathrm{Ext}_R^m(\mathrm{C}_n(D), W) &= \mathrm{H}_{-m}(\mathbf{R}\mathrm{Hom}_R(\mathrm{C}_n(D), W)) = \mathrm{H}_{-m}(\mathrm{Hom}_R(\Sigma^{-n}D_{\geqslant n}, W)) \\ &= \mathrm{H}_{-m}(\Sigma^n\mathrm{Hom}_R(D_{\geqslant n}, W)) = \mathrm{H}_{-(m+n)}(\mathrm{Hom}_R(D_{\geqslant n}, W)) \\ &= \mathrm{H}_{-(m+n)}(\mathrm{Hom}_R(D,W)_{\leqslant -n}) = \mathrm{H}_{-(m+n)}(\mathrm{Hom}_R(D,W)).\end{aligned}$$

By Lemma 2.3, $\mathbf{R}\mathrm{Hom}_R(X,W)$ is represented by $\mathrm{Hom}_R(D,W)$, so

$$\mathrm{Ext}_R^m(\mathrm{C}_n(D),W)=\mathrm{H}_{-(m+n)}(\mathbf{R}\mathrm{Hom}_R(X,W))$$

as desired. □

Lemma 2.5 *If* $D\in\mathcal{C}_{\sqsupset}^{\mathrm{D_CP}}(R)$, $U,V\in\mathcal{C}_{\sqsubset}^{\mathrm{F_C}}(R)$ *and* $U\xrightarrow{\simeq}V$, *then* $\mathrm{Hom}_R(D,U)\xrightarrow{\simeq}\mathrm{Hom}_R(D,V)$.

Proof Let A be a D_C-projective module and $\cdots\longrightarrow P_1\longrightarrow P_0\longrightarrow A\longrightarrow 0$ a projective resolution of A. Then for any C-flat module W, the sequence

$$0\longrightarrow\mathrm{Hom}_R(A,W)\longrightarrow\mathrm{Hom}_R(P_0,W)\longrightarrow\mathrm{Hom}_R(P_1,W)\longrightarrow\cdots$$

is exact. Thus we have a quasi-isomorphism $\mathrm{Hom}_R(A,W)\xrightarrow{\simeq}\mathrm{Hom}_R(P,W)$, where P is the complex $\cdots\longrightarrow P_1\longrightarrow P_0\longrightarrow 0$. Now [16, 2.7(a)] yields quasi-isomorphisms

$$\mathrm{Hom}_R(A,U)\xrightarrow{\simeq}\mathrm{Hom}_R(P,U)\ \text{ and }\ \mathrm{Hom}_R(A,V)\xrightarrow{\simeq}\mathrm{Hom}_R(P,V).$$

From the following commutative diagram

$$\begin{array}{ccc}\mathrm{Hom}_R(A,U) & \longrightarrow & \mathrm{Hom}_R(P,U)\\ \downarrow & & \downarrow\\ \mathrm{Hom}_R(A,V) & \longrightarrow & \mathrm{Hom}_R(P,V)\end{array}$$

and the fact that $\mathrm{Hom}_R(P,-)$ preserves quasi-isomorphism it follows that

$$\mathrm{Hom}_R(A,U)\xrightarrow{\simeq}\mathrm{Hom}_R(A,V),$$

and so $\mathrm{Hom}_R(D,U)\xrightarrow{\simeq}\mathrm{Hom}_R(D,V)$ by [16, 2.6(a)]. □

Now, we can achieve some characterizations of the D_C-projective dimension of complexes.

Theorem 2.6 *Let* $X\in\mathcal{C}_{(\sqsupset)}(R)$ *be a complex of finite* D_C*-projective dimension and* $n\in\mathbb{Z}$, *then the following conditions are equivalent:*

(1) X *is equivalent to a bounded complex* D *of* D_C*-projective* R*-modules with* $\sup D^{\natural}\leq n$, *and* D *can be chosen such that* $D_i=0$ *for* $i<\inf X$.

(2) $\mathrm{D}_C\text{-pd}_R X\leq n$.

(3) $\inf U-\inf\mathbf{R}\mathrm{Hom}_R(X,U)\leq n$ *for all* $0\not\simeq U\in\mathcal{C}_{\square}^{\mathrm{F_C}}(R)$.

(4) $-\inf\mathbf{R}\mathrm{Hom}_R(X,W)\leq n$ *for all* C*-flat* R*-modules* W.

(5) $\sup X\leq n$ *and the module* $\mathrm{C}_n(D)$ *is* D_C*-projective whenever* $D\in\mathcal{C}_{\sqsupset}^{\mathrm{D_CP}}(R)$ *is equivalent to* X.

Proof (1)⇒(2) and (3)⇒(4) are clear.

(2)⇒(3) Since $\mathrm{D}_C\text{-pd}_R X\leq n$, there exists a complex $D\in\mathcal{C}_{\sqsupset}^{\mathrm{D_CP}}(R)$ such that $X\simeq D$ and $D_k=0$ for $k>n$. Let $0\not\simeq U\in\mathcal{C}_{\square}^{\mathrm{F_C}}(R)$ and $\inf U=i$. Then by Lemma 2.3, $\mathbf{R}\mathrm{Hom}_R(X,U)$ can be represented by $\mathrm{Hom}_R(D,U)$. Set $\inf U^{\natural}=l$. Then $U_{\supset i}\xrightarrow{\simeq}U$ by [15, A.1.14.4] and there is an exact sequence of R-modules

$$0 \longrightarrow \mathrm{Z}_i(U) \longrightarrow U_i \longrightarrow \cdots \longrightarrow U_l \longrightarrow 0.$$

Since $\mathcal{F}_C$ is projective resolving [18, Corollary 6.4 and Proposition 3.1], $\mathrm{Z}_i(U) \in \mathcal{F}_C$. Thus $U_{\supset i} \in \mathcal{C}_{\sqsubset}^{\mathrm{F}_C}(R)$. Hence $\mathrm{Hom}_R(D, U_{\supset i}) \xrightarrow{\simeq} \mathrm{Hom}_R(D, U)$ by Lemma 2.5. So $\mathrm{Hom}_R(D, U_{\supset i})$ also represents $\mathbf{R}\mathrm{Hom}_R(X, U)$. In particular, $\inf \mathbf{R}\mathrm{Hom}_R(X, U) = \inf \mathrm{Hom}_R(D, U_{\supset i})$. For $l < i - n$ and $p \in \mathbb{Z}$, either $p > n$ or $p+l \le n+l < i$, so $\mathrm{Hom}_R(D, U_{\supset i})_l = \prod_{p\in\mathbb{Z}} \mathrm{Hom}_R(D_p, (U_{\supset i})_{p+l}) = 0$. Thus $\mathrm{H}_l(\mathrm{Hom}_R(D, U)) = 0$ for all $l < i - n$, and so $\inf \mathbf{R}\mathrm{Hom}_R(X, U) \ge i - n = \inf U - n$ as desired.

(4)⇒(5) We first prove that $\sup X \le n$. By the hypothesis, we assume that $\mathrm{D}_C\text{-pd}_R X = m < \infty$. Then there exists a $D \in \mathcal{C}_{\sqsupset}^{\mathrm{D_CP}}(R)$ such that $X \simeq D$ and $D_i = 0$ for all $i > m$. Set $s = \sup X$. Then $s \le m$. If $s = m$, then the differential $\delta_m^D : D_m \longrightarrow D_{m-1}$ is not injective since $\sup D = \sup X = m$. Since D_m is a D_C-projective module, there exists a C-projective module W and an injective homomorphism $\varphi : D_m \longrightarrow W$. Because δ_m^D is not injective, the differential $\mathrm{Hom}_R(\delta_m^D, W)$ is not surjective, otherwise $\varphi = \psi\delta_m^D$ for some $\psi \in \mathrm{Hom}_R(D_{m-1}, W)$, and so δ_m^D is injective, a contradiction. Thus $-\inf \mathrm{Hom}_R(D, W) = m = \sup X$. Hence $\sup X = -\inf \mathbf{R}\mathrm{Hom}_R(X, W) \le n$ by Lemma 2.3 and (4). Now assume that $s < m$. If $s > n$, then by Lemma 2.4 and (4), we have

$$\mathrm{Ext}_R^i(\mathrm{C}_s(D), W) = \mathrm{H}_{-(i+s)}(\mathbf{R}\mathrm{Hom}_R(X, W)) = 0$$

for any $i > 0$ and any C-flat module W. So by the exact sequence

$$0 \longrightarrow D_m \longrightarrow D_{m-1} \longrightarrow \cdots \longrightarrow D_{s+1} \longrightarrow D_s \longrightarrow \mathrm{C}_s(D) \longrightarrow 0$$

and [10, Corollary 1.15] we deduced that $\mathrm{C}_s(D)$ is D_C-projective. Hence $D_{\subset s} \in \mathcal{C}_{\square}^{\mathrm{D_CP}}(R)$. By [15, A.1.14.2], $D \simeq D_{\subset s}$. Thus $X \simeq D_{\subset s}$, and so $\mathrm{D}_C\text{-pd}_R X \le s < m$, a contradiction. Therefore $\sup X = s \le n$.

Next we show that $\mathrm{C}_n(D)$ is D_C-projective whenever $D \in \mathcal{C}_{\sqsupset}^{\mathrm{D_CP}}(R)$ is equivalent to X. By the hypothesis, $\mathrm{D}_C\text{-pd}_R X < \infty$, so there exists an $A \in \mathcal{C}_{\square}^{\mathrm{D_CP}}(R)$ such that $X \simeq A$. Assume that $\sup A^\natural = t$. Then there is an exact sequence

$$0 \longrightarrow A_t \longrightarrow \cdots \longrightarrow A_{n+1} \longrightarrow A_n \longrightarrow \mathrm{C}_n(A) \longrightarrow 0$$

since $\sup A = \sup X \le n$. By Lemma 2.4 and (4), $\mathrm{Ext}_R^i(\mathrm{C}_n(A), W) = \mathrm{H}_{-(i+n)}(\mathbf{R}\mathrm{Hom}_R(X, W)) = 0$ for any C-flat module W and any $i > 0$. Thus $\mathrm{C}_n(A)$ is D_C-projective by [10, Corollary 1.15]. To prove the assertion it is now sufficient to see that: if $P \in \mathcal{C}_{\sqsupset}^{\mathrm{P}}(R)$, $D \in \mathcal{C}_{\sqsupset}^{\mathrm{D_CP}}(R)$, and $P \simeq X \simeq D$, then the cokernel $\mathrm{C}_n(P)$ is D_C-projective if and only if $\mathrm{C}_n(D)$ is so.

Let D and P be two such complexes. Then there is a quasi-isomorphism $\pi : P \xrightarrow{\simeq} D$ by [15, A.3.6], which induces a quasi-isomorphism $\pi_{\subset n} : P_{\subset n} \xrightarrow{\simeq} D_{\subset n}$. The mapping cone

$$\mathrm{Cone}(\pi_{\subset n}) : 0 \longrightarrow \mathrm{C}_n(P) \longrightarrow P_{n-1} \oplus \mathrm{C}_n(D) \longrightarrow P_{n-2} \oplus D_{n-1} \longrightarrow \cdots$$

is a bounded exact complex, in which all modules but the two left-most ones are D_C-projective modules by [10, Propositions 1.8 and 1.11]. It follows by [10, Proposition 1.12] that $\mathrm{C}_n(P)$ is

D_C-projective if and only if $P_{n-1} \oplus \mathrm{C}_n(D)$ is D_C-projective if and only if $\mathrm{C}_n(D)$ is so.

(5) $\Rightarrow$ (1) Let $P \in \mathcal{C}_{\sqsupset}^{\mathrm{P}}(R)$ be a projective resolution of X with $P_l < 0$ for $l < \inf X$ ([15, A.3.2]). Then $P \in \mathcal{C}_{\sqsupset}^{\mathrm{D_CP}}(R)$ by [10, Propositon 1.8] and $\sup P = \sup X \leq n$. Thus $X \simeq P_{\subset n} \in \mathcal{C}_{\square}^{\mathrm{D_CP}}(R)$ as $\mathrm{C}_n(P)$ is D_C-projective. □

If we choose $C = R$ in Definition 2.1, then we have a notion of Ding projective dimension for $X \in \mathcal{C}_{(\sqsupset)}(R)$, and we denote it by $\mathrm{Dpd}_R X$. By Theorem 2.6, we get

Corollary 2.7 *Let $X \in \mathcal{C}_{(\sqsupset)}(R)$ be a complex of finite Ding projective dimension and $n \in \mathbb{Z}$, then the following conditions are equivalent:*

(1) X is equivalent to a bounded complex D of Ding projective R-modules with $\sup D^{\natural} \leq n$, and D can be chosen such that $D_i = 0$ for $i < \inf X$.

(2) $\mathrm{Dpd}_R X \leq n$.

(3) $\inf U - \inf \mathbf{R}\mathrm{Hom}_R(X, U) \leq n$ for all $0 \not\simeq U \in \mathcal{C}_{\square}^{\mathrm{F}}(R)$.

(4) $-\inf \mathbf{R}\mathrm{Hom}_R(X, W) \leq n$ for all flat R-modules W.

(5) $\sup X \leq n$ and the module $\mathrm{C}_n(D)$ is Ding projective whenever $D \in \mathcal{C}_{\sqsupset}^{\mathrm{DP}}(R)$ is equivalent to X.

Lemma 2.8 *Let M be an R-module. If $M \simeq D \in \mathcal{C}_{\sqsupset}^{\mathrm{D_CP}}(R)$, then*

$$D_{\supset 0}: \cdots \longrightarrow D_2 \longrightarrow D_1 \longrightarrow \mathrm{Z}_0(D) \longrightarrow 0$$

is a D_C-projective resolution of M.

Proof Suppose that $M \simeq D \in \mathcal{C}_{\sqsupset}^{\mathrm{D_CP}}(R)$, then $M \simeq D_{\supset 0}$ by [15, A.1.14.4] since $\inf D = \inf M = 0$, and so we have an exact sequence of R-modules

$$\cdots \longrightarrow D_2 \longrightarrow D_1 \longrightarrow \mathrm{Z}_0(D) \longrightarrow M \longrightarrow 0.$$

Set $\inf D^{\natural} = i$, and consider the exact sequence

$$0 \longrightarrow \mathrm{Z}_0(D) \longrightarrow D_0 \longrightarrow \cdots \longrightarrow D_{i+1} \longrightarrow D_i \longrightarrow 0.$$

The modules $D_0, \ldots, D_i$ are all D_C-projective, and so is $\mathrm{Z}_0(D)$ by the projective resolving properties of D_C-projective modules [10, Theorem 1.12]. Hence $D_{\supset 0}$ is a D_C-projective resolution of M. □

Corollary 2.9 ([11, Proposition 2.11; 10, Theorem 2.4]) *Let M be an R-module with finite D_C-projective dimension and $n \in \mathbb{Z}$, then the following conditions are equivalent:*

(1) $\mathrm{D}_C\text{-pd}_R M \leq n$.

(2) $\mathrm{Ext}_R^i(M, N) = 0$ for all $i > n$ and all R-module N with finite C-flat dimension.

(3) $\mathrm{Ext}_R^i(M, N) = 0$ for all $i > n$ and all C-flat R-modules N.

(4) For any D_C-projective resolution

$$\cdots \longrightarrow D_2 \longrightarrow D_1 \longrightarrow D_0 \longrightarrow M \longrightarrow 0$$

of M, the Kernel $K_n = \ker(\ D_{n-1} \longrightarrow D_{n-2}\ )$ is a D_C-projective module.

Proof It follows from Theorem 2.6 and Lemma 2.8. □

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