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The Riccati Equation Method Combined with the Generalized Extended (G'/G)-Expansion Method for Solving the Nonlinear KPP Equation

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Abstract An analytic study of the nonlinear Kolmogorov-Petrovskii-Piskunov (KPP) equation is presented in this paper. The Riccati equation method combined with the generalized extended (G'/G)-expansion method is an interesting approach to find more general exact solutions of the nonlinear evolution equations in mathematical physics. We obtain the traveling wave solutions involving parameters, which are expressed by the hyperbolic and trigonometric function solutions. When the parameters are taken as special values, the solitary and periodic wave solutions are given. Comparison of our new results in this paper with the well-known results are given.

Keywords the generalized extended (G'/G)-expansion method; the Riccati equation method; the nonlinear KPP equation; exact solutions; solitary wave solutions; periodic wave solutions

MR(2010) Subject Classification 35K99; 35P05; 35P99; 35C05

1. Introduction

In the recent years, investigations of exact solutions to nonlinear evolution equations play an important role in the study of nonlinear physical phenomena in such as fluid mechanics, optics, hydrodynamics, plasma physics, biology, solid state physics and so on. Several methods for finding the exact solutions to nonlinear equations in mathematical physics have been presented, such as the homogeneous balance method [1,2], the tanh-function method [3,4], the (G'/G)-expansion method [5–9], the Exp-function method [10–12], the multiple exp-function method [13,14], the symmetry method [15,16], the modified simple equation method [17–19], the improved (G'/G)-expansion method [20], the Jacobi elliptic function expansion method [21], the Bäcklund transform method [22,23], the generalized Riccati equation method [24,25], the auxiliary equation method [26,27], the first integral method [28,29], the extended (G'/G)-expansion method [30], the generalized Kudryashov method [31–38], the Riccati equation method combined with the (G'/G)-expansion method [39,40], the modified extended Fan sub equation method [41], the transformed rational function method [42], the invariant subspaces method [43], the refined invariant subspace method [44] and so on.

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The objective of this paper is to apply the Riccati equation method combined with the generalized extended (G'/G)-expansion method to find new exact solutions of the following nonlinear KPP equation:

$$u_t - u_{xx} + \mu u + \gamma u^2 + \delta u^3 = 0, \tag{1.1}$$

where μ, γ, δ are real constants. Eq. (1.1) is important in the physical fields, and includes the Fisher equation, the Huxley equation, the Burgers-Huxley equation, the Chaffee-Infanfe equation and the Fitzhugh-Nagumo equation. Eq. (1.1) has been investigated in [45] using the (G'/G)-expansion method, in [46] using the modified simple equation method, and in [25] using two ansätze, via the Cole-Hopf transformation and the Bäcklund transformation methods.

2. Description of the Riccati equation method combined with the generalized extended (G'/G)-expansion method

Suppose that we have the following nonlinear evolution equation:

$$F(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0, \tag{2.1}$$

where F is a polynomial in $u(x,t)$ and its partial derivatives, in which the highest order derivatives and the nonlinear term, are involved. In the following, we give the main steps of the Riccati equation method combined with the generalized extended (G'/G)-expansion method:

Step 1. We use the traveling wave transformation:

$$u(x,t) = U(\xi), \quad \xi = kx + wt, \tag{2.2}$$

where k, w are constants, to reduce Eq. (1.1) to the following ordinary differential equation (ODE):

$$P(U, U', U'', \ldots) = 0, \tag{2.3}$$

where P is a polynomial in $U(\xi)$ and its total derivatives, while the dashes denote the derivatives with respect to ξ.

Step 2. We assume that Eq. (2.3) has the formal solution:

$$U(\xi) = \sum_{i=-n}^{n} \alpha_i [f(\xi)]^i, \tag{2.4}$$

where α_i $(i = -n, \ldots, n)$ are constants to be determined later $\alpha_n \neq 0$ or $\alpha_{-n} \neq 0$, while $f(\xi)$ satisfies the generalized Riccati equation:

$$f'(\xi) = p + rf(\xi) + qf^2(\xi), \tag{2.5}$$

where p, r and q are real constants, such that $q \neq 0$.

Step 3. The positive integer n in Eq. (2.4) can be determined by balancing the highest-order derivatives with the nonlinear terms appearing in Eq. (2.3).

Step 4. We determine the solution $f(\xi)$ of Eq. (2.5) using the generalized extended (G'/G)-

expansion method, by assuming that its formal solution has the form:

$$f(\xi)=a_0+\sum_{i=1}^{m}\Big\{a_i(\frac{G'}{G})^i+b_i(\frac{G'}{G})^{i-1}\sqrt{\sigma(1+\frac{1}{\mu}(\frac{G'}{G})^2)}+c_i(\frac{G'}{G})^{-i}+d_i\frac{(\frac{G'}{G})^{1-i}}{\sqrt{\sigma(1+\frac{1}{\mu}(\frac{G'}{G})^2)}}\Big\} \quad (2.6)$$

where $G=G(\xi)$ satisfies the following second-order linear ODE:

$$G''(\xi)+\mu G(\xi)=0, \quad (2.7)$$

while a_i, b_i, c_i, d_i $(i=1,\dots,m)$ and a_0 are constants to be determined later, such that $\sigma=\pm 1$.

Step 5. The positive integer m in Eq. (2.6) can be determined by balancing $f'(\xi)$ and $f^2(\xi)$ in Eq. (2.5) to get $m=1$. Thus, the solution (2.6) reduces to

$$f(\xi)=a_0+a_1(\frac{G'}{G})+b_1\sqrt{\sigma(1+\frac{1}{\mu}(\frac{G'}{G})^2)}+c_1(\frac{G'}{G})^{-1}+\frac{d_1}{\sqrt{\sigma(1+\frac{1}{\mu}(\frac{G'}{G})^2)}} \quad (2.8)$$

where a_0, a_1, b_1, c_1 and d_1 are constants to be determined. Substituting Eq. (2.8) along with Eq. (2.7) into Eq. (2.5), collecting all terms with the same powers of $(\frac{G'}{G})^j$, $(\frac{G'}{G})^j\sqrt{\sigma(1+\frac{1}{\mu}(\frac{G'}{G})^2)}$, and setting them to zero, yields a set of algebraic equations, which can be solved to get the following three cases:

Case 1 $\mu=-\frac{1}{4}\Delta$, $a_0=\frac{-r}{2q}$, $a_1=\frac{-1}{q}$, $b_1=c_1=d_1=0$, $q\neq 0$, where $\Delta=r^2-4pq$. In this case, the solution of Eq. (2.5) has the form

$$f(\xi)=\frac{-1}{2q}[r+2(\frac{G'}{G})]. \quad (2.9)$$

Case 2 $\mu=-\Delta$, $a_0=\frac{-r}{2q}$, $a_1=\frac{-1}{2q}$, $b_1=\pm\frac{1}{2q}\sqrt{\frac{-\Delta}{\sigma}}$, $c_1=d_1=0$, $q\neq 0$, where $\Delta=r^2-4pq$. In this case:

(i) If $\sigma=1$, then $\Delta<0$, the solution of Eq. (2.5) has the form

$$f(\xi)=\frac{-1}{2q}\big[r+(\frac{G'}{G})\mp\sqrt{-\Delta}\sqrt{1-\frac{1}{\Delta}(\frac{G'}{G})^2}\big]. \quad (2.10)$$

(ii) If $\sigma=-1$, then $\Delta>0$, the solution of Eq. (2.5) has the form

$$f(\xi)=\frac{-1}{2q}\big[r+(\frac{G'}{G})\mp\sqrt{\Delta}\sqrt{\frac{1}{\Delta}(\frac{G'}{G})^2-1}\big]. \quad (2.11)$$

Case 3 $\mu=-\frac{1}{16}\Delta$, $a_0=\frac{-r}{2q}$, $a_1=\frac{-1}{q}$, $c_1=-\frac{1}{16q}\Delta$, $b_1=d_1=0$, $q\neq 0$, where $\Delta=r^2-4pq$. In this case, the solution of Eq. (2.5) has the form

$$f(\xi)=\frac{-1}{2q}\big[r+2(\frac{G'}{G})+\frac{1}{8}\Delta(\frac{G'}{G})^{-1}\big]. \quad (2.12)$$

On solving Eq. (2.7) we deduce that (G'/G) has the form

$$\frac{G'(\xi)}{G(\xi)}=\begin{cases}\sqrt{-\mu}\big[\frac{A\sinh(\xi\sqrt{-\mu})+B\cosh(\xi\sqrt{-\mu})}{A\cosh(\xi\sqrt{-\mu})+B\sinh(\xi\sqrt{-\mu})}\big], & \text{if } \mu<0, \quad (2.13)\\ \sqrt{\mu}\big[\frac{A\cos(\xi\sqrt{\mu})-B\sin(\xi\sqrt{\mu})}{A\sin(\xi\sqrt{\mu})+B\cos(\xi\sqrt{\mu})}\big], & \text{if } \mu>0, \quad (2.14)\end{cases}$$

where A and B are arbitrary constants.

Step 6. Substituting Eq. (2.4) along with Eq. (2.5) into Eq. (2.3) and equating the coefficient of the same powers of $f(\xi)$ to zero, we obtain a system of algebraic equations, which can be solved using the Maple or Mathematica to get the values of α_i, k and w.

Step 7. Substituting the values of α_i, k and w as well as the solution $f(\xi)$ given by Eq. (2.9) or Eq. (2.10) or Eq. (2.11) into Eq. (2.4), we finally obtain the exact solutions of Eq. (2.1) for the all Cases 1–3.

On the other hand, it is well-known [25] that the generalized Riccati equation (2.5) has the solutions:

(i) If $\Delta = r^2 - 4pq > 0$, then

$$f(\xi) = \begin{cases} \frac{-\sqrt{\Delta}}{2q}\tanh(\frac{1}{2}\sqrt{\Delta}\xi - \frac{\epsilon \ln \xi_0}{2}) - \frac{r}{2q}, & \text{if } \xi_0 > 0, \quad (2.15) \\ \frac{-\sqrt{\Delta}}{2q}\coth(\frac{1}{2}\sqrt{\Delta}\xi - \frac{\epsilon \ln(-\xi_0)}{2}) - \frac{r}{2q}, & \text{if } \xi_0 < 0. \quad (2.16) \end{cases}$$

(ii) If $\Delta = r^2 - 4pq < 0$, then

$$f(\xi) = \begin{cases} \frac{\sqrt{-\Delta}}{2q}\tan(\frac{1}{2}\sqrt{-\Delta}\xi + \xi_0) - \frac{r}{2q}, & (2.17) \\ \frac{-\sqrt{-\Delta}}{2q}\cot(\frac{1}{2}\sqrt{-\Delta}\xi + \xi_0) - \frac{r}{2q}, & (2.18) \end{cases}$$

where ξ_0 is a constant and $\epsilon = \pm 1$.

3. An application

Here we apply the method described in Section 2 to construct new exact solutions of the nonlinear KPP equation (1.1). To the aim, we use the wave transformation (2.2) to reduce Eq. (1.1) to the following ODE:

$$wU' - k^2U'' + \mu U + \gamma U^2 + \delta U^3 = 0. \tag{3.1}$$

By balancing U'' with U^3, we have $n = 1$. Consequently, we have the formal solution:

$$U(\xi) = \alpha_0 + \alpha_1 f(\xi) + \alpha_{-1}[f(\xi)]^{-1}, \tag{3.2}$$

where α_0, α_1 and α_{-1} are parameters to be determined later, such that $\alpha_1 \neq 0$ or $\alpha_{-1} \neq 0$.

Substituting Eq. (3.2) along with Eq. (2.5) into Eq. (3.1) and equating the coefficients of all powers of $f(\xi)$ to zero, we get the following system of algebraic equations:

$f^3 : -2k^2q^2\alpha_1 + \delta\alpha_1^3 = 0,$

$f^2 : 3\delta\alpha_0\alpha_1^2 + wq\alpha_1 + \gamma\alpha_1^2 - 3k^2rq\alpha_1 = 0,$

$f^1 : wr\alpha_1 - k^2r^2\alpha_1 + \mu\alpha_1 + 3\delta\alpha_0^2\alpha_1 + 2\gamma\alpha_0\alpha_1 - 2k^2pq\alpha_1 + 3\delta\alpha_1^2\alpha_{-1} = 0,$

$f^0 : 2\gamma\alpha_1\alpha_{-1} - wq\alpha_{-1} + \mu\alpha_0 - k^2qr\alpha_{-1} + \gamma\alpha_0^2 - k^2pr\alpha_1 + 6\delta\alpha_0\alpha_1\alpha_{-1} + \delta\alpha_0^3 + wp\alpha_1 = 0,$

$f^{-1} : -wr\alpha_{-1} - k^2r^2\alpha_{-1} + \mu\alpha_{-1} + 3\delta\alpha_0^2\alpha_{-1} + 2\gamma\alpha_0\alpha_{-1} - 2k^2pq\alpha_{-1} + 3\delta\alpha_{-1}^2\alpha_1 = 0,$

$f^{-2} : 3\delta\alpha_0\alpha_{-1}^2 - wp\alpha_{-1} + \gamma\alpha_{-1}^2 - 3k^2rp\alpha_{-1} = 0,$

$f^{-3} : -2k^2p^2\alpha_{-1} + \delta\alpha_{-1}^3 = 0.$

By solving the above algebraic equations with the aid of Maple, we have the following results:

Result 1

$$w = \pm\frac{\mu - k^2\Delta}{\sqrt{\Delta}},\ \gamma = -\frac{(\sqrt{\Delta}\mp r)(2k^2\Delta+\mu)}{2\alpha_0\sqrt{\Delta}},\ \delta = \frac{k^2(\sqrt{\Delta}\mp r)^2}{2\alpha_0^2},$$
$$\alpha_{-1} = \alpha_0(\frac{r\pm\sqrt{\Delta}}{2q}),\ \alpha_1 = 0,\ \alpha_0 = \alpha_0,$$

where $\Delta = r^2 - 4pq > 0$.

Now, the solution for the result 1 becomes:

$$U(\xi) = \alpha_0 + \alpha_0(\frac{r\pm\sqrt{\Delta}}{2q})[f(\xi)]^{-1}. \tag{3.3}$$

For Case 1, we substitute (2.9) into (3.3) and using (2.13), we have the exact solution:

$$u(x,t) = \alpha_0 - \alpha_0(r\pm\sqrt{\Delta})\{r+\sqrt{\Delta}[\frac{A\sinh(\frac{1}{2}\sqrt{\Delta}\xi)+B\cosh(\frac{1}{2}\sqrt{\Delta}\xi)}{A\cosh(\frac{1}{2}\sqrt{\Delta}\xi)+B\sinh(\frac{1}{2}\sqrt{\Delta}\xi)}]\}^{-1}. \tag{3.4}$$

It is well-known [47] that if $\Delta = r^2 - 4pq > 0$, then

$$\frac{A\sinh(\frac{1}{2}\sqrt{\Delta}\xi)+B\cosh(\frac{1}{2}\sqrt{\Delta}\xi)}{A\cosh(\frac{1}{2}\sqrt{\Delta}\xi)+B\sinh(\frac{1}{2}\sqrt{\Delta}\xi)} = \begin{cases} \tanh(\frac{1}{2}\sqrt{\Delta}\xi + \operatorname{sgn}(AB)\psi_1), & \text{if } |A|>|B|\neq 0,\\ \coth(\frac{1}{2}\sqrt{\Delta}\xi + \operatorname{sgn}(AB)\psi_2), & \text{if } |B|>|A|\neq 0,\\ \tanh(\frac{1}{2}\sqrt{\Delta}\xi), & \text{if } |A|>|B|=0,\\ \coth(\frac{1}{2}\sqrt{\Delta}\xi), & \text{if } |B|>|A|=0,\end{cases} \tag{3.5}$$

where $\psi_1 = \tanh^{-1}(|B|/|A|)$, $\psi_2 = \coth^{-1}(|B|/|A|)$ and $\operatorname{sgn}(AB)$ is the well-known sign function.

Substituting the formulas (3.5) into (3.4), we respectively have the following solitary wave solutions of Eq. (1.1):

$$u(x,t) = \alpha_0 - \alpha_0(r\pm\sqrt{\Delta})\big[r+\sqrt{\Delta}\tanh(\frac{1}{2}\sqrt{\Delta}\xi+\operatorname{sgn}(AB)\psi_1)\big]^{-1}, \tag{3.6}$$

$$u(x,t) = \alpha_0 - \alpha_0(r\pm\sqrt{\Delta})\big[r+\sqrt{\Delta}\coth(\frac{1}{2}\sqrt{\Delta}\xi+\operatorname{sgn}(AB)\psi_2)\big]^{-1}, \tag{3.7}$$

$$u(x,t) = \alpha_0 - \alpha_0(r\pm\sqrt{\Delta})\big[r+\sqrt{\Delta}\tanh(\frac{1}{2}\sqrt{\Delta}\xi)\big]^{-1}, \tag{3.8}$$

$$u(x,t) = \alpha_0 - \alpha_0(r\pm\sqrt{\Delta})\big[r+\sqrt{\Delta}\coth(\frac{1}{2}\sqrt{\Delta}\xi)\big]^{-1}. \tag{3.9}$$

Comparing the general solution (3.4) and the particular solutions (3.6)–(3.9), with the results (19)–(22) obtained in [40], we deduce that they are exactly the same.

Substituting (2.15) and (2.16) into (3.3), we respectively have the solutions:

$$u(x,t) = \alpha_0 - \alpha_0(r\pm\sqrt{\Delta})\big[r+\sqrt{\Delta}\tanh(\frac{1}{2}\sqrt{\Delta}\xi-\frac{\epsilon\ln\xi_0}{2})\big]^{-1}, \tag{3.10}$$

and

$$u(x,t) = \alpha_0 - \alpha_0(r\pm\sqrt{\Delta})\big[r+\sqrt{\Delta}\coth(\frac{1}{2}\sqrt{\Delta}\xi-\frac{\epsilon\ln(-\xi_0)}{2})\big]^{-1}, \tag{3.11}$$

which are in agreement with the results (3.6) and (3.7), where $-\frac{\epsilon\ln\xi_0}{2} = \operatorname{sgn}(AB)\psi_1$ and $-\frac{\epsilon\ln(-\xi_0)}{2} = \operatorname{sgn}(AB)\psi_2$.

In particular, if we set $\xi_0 = 1$ in (3.10), we get the solution (3.8), while if we set $\xi_0 = -1$ in (3.11), we get the solution (3.9).

For Case 2, we substitute (2.11) into (3.3) and using (2.13), we have the exact solution:

$$u(x,t) = \alpha_0 - \alpha_0(r \pm \sqrt{\Delta})\Big\{r + \sqrt{\Delta}[\frac{A\sinh(\sqrt{\Delta}\xi) + B\cosh(\sqrt{\Delta}\xi)}{A\cosh(\sqrt{\Delta}\xi) + B\sinh(\sqrt{\Delta}\xi)}] \mp \sqrt{\Delta}(\sqrt{[\frac{A\sinh(\sqrt{\Delta}\xi) + B\cosh(\sqrt{\Delta}\xi)}{A\cosh(\sqrt{\Delta}\xi) + B\sinh(\sqrt{\Delta}\xi)}]^2 - 1})\Big\}^{-1}. \quad (3.12)$$

Substituting (3.5) into (3.12), we respectively have the following solitary wave solutions of Eq. (1.1):

$$u(x,t) = \alpha_0 - \alpha_0(r \pm \sqrt{\Delta})\big[r + \sqrt{\Delta}\tanh(\sqrt{\Delta}\xi + \operatorname{sgn}(AB)\psi_1) \pm i\sqrt{\Delta}\operatorname{sech}(\sqrt{\Delta}\xi + \operatorname{sgn}(AB)\psi_1)\big]^{-1}, \quad i = \sqrt{-1}, \quad (3.13)$$

$$u(x,t) = \alpha_0 - \alpha_0(r \pm \sqrt{\Delta})\big[r + \sqrt{\Delta}\coth(\sqrt{\Delta}\xi + \operatorname{sgn}(AB)\psi_2) \mp \sqrt{\Delta}\operatorname{csch}(\sqrt{\Delta}\xi + \operatorname{sgn}(AB)\psi_2)\big]^{-1}, \quad (3.14)$$

$$u(x,t) = \alpha_0 - \alpha_0(r \pm \sqrt{\Delta})\big\{r + \sqrt{\Delta}[\tanh(\sqrt{\Delta}\xi) \mp i\operatorname{sech}(\sqrt{\Delta}\xi)]\big\}^{-1}. \quad (3.15)$$

$$u(x,t) = \alpha_0 - \alpha_0(r \pm \sqrt{\Delta})\big\{r + \sqrt{\Delta}[\coth(\sqrt{\Delta}\xi) \mp \operatorname{csch}(\sqrt{\Delta}\xi)]\big\}^{-1}. \quad (3.16)$$

For Case 3, we substitute (2.12) into (3.3) and using (2.13), we have the hyperbolic wave solution of Eq. (1.1) as follows:

$$u(x,t) = \alpha_0 - \alpha_0(r \pm \sqrt{\Delta})\Big\{r + \frac{1}{2}\sqrt{\Delta}[\frac{A\sinh(\frac{1}{4}\sqrt{\Delta}\xi) + B\cosh(\frac{1}{4}\sqrt{\Delta}\xi)}{A\cosh(\frac{1}{4}\sqrt{\Delta}\xi) + B\sinh(\frac{1}{4}\sqrt{\Delta}\xi)}] + \frac{1}{2}\sqrt{\Delta}[\frac{A\sinh(\frac{1}{4}\sqrt{\Delta}\xi) + B\cosh(\frac{1}{4}\sqrt{\Delta}\xi)}{A\cosh(\frac{1}{4}\sqrt{\Delta}\xi) + B\sinh(\frac{1}{4}\sqrt{\Delta}\xi)}]^{-1}\Big\}^{-1}. \quad (3.17)$$

Substituting (3.5) into (3.17), we respectively have the following solitary wave solutions of Eq. (1.1):

$$u(x,t) = \alpha_0 - \alpha_0(r \pm \sqrt{\Delta})\big[r + \frac{1}{2}\sqrt{\Delta}\tanh(\frac{1}{4}\sqrt{\Delta}\xi + \operatorname{sgn}(AB)\psi_1) + \frac{1}{2}\sqrt{\Delta}\coth(\frac{1}{4}\sqrt{\Delta}\xi + \operatorname{sgn}(AB)\psi_1)\big]^{-1}, \quad (3.18)$$

$$u(x,t) = \alpha_0 - \alpha_0(r \pm \sqrt{\Delta})\big[r + \frac{1}{2}\sqrt{\Delta}\coth(\frac{1}{4}\sqrt{\Delta}\xi + \operatorname{sgn}(AB)\psi_2) + \frac{1}{2}\sqrt{\Delta}\tanh(\frac{1}{4}\sqrt{\Delta}\xi + \operatorname{sgn}(AB)\psi_2)\big]^{-1}, \quad (3.19)$$

$$u(x,t) = \alpha_0 - \alpha_0(r \pm \sqrt{\Delta})\big\{r + \frac{1}{2}\sqrt{\Delta}[\tanh(\frac{1}{4}\sqrt{\Delta}\xi) + \coth(\frac{1}{4}\sqrt{\Delta}\xi)]\big\}^{-1}, \quad (3.20)$$

$$u(x,t) = \alpha_0 - \alpha_0(r \pm \sqrt{\Delta})\big\{r + \frac{1}{2}\sqrt{\Delta}[\coth(\frac{1}{4}\sqrt{\Delta}\xi) + \tanh(\frac{1}{4}\sqrt{\Delta}\xi)]\big\}^{-1}, \quad (3.21)$$

where $\xi = kx \pm \frac{\mu - k^2\Delta}{\sqrt{\Delta}}t$.

Result 2

$$w=\pm\frac{k^2\Delta-\mu}{\sqrt{\Delta}},\ \gamma=-\frac{(\sqrt{\Delta}\mp r)(2k^2\Delta+\mu)}{2\alpha_0\sqrt{\Delta}},\ \delta=\frac{k^2(\sqrt{\Delta}\mp r)^2}{2\alpha_0^2},$$
$$\alpha_1=\frac{2q\alpha_0}{r\mp\sqrt{\Delta}},\ \alpha_{-1}=0,\ \alpha_0=\alpha_0,$$

where $\Delta=r^2-4pq>0$.

Now, the solution for the result 2 becomes:

$$U(\xi)=\alpha_0+\frac{2q\alpha_0}{r\mp\sqrt{\Delta}}\left[f(\xi)\right]. \tag{3.22}$$

For Case 1, we substitute (2.9) into (3.22) and using (2.13), we have the hyperbolic wave solution of Eq. (1.1) as follows:

$$u(x,t)=\alpha_0-\frac{\alpha_0}{r\mp\sqrt{\Delta}}\big[r+\sqrt{\Delta}(\frac{A\sinh(\frac{1}{2}\sqrt{\Delta}\xi)+B\cosh(\frac{1}{2}\sqrt{\Delta}\xi)}{A\cosh(\frac{1}{2}\sqrt{\Delta}\xi)+B\sinh(\frac{1}{2}\sqrt{\Delta}\xi)})\big]. \tag{3.23}$$

For Case 2, we substitute (2.10) into (3.22) and using (2.13), we have the hyperbolic wave solution of Eq. (1.1) as follows:

$$\begin{aligned}u(x,t)=&\alpha_0-\frac{\alpha_0}{r\mp\sqrt{\Delta}}\big\{r+\sqrt{\Delta}(\frac{A\sinh(\sqrt{\Delta}\xi)+B\cosh(\sqrt{\Delta}\xi)}{A\cosh(\sqrt{\Delta}\xi)+B\sinh(\sqrt{\Delta}\xi)})\mp\\&\sqrt{\Delta}(\sqrt{[\frac{A\sinh(\sqrt{\Delta}\xi)+B\cosh(\sqrt{\Delta}\xi)}{A\cosh(\sqrt{\Delta}\xi)+B\sinh(\sqrt{\Delta}\xi)}]^2-1})\big\}.\end{aligned} \tag{3.24}$$

For Case 3, we substitute (2.12) into (3.22) and using (3.13), we have the hyperbolic wave solution of Eq. (1.1) as follows:

$$\begin{aligned}u(x,t)=&\alpha_0-\frac{\alpha_0}{r\mp\sqrt{\Delta}}\big\{r+\frac{1}{2}\sqrt{\Delta}[\frac{A\sinh(\frac{1}{4}\sqrt{\Delta}\xi)+B\cosh(\frac{1}{4}\sqrt{\Delta}\xi)}{A\cosh(\frac{1}{4}\sqrt{\Delta}\xi)+B\sinh(\frac{1}{4}\sqrt{\Delta}\xi)}]+\\&\frac{1}{2}\sqrt{\Delta}[\frac{A\sinh(\frac{1}{4}\sqrt{\Delta}\xi)+B\cosh(\frac{1}{4}\sqrt{\Delta}\xi)}{A\cosh(\frac{1}{4}\sqrt{\Delta}\xi)+B\sinh(\frac{1}{4}\sqrt{\Delta}\xi)}]^{-1}\big\},\end{aligned} \tag{3.25}$$

where $\xi=kx\pm\frac{k^2\Delta-\mu}{\sqrt{\Delta}}t$.

Result 3

$$w=\pm k\sqrt{k^2\Delta+2\mu},\ \gamma=-\frac{(\sqrt{k^2\Delta+2\mu}\pm kr)\sqrt{k^2\Delta+2\mu}}{\alpha_0},$$
$$\delta=\frac{(\sqrt{k^2\Delta+2\mu}\pm kr)^2}{2\alpha_0^2},\ \alpha_{-1}=\frac{2pk\alpha_0}{kr\pm\sqrt{k^2\Delta+2\mu}},\ \alpha_1=0,\ \alpha_0=\alpha_0,$$

where $\Delta=(r^2-4pq)$, provided that $k^2\Delta+2\mu>0$.

Now, the solution for the result 3 becomes:

$$U(\xi)=\alpha_0+\frac{2pk\alpha_0}{kr\pm\sqrt{k^2\Delta+2\mu}}[f(\xi)]^{-1}. \tag{3.26}$$

For Case 1, we substitute (2.9) into (3.26) and using (2.13), (2.14), we respectively have the exact solutions of Eq. (1.1) as follows:

(i) If $\Delta > 0$, we have the hyperbolic wave solution:

$$u(x,t) = \alpha_0 - \frac{4pqk\alpha_0}{kr \pm \sqrt{k^2\Delta + 2\mu}}\{r + \sqrt{\Delta}[\frac{A\sinh(\frac{1}{2}\sqrt{\Delta}\xi) + B\cosh(\frac{1}{2}\sqrt{\Delta}\xi)}{A\cosh(\frac{1}{2}\sqrt{\Delta}\xi) + B\sinh(\frac{1}{2}\sqrt{\Delta}\xi)}]\}^{-1}. \quad (3.27)$$

(ii) If $\Delta < 0$, we have the trigonometric wave solution:

$$u(x,t) = \alpha_0 - \frac{4pqk\alpha_0}{kr \pm \sqrt{k^2\Delta + 2\mu}}\{r + \sqrt{-\Delta}[\frac{A\cos(\frac{1}{2}\sqrt{-\Delta}\xi) - B\sin(\frac{1}{2}\sqrt{-\Delta}\xi)}{A\sin(\frac{1}{2}\sqrt{-\Delta}\xi) + B\cos(\frac{1}{2}\sqrt{-\Delta}\xi)}]\}^{-1}. \quad (3.28)$$

Now, we can simplify Eq. (3.28) to get the following periodic wave solutions:

$$u(x,t) = \alpha_0 - \frac{4pqk\alpha_0}{kr \pm \sqrt{k^2\Delta + 2\mu}}\big[r - \sqrt{-\Delta}\tan(\frac{1}{2}\sqrt{-\Delta}\xi - \xi_1)\big]^{-1}, \quad (3.29)$$

and

$$u(x,t) = \alpha_0 - \frac{4pqk\alpha_0}{kr \pm \sqrt{k^2\Delta + 2\mu}}\big[r + \sqrt{-\Delta}\cot(\frac{1}{2}\sqrt{-\Delta}\xi + \xi_2)\big]^{-1}. \quad (3.30)$$

Comparing our solutions (3.27) and (3.28)–(3.30), with the results (41) and (46)–(48) obtained in [40], we deduce that they are exactly the same.

Substituting (2.17) and (2.18) into (3.26), we respectively have the solutions:

$$u(x,t) = \alpha_0 - \frac{4pqk\alpha_0}{kr \pm \sqrt{k^2\Delta + 2\mu}}\big[r - \sqrt{-\Delta}\tan(\frac{1}{2}\sqrt{-\Delta}\xi + \xi_0)\big]^{-1}, \quad (3.31)$$

and

$$u(x,t) = \alpha_0 - \frac{4pqk\alpha_0}{kr \pm \sqrt{k^2\Delta + 2\mu}}\big[r + \sqrt{-\Delta}\cot(\frac{1}{2}\sqrt{-\Delta}\xi + \xi_0)\big]^{-1}. \quad (3.32)$$

Note that the solution (3.31) is equivalent to the solution (3.29) if $\xi_1 = -\xi_0$ while, the solution (3.32) is equivalent to the solution (3.30) if $\xi_2 = \xi_0$.

For Case 2, we have the following results:

(i) If $\Delta > 0$, substituting (2.11) into (3.26) and using (2.13), we have the exact solution of Eq. (1.1):

$$\begin{aligned} u(x,t) =&\alpha_0 - \frac{4pqk\alpha_0}{kr \pm \sqrt{k^2\Delta + 2\mu}}\{r + \sqrt{\Delta}[\frac{A\sinh(\sqrt{\Delta}\xi) + B\cosh(\sqrt{\Delta}\xi)}{A\cosh(\sqrt{\Delta}\xi) + B\sinh(\sqrt{\Delta}\xi)}]\mp \\ &\sqrt{\Delta}(\sqrt{[\frac{A\sinh(\sqrt{\Delta}\xi) + B\cosh(\sqrt{\Delta}\xi)}{A\cosh(\sqrt{\Delta}\xi) + B\sinh(\sqrt{\Delta}\xi)}]^2 - 1})\}^{-1}. \end{aligned} \quad (3.33)$$

(ii) If $\Delta < 0$, substituting (2.10) into (3.26) and using (2.14), we have the exact solution of Eq. (1.1):

$$\begin{aligned} u(x,t) =&\alpha_0 - \frac{4pqk\alpha_0}{kr \pm \sqrt{k^2\Delta + 2\mu}}\big[r + \sqrt{-\Delta}[\frac{A\cos(\sqrt{-\Delta}\xi) - B\sin(\sqrt{-\Delta}\xi)}{A\sin(\sqrt{-\Delta}\xi) + B\cos(\sqrt{-\Delta}\xi)}]\mp \\ &\sqrt{-\Delta}(\sqrt{1 + [\frac{A\cos(\sqrt{-\Delta}\xi) - B\sin(\sqrt{-\Delta}\xi)}{A\sin(\sqrt{-\Delta}\xi) + B\cos(\sqrt{-\Delta}\xi)}]^2})\big]^{-1}. \end{aligned} \quad (3.34)$$

Now, we can simplify Eq. (3.34) to get the following periodic wave solutions:

$$u(x,t) = \alpha_0 - \frac{4pqk\alpha_0}{kr \pm \sqrt{k^2\Delta + 2\mu}}\{r - \sqrt{-\Delta}[\tan(\sqrt{-\Delta}\xi - \xi_1) \mp \sec(\sqrt{-\Delta}\xi - \xi_1)\}^{-1}, \quad (3.35)$$

and

$$u(x,t) = \alpha_0 - \frac{4pqk\alpha_0}{kr \pm \sqrt{k^2\Delta + 2\mu}}\{r + \sqrt{-\Delta}[\cot(\sqrt{-\Delta}\xi + \xi_2) \mp \csc(\sqrt{-\Delta}\xi + \xi_2)]\}^{-1}. \quad (3.36)$$

For Case 3, we substitute (2.12) into (3.26) and using (2.13), (2.14), we respectively have the exact solutions of Eq. (1.1) as follows:

(i) If $\Delta > 0$, we have the hyperbolic wave solution:

$$\begin{aligned} u(x,t) =& \alpha_0 - \frac{4pqk\alpha_0}{kr \pm \sqrt{k^2\Delta + 2\mu}}\{r + \frac{1}{2}\sqrt{\Delta}[\frac{A\sinh(\frac{1}{4}\sqrt{\Delta}\xi) + B\cosh(\frac{1}{4}\sqrt{\Delta}\xi)}{A\cosh(\frac{1}{4}\sqrt{\Delta}\xi) + B\sinh(\frac{1}{4}\sqrt{\Delta}\xi)}] + \\ & \frac{1}{2}\sqrt{\Delta}[\frac{A\sinh(\frac{1}{4}\sqrt{\Delta}\xi) + B\cosh(\frac{1}{4}\sqrt{\Delta}\xi)}{A\cosh(\frac{1}{4}\sqrt{\Delta}\xi) + B\sinh(\frac{1}{4}\sqrt{\Delta}\xi)}]^{-1}\}^{-1}. \end{aligned} \quad (3.37)$$

(ii) If $\Delta < 0$, we have the trigonometric wave solution:

$$\begin{aligned} u(x,t) =& \alpha_0 - \frac{4pqk\alpha_0}{kr \pm \sqrt{k^2\Delta + 2\mu}}\{r + \frac{1}{2}\sqrt{-\Delta}[\frac{A\cos(\frac{1}{4}\sqrt{-\Delta}\xi) - B\sin(\frac{1}{4}\sqrt{-\Delta}\xi)}{A\sin(\frac{1}{4}\sqrt{-\Delta}\xi) + B\cos(\frac{1}{4}\sqrt{-\Delta}\xi)}] + \\ & \frac{1}{2}\sqrt{-\Delta}[\frac{A\cos(\frac{1}{4}\sqrt{-\Delta}\xi) - B\sin(\frac{1}{4}\sqrt{-\Delta}\xi)}{A\sin(\frac{1}{4}\sqrt{-\Delta}\xi) + B\cos(\frac{1}{4}\sqrt{-\Delta}\xi)}]^{-1}\}^{-1}. \end{aligned} \quad (3.38)$$

Now, we can simplify Eq. (3.38) to get the following periodic wave solutions:

$$u(x,t) = \alpha_0 - \frac{4pqk\alpha_0}{kr \pm \sqrt{k^2\Delta + 2\mu}}\{r - \frac{1}{2}\sqrt{-\Delta}[\tan(\frac{1}{4}\sqrt{-\Delta}\xi - \xi_1) + \cot(\frac{1}{4}\sqrt{-\Delta}\xi - \xi_1)]\}^{-1}, \quad (3.39)$$

or

$$u(x,t) = \alpha_0 - \frac{4pqk\alpha_0}{kr \pm \sqrt{k^2\Delta + 2\mu}}\{r + \frac{1}{2}\sqrt{-\Delta}[\cot(\frac{1}{4}\sqrt{-\Delta}\xi + \xi_2) + \tan(\frac{1}{4}\sqrt{-\Delta}\xi + \xi_2)]\}^{-1}, \quad (3.40)$$

where $\xi = kx \pm k\sqrt{k^2\Delta + 2\mu}\ t$, and $\xi_1 = \tan^{-1}(A/B)$, $\xi_2 = \cot^{-1}(A/B)$.

Result 4

$$w = \pm k\sqrt{k^2\Delta + 2\mu},\ \gamma = -\frac{(\sqrt{k^2\Delta + 2\mu} \mp kr)\sqrt{k^2\Delta + 2\mu}}{\alpha_0},$$
$$\delta = \frac{(\sqrt{k^2\Delta + 2\mu} \mp kr)^2}{2\alpha_0^2},\ \alpha_1 = \frac{2qk\alpha_0}{kr \mp \sqrt{k^2\Delta + 2\mu}},\ \alpha_{-1} = 0,\ \alpha_0 = \alpha_0,$$

where $\Delta = r^2 - 4pq$, provided that $k^2\Delta + 2\mu > 0$.

Now, the solution for the result 4 becomes:

$$U(\xi) = \alpha_0 + \frac{2qk\alpha_0}{kr \mp \sqrt{k^2\Delta + 2\mu}}[f(\xi)]. \quad (3.41)$$

For Case 1, we substitute (2.9) into (3.41) and using (2.13), (2.14), we respectively have the exact solutions of Eq. (1.1) as follows:

(i) If $\Delta > 0$, then we have the hyperbolic wave solution:

$$u(x,t) = \alpha_0 - \frac{k\alpha_0}{kr \mp \sqrt{k^2\Delta + 2\mu}}\{r + \sqrt{\Delta}[\frac{A\sinh(\frac{1}{2}\sqrt{\Delta}\xi) + B\cosh(\frac{1}{2}\sqrt{\Delta}\xi)}{A\cosh(\frac{1}{2}\sqrt{\Delta}\xi) + B\sinh(\frac{1}{2}\sqrt{\Delta}\xi)}]\}, \quad (3.42)$$

(ii) If $\Delta < 0$, then we have the trigonometric wave solution:

$$u(x,t) = \alpha_0 - \frac{k\alpha_0}{kr \mp \sqrt{k^2\Delta + 2\mu}}\big[r + \sqrt{-\Delta}(\frac{A\cos(\frac{1}{2}\sqrt{-\Delta}\xi) - B\sin(\frac{1}{2}\sqrt{-\Delta}\xi)}{A\sin(\frac{1}{2}\sqrt{-\Delta}\xi) + B\cos(\frac{1}{2}\sqrt{-\Delta}\xi)})\big], \tag{3.43}$$

Now, we can simplify Eq. (3.43) to get the following periodic wave solutions:

$$u(x,t) = \alpha_0 - \frac{k\alpha_0}{kr \mp \sqrt{k^2\Delta + 2\mu}}\big[r - \sqrt{-\Delta}\tan(\frac{1}{2}\sqrt{-\Delta}\xi - \xi_1)\big], \tag{3.44}$$

or

$$u(x,t) = \alpha_0 - \frac{k\alpha_0}{kr \mp \sqrt{k^2\Delta + 2\mu}}\big[r + \sqrt{-\Delta}\cot(\frac{1}{2}\sqrt{-\Delta}\xi + \xi_2)\big]. \tag{3.45}$$

Note that if we substitute (2.17) and (2.18) into (3.41), then we have the same solutions (3.44) and (3.45) if we set $\xi_1 = -\xi_0$ and $\xi_2 = \xi_0$, respectively.

For Case 2, we have the following results:

(i) If $\Delta > 0$, substituting (2.11) into (3.41) and using (2.13), we have the exact solution of Eq. (1.1):

$$\begin{aligned} u(x,t) =& \alpha_0 - \frac{k\alpha_0}{kr \mp \sqrt{k^2\Delta + 2\mu}}\big\{r + \sqrt{\Delta}[\frac{A\sinh(\sqrt{\Delta}\xi) + B\cosh(\sqrt{\Delta}\xi)}{A\cosh(\sqrt{\Delta}\xi) + B\sinh(\sqrt{\Delta}\xi)}] \mp \\ & \sqrt{\Delta}(\sqrt{[\frac{A\sinh(\sqrt{\Delta}\xi) + B\cosh(\sqrt{\Delta}\xi)}{A\cosh(\sqrt{\Delta}\xi) + B\sinh(\sqrt{\Delta}\xi)}]^2 - 1})\big\}. \end{aligned} \tag{3.46}$$

(ii) If $\Delta < 0$, substituting (2.10) into (3.41) and using (2.14), we have the exact solution of Eq. (1.1):

$$\begin{aligned} u(x,t) =& \alpha_0 - \frac{k\alpha_0}{kr \mp \sqrt{k^2\Delta + 2\mu}}\big\{r + \sqrt{-\Delta}[\frac{A\cos(\sqrt{-\Delta}\xi) - B\sin(\sqrt{-\Delta}\xi)}{A\sin(\sqrt{-\Delta}\xi) + B\cos(\sqrt{-\Delta}\xi)}] \mp \\ & \sqrt{-\Delta}(\sqrt{1 + [\frac{A\cos(\sqrt{-\Delta}\xi) - B\sin(\sqrt{-\Delta}\xi)}{A\sin(\sqrt{-\Delta}\xi) + B\cos(\sqrt{-\Delta}\xi)}]^2})\big\}, \end{aligned} \tag{3.47}$$

Now, we can simplify Eq. (3.47) to get the following periodic wave solutions:

$$u(x,t) = \alpha_0 - \frac{k\alpha_0}{kr \mp \sqrt{k^2\Delta + 2\mu}}\big\{r - \sqrt{-\Delta}[\tan(\sqrt{-\Delta}\xi - \xi_1) \mp \sec(\sqrt{-\Delta}\xi - \xi_1)]\big\}, \tag{3.48}$$

and

$$u(x,t) = \alpha_0 - \frac{k\alpha_0}{kr \mp \sqrt{k^2\Delta + 2\mu}}\big\{r + \sqrt{-\Delta}[\cot(\sqrt{-\Delta}\xi + \xi_2) \mp \csc(\sqrt{-\Delta}\xi + \xi_2)]\big\}. \tag{3.49}$$

For Case 3, we substitute (2.12) into (3.41) and using (2.13), (2.14), we respectively have the exact solutions of Eq. (1.1) as follows:

(i) If $\Delta > 0$, we have the hyperbolic wave solution:

$$\begin{aligned} u(x,t) =& \alpha_0 - \frac{k\alpha_0}{kr \mp \sqrt{k^2\Delta + 2\mu}}\big\{r + \frac{1}{2}\sqrt{\Delta}[\frac{A\sinh(\frac{1}{4}\sqrt{\Delta}\xi) + B\cosh(\frac{1}{4}\sqrt{\Delta}\xi)}{A\cosh(\frac{1}{4}\sqrt{\Delta}\xi) + B\sinh(\frac{1}{4}\sqrt{\Delta}\xi)}] + \\ & \frac{1}{2}\sqrt{\Delta}[\frac{A\sinh(\frac{1}{4}\sqrt{\Delta}\xi) + B\cosh(\frac{1}{4}\sqrt{\Delta}\xi)}{A\cosh(\frac{1}{4}\sqrt{\Delta}\xi) + B\sinh(\frac{1}{4}\sqrt{\Delta}\xi)}]^{-1}\big\}. \end{aligned} \tag{3.50}$$

(ii) If $\Delta < 0$, we have the trigonometric wave solution:

$$\begin{aligned} u(x,t) =& \alpha_0 - \frac{k\alpha_0}{kr \mp \sqrt{k^2\Delta + 2\mu}}\{r + \frac{1}{2}\sqrt{-\Delta}[\frac{A\cos(\frac{1}{4}\sqrt{-\Delta}\xi) - B\sin(\frac{1}{4}\sqrt{-\Delta}\xi)}{A\sin(\frac{1}{4}\sqrt{-\Delta}\xi) + B\cos(\frac{1}{4}\sqrt{-\Delta}\xi)}]+ \\ & \frac{1}{2}\sqrt{-\Delta}[\frac{A\cos(\frac{1}{4}\sqrt{-\Delta}\xi) - B\sin(\frac{1}{4}\sqrt{-\Delta}\xi)}{A\sin(\frac{1}{4}\sqrt{-\Delta}\xi) + B\cos(\frac{1}{4}\sqrt{-\Delta}\xi)}]^{-1}\}, \end{aligned} \tag{3.51}$$

Now, we can simplify Eq. (3.51) to get the following periodic wave solutions:

$$u(x,t) = \alpha_0 - \frac{k\alpha_0}{kr \mp \sqrt{k^2\Delta + 2\mu}}\{r - \frac{1}{2}\sqrt{-\Delta}[\tan(\frac{1}{4}\sqrt{-\Delta}\xi - \xi_1) + \cot(\frac{1}{4}\sqrt{-\Delta}\xi - \xi_1)]\}, \tag{3.52}$$

and

$$u(x,t) = \alpha_0 - \frac{k\alpha_0}{kr \mp \sqrt{k^2\Delta + 2\mu}}\{r + \frac{1}{2}\sqrt{-\Delta}[\cot(\frac{1}{4}\sqrt{-\Delta}\xi + \xi_2) + \tan(\frac{1}{4}\sqrt{-\Delta}\xi + \xi_2)]\}, \tag{3.53}$$

where $\xi = kx \pm k\sqrt{k^2\Delta + 2\mu}t$, and $\xi_1 = \tan^{-1}(A/B)$, $\xi_2 = \cot^{-1}(A/B)$.

Finally, we mention here that (3.5) can be used in our exact solutions (3.23), (3.24), (3.25), (3.27), (3.33), (3.37), (3.42), (3.46) and (3.50) to get many solitary wave solutions of Eq. (1.1) in simple forms which are omitted for simplicity.

4. Some graphical representations for some solutions

In this section, we will illustrate the application of the some results established above. Exact solutions of the results describe different nonlinear waves. For the established exact hyperbolic wave solutions and trigonometric wave solutions.

Let us now examine Figures 1–4 as it illustrates some of our results obtained in this paper. To this aim, we select some special values of the parameters obtained, for example, in some Figures of the solutions (3.4), (3.25), (3.34) and (3.51) of the nonlinear Kolmogorov-Petrovskii-Piskunov (KPP) equation (1.1) with $-10 < x, t < 10$, respectively. For more convenience the graphical representations of these solutions are shown in the following figures:

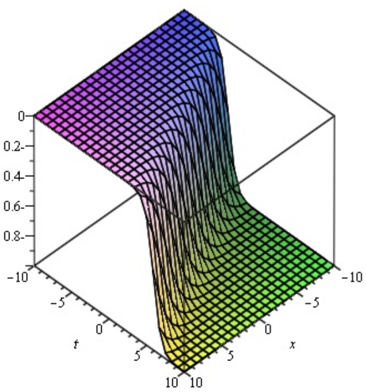

Figure 1 Plot solution (3.4) with $r = 3, p = 2$, $q = k = \alpha_0 = B = 1, \mu = -1, A = 2$

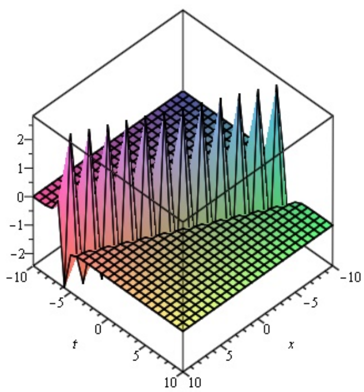

Figure 2 Plot solution (3.25) with $r = 3, p = 2$, $q = k = \alpha_0 = B = 1, \mu = -1, A = 2$

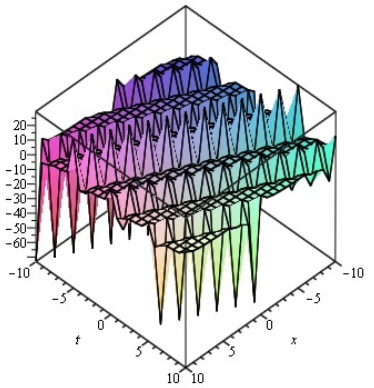

Figure 3 Plot solution (3.34) with $r = p = q = B = 2, k = \alpha_0 = A = 1, \mu = 8$

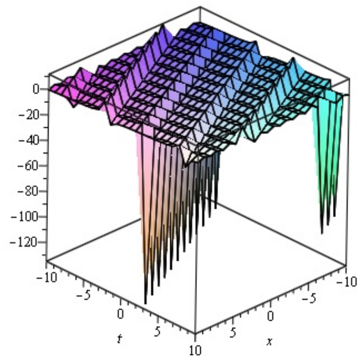

Figure 4 Plot solution (3.51) with $r = p = q = B = 2, k = \alpha_0 = A = 1, \mu = 8$

5. Conclusions

The purpose of the study is to show that exact travelling wave solutions of the nonlinear KPP equation can be obtained by the Riccati equation method combined with the generalized extended (G'/G)-expansion method. These solutions include hyperbolic and trigonometric function solutions. Our new results obtained in this paper have been compared with the well-known results obtained in [25,40,45,46]. We conclude that most of our results for Eq. (1.1) are new and not published elsewhere. Overall, the results reveal that the Riccati equation method combined with the generalized extended (G'/G)-expansion method is a powerful mathematical tool to solve nonlinear partial differential equations. Finally, our new solutions have been checked with the aid of the Maple by putting them back into the original equation (1.1).

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