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The Representation Theorems of Conjugate Spaces of Some $l^0(\{X_i\})$ Type F-Normed Spaces

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Abstract In a paper published in *Acta Mathematica Sinica* (2016, 59(4)) we obtained some representation theorems for the conjugate spaces of some l^0 type F-normed spaces. In this paper, for a sequence of normed spaces $\{X_i\}$, we study the representation problems of conjugate spaces of some $l^0(\{X_i\})$ type F-normed spaces, obtain the algebraic representation continued equalities

$$(l^0(\{X_i\}))^* \overset{A}{=} (c^0_{00}(\{X_i\}))^* \overset{A}{=} c_{00}(\{X_i^*\}),$$

$$(l^0(X))^* \overset{A}{=} (c^0(X))^* \overset{A}{=} (c^0_0(X))^* \overset{A}{=} (c^0_{00}(X))^* \overset{A}{=} c_{00}(X^*),$$

and the topological representation $((c^0_{00}(\{X_i\}))^*, sw^*) = c^0_{00}(\{X_i^*\})$, where sw^* is the sequential weak star topology. For the sequences of inner product spaces and number fields with the usual topology, the concrete forms of the basic representation theorems are obtained at last.

Keywords $l^0(\{X_i\})$ type F-normed space; locally convex space; locally bounded space; sequential weak star topology; representation theorem

MR(2010) Subject Classification 46A04; 46A20

1. Introduction

Representation theory is one of core problems for many branches of mathematics [1–3]. For some l^0 type F-normed scalar-valued sequence spaces, we obtained in [3] some representation theorems of their conjugate spaces. Extending scalar-valued sequence spaces to vector-valued sequence spaces, this paper studies the representation problem of conjugate spaces of some $l^0(\{X_i\})$ type F-normed vector-valued sequence spaces.

Let X be a vector space over number field $\mathbf{K}$ ($\mathbf{R}$ or $\mathbf{C}$). An F-norm on X is a function $\|\cdot\| : X \to \mathbf{R}_+$ satisfying the following conditions:

(n_1) $\|x\| = 0 \Leftrightarrow x = \theta$ (zero element);

(n_2) $\|ax\| \le \|x\|$, $x \in X$, $a \in \mathbf{K}$, $|a| \le 1$;

(n_3) $\|x + y\| \le \|x\| + \|y\|$, $x, y \in X$;

(n_4) $\lim_{a\to 0} \|ax\| \to 0$, $x \in X$.

If $\|\cdot\|$ is an F-norm, then it induces on X a metrizable vector topology, and $(X, \|\cdot\|)$ is called an F-normed space. If the monotonicity (n_2) is replaced by the absolute homogeneity

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(n_2') $\|ax\| = |a|\|x\|$, $x \in X$, $a \in \mathbf{K}$,

then $\|\cdot\|$ is called a norm and $(X, \|\cdot\|)$ a normed space. If condition (n_2) is replaced by the p-absolute homogeneity $(0 < p \leq 1)$

(n_2'') $\|ax\| = |a|^p\|x\|$, $x \in X$, $a \in \mathbf{K}$,

then $\|\cdot\|$ is called a p-norm and $(X, \|\cdot\|)$ a p-normed space.

The difference between an F-norm and a norm is just the absolute homogeneity. However, this small gap makes F-normed spaces much more complicated than normed spaces.

Let $(X_i, \|\cdot\|_i)$ be a sequence of normed spaces over $\mathbf{K}$. Then on the Cartesian product $\prod_{i=1}^{\infty} X_i$, the function

$$\|x\|_0 = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|\xi_i\|_i}{1 + \|\xi_i\|_i}, \quad x = (\xi_i) \in \prod_{i=1}^{\infty} X_i \tag{1.1}$$

satisfies the conditions (n_1), (n_2) and (n_4) clearly. From the inequality

$$\frac{u+v}{1+u+v} \leq \frac{u}{1+u} + \frac{v}{1+v}, \quad u, v \in \mathbf{R}_+$$

and the monotonicity of $\frac{u}{1+u}$ on $\mathbf{R}_+$, the function $\|\cdot\|_0$ also satisfies the inequality $\|x+y\|_0 \leq \|x\|_0 + \|y\|_0$ for any $x,\ y \in \prod_{i=1}^{\infty} X_i$, so $\|\cdot\|_0$ is an F-norm on $\prod_{i=1}^{\infty} X_i$. From now on, the symbol $l^0(\{X_i\})$ is used to denote the vector space $\prod_{i=1}^{\infty} X_i$ with the F-norm $\|\cdot\|_0$, i.e.,

$$l^0(\{X_i\}) = \Big(\prod_{i=1}^{\infty} X_i, \|\cdot\|_0\Big). \tag{1.2}$$

Let us make clear the meanings of other symbols used in the following. For a sequence $(X_i, \|\cdot\|_i)$ of normed space, let

$$c_{00}(\{X_i\}) = \Big\{x = (\xi_i) \in \prod_{i=1}^{\infty} X_i :\ \exists n \in \mathbf{N} \text{ such that } \xi_i = \theta \text{ for } i > n\Big\}.$$

When $(X_i, \|\cdot\|_i) \equiv (X, \|\cdot\|)$, a same normed space, we use $l^0(X)$ to denote $l^0(\{X_i\})$, and let

$$c(X) = \left\{x = (\xi_i) \in X^{\mathbf{N}} :\ (\xi_i) \text{ is convergent in } X\right\},$$

$$c_0(X) = \left\{x = (\xi_i) \in X^{\mathbf{N}} :\ (\xi_i) \text{ is convergent to } \theta \text{ in } X\right\},$$

$$c_{00}(X) = \left\{x = (\xi_i) \in X^{\mathbf{N}} :\ \exists n \in \mathbf{N} \text{ such that } \xi_i = \theta \text{ for } i > n\right\}.$$

Now as vector spaces, we have the natural inclusion relations $c_{00}(\{X_i\}) \subset l^0(\{X_i\})$ and

$$c_{00}(X) \subset c_0(X) \subset c(X) \subset l^0(X);$$

as topological vector spaces, the symbols $c_{00}(\{X_i\})$, $c_{00}(X)$, $c_0(X)$ and $c(X)$ are used to denote the corresponding normed spaces with the norm defined by

$$\|x\|_\infty = \sup_i \|\xi_i\|, \quad x = (\xi_i). \tag{1.3}$$

The symbols $l^0(\{X_i\})$, $c_{00}^0(\{X_i\})$ and $l^0(X)$, $c_{00}^0(X)$, $c_0^0(X)$, $c^0(X)$ are used to denote the corresponding F-normed spaces with the F-norm $\|\cdot\|_0$, respectively, referred to as $l^0(\{X_i\})$ type spaces. When $(X_i, \|\cdot\|_i) \equiv (\mathbf{K}, |\cdot|)$, the symbols l^0, c^0, c_0^0, c_{00}^0 stand for the corresponding scalar-valued sequence spaces with F-norm $\|\cdot\|_0$, the symbols c, c_0, c_{00} denote the corresponding vector spaces or normed spaces with norm $\|\cdot\|_\infty$, respectively.

If $(X_i^*, \|\cdot\|_i)$ is used to denote the conjugate space of $(X_i, \|\cdot\|_i)$ with the norm defined by

$$\|f\|_i = \sup_{\|\xi_i\|_i \le 1} \|f(\xi_i)\|, \quad f \in X_i^*,$$

then $c_{00}^0(\{X_i^*\})$ is the F-normed space with F-norm $\|\cdot\|_0$ and $c_{00}(\{X_i^*\})$ is the corresponding normed space with norm $\|\cdot\|_\infty$, respectively. The meanings of $c_{00}^0(X^*)$ and $c_{00}(X^*)$ are self-evident.

By the same arguments used in [4,5] it is easy to obtain:

Proposition 1.1 *The convergence in $l^0(\{X_i\})$ type spaces is equivalent to coordinate-wise convergence, i.e., for $x^{(m)} = (\xi_i^{(m)})$, $x^{(0)} = (\xi_i^{(0)}) \in l^0(\{X_i\})$ (or $c_{00}^0(\{X_i\})$, etc.),*

$$\lim_{m\to\infty} x^{(m)} = x^{(0)} \Leftrightarrow \lim_{m\to\infty} \xi_i^{(m)} = \xi_i^{(0)}, \quad i = 1, 2, \ldots. \tag{1.4}$$

The following proposition reveals the intrinsic properties of $l^0(\{X_i\})$ type spaces.

Proposition 1.2 *Every $l^0(\{X_i\})$ type space is locally convex, but non-locally bounded.*

Proof It follows from Proposition 1.1 that the topology on $l^0(\{X_i\})$ is just the product topology on $\prod_{i=1}^\infty X_i$. Then by the local convexity of X_i and the fact that the product space of locally convex spaces is still locally convex [5, p.52] we know that $l^0(\{X_i\})$ is also locally convex, and the family of convex sets

$$\Big\{ \prod_{j=1}^n D_{i_j} \times \prod_{k\ne i_j} X_k : \ n \in \mathbf{N},\ D_{i_j} \text{ is some convex } \theta\text{-neighborhood in } X_{i_j} \Big\} \tag{1.5}$$

constitutes the θ-neighborhood basis of $l^0(\{X_i\})$.

The image of any bounded set under continuous linear mapping is also bounded. If B is a bounded convex set in $\prod_{i=1}^\infty X_i$, then its image $P_i(B)$ under any natural projection P_i : $\prod_{i=1}^\infty X_i \to X_i$ is also bounded, so for each $i = 1, 2, \ldots$, there exists a bounded set B_i in X_i such that $B \subset \prod_{i=1}^\infty B_i$. Thus by (1.5) we know that there is not any bounded θ-neighborhood in $\prod_{i=1}^\infty X_i$, so $l^0(\{X_i\})$, or $\prod_{i=1}^\infty X_i$ equipped with the product topology, is non-locally bounded. The same arguments can be used to verify the same conclusion for other $l^0(\{X_i\})$ type spaces. □

For a p-normed space $(X, \|\cdot\|)$, if its conjugate space X^* is nontrivial, then the function

$$\|f\| = \sup_{\|x\|\le 1} |f(x)|, \quad f \in X^* \tag{1.6}$$

makes $(X^*, \|\cdot\|)$ into a new normed space. In this case, it is very important to study the isometric representation of $(X^*, \|\cdot\|)$. For $l^0(\{X_i\})$ type spaces, the local convexity makes their conjugate spaces large enough to separate the points of them [4,6], but the non-local boundedness means that it is impossible to equip them with any p-norm [5, p.73]. Thus it is also impossible to endow their conjugate spaces with any (F-)norm via equation (1.6), so there is no sense in studying the isometrical representation of their conjugate spaces. But now we can do the following two things: (a) looking for the algebraic representation of their conjugate spaces; (b) looking for the topological representation of their conjugate spaces equipped with some natural topologies.

On the conjugate spaces of $l^0(\{X_i\})$ type spaces, there are two most natural vector topologies, one is the weak star topology w^* induced by the pointwise convergence of nets

$$f_\lambda \stackrel{w^*}{\to} f \Leftrightarrow f_\lambda(x) \to f(x), \quad \forall x \in l^0(\{X_i\})(\text{etc.}), \tag{1.7}$$

where $f_\lambda, f \in (l^0(\{X_i\}))^*(\text{etc.})$ $(\lambda \in \Lambda)$, the other is the sequential weak star topology sw^* induced by the pointwise convergence of sequences

$$f_n \stackrel{sw^*}{\to} f \Leftrightarrow f_n(x) \to f(x), \quad \forall x \in l^0(\{X_i\})(\text{etc.}) \tag{1.8}$$

where $f_n, f \in (l^0(\{X_i\}))^*(\text{etc.})$ $(n \in \mathbf{N})$. The weak star topology w^* is the most common vector topology on conjugate spaces [4–6]. On the conjugate spaces of $l^0(\{X_i\})$ type spaces, it is not difficult to verify that the family of sets

$$\mathcal{B} = \{((f_n), f) : f_n, f \in (l^0(\{X_i\}))^*(\text{etc.}),\ f_n(x) \to f(x), \quad \forall x \in l^0(\{X_i\})(\text{etc.})\}$$

satisfies the axiom of convergence classes, so by [7, chap.2, Theorem 9] there exits a unique topology sw^* on $(l^0(\{X_i\}))^*$ (etc.) such that the sequence $f_n \stackrel{sw^*}{\to} f$ if and only if $((f_n), f) \in \mathcal{B}$. It is not difficult to see that the sequential weak star topology sw^* is a vector topology with countable θ-neighborhood basis, so the continuity of operators with respect to this topology could be dealt with via sequences. We ought to note that the weak star topology has no such advantage.

In the next section, we study the algebraic representation problems of conjugate spaces of $l^0(\{X_i\})$ type spaces, obtain the algebraic representation continued equalities

$$(l^0(\{X_i\}))^* \stackrel{A}{=} (c_{00}^0(\{X_i\}))^* \stackrel{A}{=} c_{00}(\{X_i^*\})$$

and

$$(l^0(X))^* \stackrel{A}{=} (c^0(X))^* \stackrel{A}{=} (c_0^0(X))^* \stackrel{A}{=} (c_{00}^0(X))^* \stackrel{A}{=} c_{00}(X^*).$$

In the third section, with respect to the sequential weak star topology sw^*, we obtain the topological representation $((c_{00}^0(\{X_i\}))^*, sw^*) = c_{00}^0(\{X_i^*\})$. For the sequences of inner product spaces and number fields with the usual topology, the concrete forms of the basic representation theorems are obtained at last.

2. The algebraic representation theorems

Theorem 2.1 *Let $(X_i, \|\cdot\|_i)$ be a sequence of normed spaces. Then the conjugate space $(l^0(\{X_i\}))^*$ is algebraically isomorphic to $c_{00}(\{X_i^*\})$, i.e., we have the algebraic representation*

$$(l^0(\{X_i\}))^* \stackrel{A}{=} c_{00}(\{X_i^*\}). \tag{2.1}$$

Proof Note that the space $l^0(\{X_i\})$ is the Cartesian product $\prod_{i=1}^\infty X_i$ with the F-norm $\|\cdot\|_0$. For the standard basis sequence

$$e_i = (0, \ldots, 0, \overset{i\text{th}}{1}, 0, \ldots), \quad i \in \mathbf{N},$$

of l^p $(0 < p < \infty)$, and $\xi_i \in X_i$, let

$$\xi_i e_i = (\theta, \ldots, \theta, \overset{i\text{th}}{\xi_i}, \theta, \ldots) \in \prod_{i=1}^{\infty} X_i, \quad i = 1, 2, \ldots.$$

For every $x = (\xi_i) \in l^0(\{X_i\})$, as

$$\begin{aligned}\|x - \sum_{i=1}^{n} \xi_i e_i\|_0 &= \|(\theta, \ldots, \theta, \xi_{n+1}, \xi_{n+2}, \ldots)\|_0 \\ &= \sum_{i=n+1}^{\infty} \frac{1}{2^i} \frac{\|\xi_i\|_i}{1 + \|\xi_i\|_i} \le \sum_{i=n+1}^{\infty} \frac{1}{2^i} \to 0, \quad n \to \infty, \end{aligned} \tag{2.2}$$

it could be represented as the series

$$x = \lim_{n\to\infty} \sum_{i=1}^{n} \xi_i e_i = \sum_{i=1}^{\infty} \xi_i e_i. \tag{2.3}$$

Let

$$\widehat{X}_i = \{\xi_i e_i = (\theta, \ldots, \theta, \overset{i\text{th}}{\xi_i}, \theta \ldots) : \ \xi_i \in X_i\}$$

be the subspace of $l^0(\{X_i\}) = \prod_{i=1}^{\infty} X_i$ corresponding to X_i. Then the canonical projection P_i is a topological isomorphism between $(\widehat{X}_i, \|\cdot\|_0)$ and $(X_i, \|\cdot\|_i)$ (not isometric isomorphism). Now for every $f \in (l^0(\{X_i\}))^*$, by the continuity of f we have

$$f(x) = \sum_{i=1}^{\infty} f(\xi_i e_i) = \sum_{i=1}^{\infty} f_i(\xi_i), \quad x = \sum_{i=1}^{\infty} \xi_i e_i \in l^0(\{X_i\}), \tag{2.4}$$

where $f_i = f \circ P_i^{-1} \in X_i^*$ is uniquely determined by f. We assert that the sequence $(f_i) \in c_{00}(\{X_i^*\})$. If not, then there is a sequence of strictly monotone natural numbers $i_j \to \infty$ such that $f_{i_j} \neq \theta$. Take $\xi_{i_j} \in X_{i_j}$ such that $f_{i_j}(\xi_{i_j}) = 1$ for each i_j, and $\xi_i = \theta \in X_i$ if $i \neq i_j$, let $x = (\xi_i) \in l^0(\{X_i\})$, then by (2.4) there exists the contradiction

$$f(x) = \sum_{j=1}^{\infty} f_{i_j}(\xi_{i_j}) = +\infty,$$

so $(f_i) \in c_{00}(\{X_i^*\})$. Now define $T : (l^0(\{X_i\}))^* \to c_{00}(\{X_i^*\})$ by

$$T(f) = (f_i), \quad f \in (l^0(\{X_i\}))^*, \tag{2.5}$$

then T is a linear mapping. If $f, g \in (l^0(\{X_i\}))^*$ and $f \neq g$, then there is an i such that $f_i \neq g_i$, so T is an injection from $(l^0(\{X_i\}))^*$ to $c_{00}(\{X_i^*\})$.

On the other hand, for any $F = (f_1, f_2, \ldots, \overset{n\text{th}}{f_n}, \theta, \ldots) \in c_{00}(\{X_i^*\})$, where $f_n \neq \theta$, define

$$f_F(x) = \sum_{i=1}^{n} f_i(\xi_i), \quad x = (\xi_i) \in l^0(\{X_i\}), \tag{2.6}$$

then f_F is a linear functional on $l^0(\{X_i\})$. If a sequence $x^{(m)} = (\xi_i^{(m)}) \to \theta$ in $l^0(\{X_i\})$, then $\xi_i^{(m)} \to \theta$ $(m \to \infty)$ for any i by Proposition 1.1, so

$$\lim_{m\to\infty} f_F(x^{(m)}) = \lim_{m\to\infty} \sum_{i=1}^{n} f_i(\xi_i^{(m)}) = 0,$$

i.e., f_F is continuous or $f_F \in (l^0(\{X_i\}))^*$. Finally, by $f_F \circ P_i^{-1} = f_i$ we know that $Tf_F = F$, namely, T is also a surjection. This proves that the mapping T defined by (2.5) is an algebra isomorphism between $(l^0(\{X_i\}))^*$ and $c_{00}(\{X_i^*\})$, or we have the algebraic representation (2.1). $\square$

Theorem 2.2 *Let $(X, \|\cdot\|)$ be a normed space. Then we have the algebraic representation continued equalities*

$$(l^0(X))^* \stackrel{A}{=} (c^0(X))^* \stackrel{A}{=} (c_0^0(X))^* \stackrel{A}{=} (c_{00}^0(X))^* \stackrel{A}{=} c_{00}(X^*). \tag{2.7}$$

Proof The continued equalities (2.7) is clearly equivalent to

$$c_{00}(X^*) = (l^0(X))^* \subset (c^0(X))^* \subset (c_0^0(X))^* \subset (c_{00}^0(X))^* \subset c_{00}(X^*). \tag{2.8}$$

The equality on the left-hand side of (2.8) follows Theorem 2.1. If $E \subset F$ and $f \in F^*$, then its restriction $f|_E \in E^*$, and in this sense we have $F^* \subset E^*$. Thus by the natural inclusion relations

$$l^0(X) \supset c^0(X) \supset c_0^0(X) \supset c_{00}^0(X) \tag{2.9}$$

we have the three inclusion relations in the middle of (2.8). For any $f \in (c_{00}^0(X))^*$, we assert that the sequence $T(f) = (f_i) = (f \circ P_i^{-1}) \in c_{00}(X^*)$, so we have the inclusion relation at the right end of (2.8). If not, there is an infinite subsequence $\theta \neq f_{i_j} \in X^*$. For each f_{i_j}, take $\xi_{i_j} \in X$ such that $f_{i_j}(\xi_{i_j}) = 1$. Consider the sequence

$$x^{(j)} = (\theta, \ldots, \theta, \overset{i_j\text{th}}{\xi_{i_j}}, \theta, \ldots) \in c_{00}^0(X), \quad j = 1, 2, \ldots,$$

one has $x^{(j)} \to \theta$ $(j \to \infty)$ in $c_{00}^0(X)$ (under the F-norm $\|\cdot\|_0$), but

$$f(x^{(j)}) = f_{i_j}(\xi_{i_j}) = 1 \not\to 0, \quad j \to \infty,$$

which contradicts the continuity of f on $c_{00}^0(X)$. This proves the relation $(c_{00}^0(X))^* \subset c_{00}(X^*)$ and the continued equalities (2.7). $\square$

For a general sequence of normed spaces $(X_i, \|\cdot\|_i)$, removing the objects $(c^0(\{X_i\}))^*$ and $(c_0^0(\{X_i\}))^*$ in the continued equalities (2.7) that may have no sense, using the same arguments we can show:

Theorem 2.3 *Let $(X_i, \|\cdot\|_i)$ be a sequence of normed spaces. Then we have the algebraic representation continued equalities*

$$(l^0(\{X_i\}))^* \stackrel{A}{=} (c_{00}^0(\{X_i\}))^* \stackrel{A}{=} c_{00}(\{X_i^*\}). \tag{2.10}$$

3. The topological representation theorems

By Theorem 2.3 we know that under the linear mapping T, the conjugate spaces $(l^0(\{X_i\}))^*$ and $(c_{00}^0(\{X_i\}))^*$ are algebraically isomorphic to a same vector space $c_{00}(\{X_i^*\})$. As topological vector spaces, the symbol $c_{00}(\{X_i^*\})$ denotes the normed space with the norm $\|\cdot\|_\infty$, $c_{00}^0(\{X_i^*\})$ the F-normed space with the F-norm $\|\cdot\|_0$. In this section, for the linear mapping T from

$((l^0(\{X_i\}))^*, sw^*)$ and $((c_{00}^0(\{X_i\}))^*, sw^*)$ to $c_{00}(\{X_i^*\})$ and $c_{00}^0(\{X_i^*\})$, we study the continuity of T and T^{-1}, to find the conditions that make T a topological isomorphism.

Theorem 3.1 *Let $(X_i, \|\cdot\|_i)$ be a sequence of finite dimensional normed spaces. Then the mapping $T : ((l^0(\{X_i\}))^*, sw^*) \to c_{00}(\{X_i^*\})$ defined by (2.5) is continuous, but its inverse T^{-1} is not.*

Proof Let us show the continuity of T first. Suppose $f^{(m)} \in (l^0(\{X_i\}))^*$ is a non-zero sequence satisfying $f^{(m)} \overset{sw^*}{\to} \theta$, we need to show that its image

$$F^{(m)} = Tf^{(m)} = (f_1^{(m)}, \dots, f_{n_m}^{(m)}, \theta, \dots), \quad \text{where } f_{n_m}^{(m)} \neq \theta$$

converges to θ in $c_{00}(\{X_i^*\})$, i.e., $\lim_{m\to\infty} \|F^{(m)}\|_\infty = 0$. If not, there is a number $\varepsilon_0 > 0$ and a strictly monotone sequence m_k of natural numbers such that

$$\|F^{(m_k)}\|_\infty > 2\varepsilon_0, \quad k = 1, 2, \dots.$$

As $f^{(m_k)} \overset{sw^*}{\to} \theta (k \to \infty)$, assume without loss of generality that

$$\|F^{(m)}\|_\infty > 2\varepsilon_0, \quad m = 1, 2, \dots. \tag{3.1}$$

We will construct two strictly monotone sequences (m_k) and (i_k) of natural numbers to find contradictions.

(i) Take $m_1 = 1$. By $\|F^{(m_1)}\|_\infty > 2\varepsilon_0$, there is a natural number i_1 with $m_1 \leq i_1 \leq n_{m_1}$ such that

$$\|f_{i_1}^{(m_1)}\|_{i_1} > 2\varepsilon_0.$$

(ii) By $f^{(m)} \overset{sw^*}{\to} \theta$ we have

$$\lim_{m\to\infty} f_i^{(m)}(\xi_i) = \lim_{m\to\infty} f^{(m)}(\xi_i e_i) = 0 \tag{3.2}$$

for every $i = 1, 2, \dots, n_{m_1}$ and $\xi_i \in X_i$. The fact of $\dim X_i < \infty$ implies that the conjugate space X_i^* is also finite-dimensional; the reflexivity of X_i means that the weak star topology and the weak topology on X_i^* are equivalent, so by [8, p.215] we know that the norm topology and the weak star topology on X_i^* are equivalent (This property of finite dimensional space will also be used later). Thus the equality (3.2) means that the equality $\lim_{m\to\infty} \|f_i^{(m)}\|_i = 0$ holds uniformly for every $i = 1, 2, \dots, n_{m_1}$. Then there is a natural number $m_2 > m_1$ such that

$$\sum_{i=1}^{n_{m_1}} \|f_i^{(m_2)}\|_i < \varepsilon_0.$$

By $\|F^{(m_2)}\|_\infty > 2\varepsilon_0$, there is a natural number i_2 with $n_{m_1} < i_2 \leq n_{m_2}$ such that

$$\|f_{i_2}^{(m_2)}\|_{i_2} > 2\varepsilon_0.$$

(iii) By $f^{(m)} \overset{sw^*}{\to} \theta$ and $\dim X_i < \infty$ we know that $\lim_{m\to\infty} \|f_i^{(m)}\|_i = 0$ holds uniformly for every $i = 1, 2, \dots, n_{m_2}$, so there is a natural number $m_3 > m_2$ such that

$$\sum_{i=1}^{n_{m_2}} \|f_i^{(m_3)}\|_i < \varepsilon_0.$$

Again by $\|F^{(m_3)}\|_\infty > 2\varepsilon_0$, there is a natural number i_3 with $n_{m_2} < i_3 \le n_{m_3}$ such that

$$\|f_{i_3}^{(m_3)}\|_{i_3} > 2\varepsilon_0.$$

Via mathematical induction we can construct two strictly monotone sequences (m_k) ($m_1 = 1$) and (i_k) of natural numbers with

$$n_{m_{k-1}} < i_k \le n_{m_k}, \quad k = 1, 2, \dots \tag{3.3}$$

such that

$$\sum_{i=1}^{n_{m_{k-1}}} \|f_i^{(m_k)}\|_i < \varepsilon_0, \tag{3.4}$$

and

$$\|f_{i_k}^{(m_k)}\|_{i_k} > 2\varepsilon_0 \tag{3.5}$$

hold at the same time. For each i_k, by inequality (3.5) there is a $\xi_{i_k} \in X_{i_k}$ with $\|\xi_{i_k}\|_{i_k} = 1$ such that

$$|f_{i_k}^{(m_k)}(\xi_{i_k})| > 2\varepsilon_0.$$

Take $x^{(0)} = (\xi_i) \in l^0(\{X_i\})$, here

$$\xi_i = \begin{cases} \xi_{i_k}, & i = i_k, \\ \theta, & i \ne i_k, \end{cases}$$

then for any m_k we have

$$\begin{aligned} |f^{(m_k)}(x^{(0)})| &= \Big|\sum_{i=1}^{n_{m_k}} f_i^{(m_k)}(\xi_i)\Big| \ge |f_{i_k}^{(m_k)}(\xi_{i_k})| - \sum_{i=1}^{n_{m_{k-1}}} |f_i^{(m_k)}(\xi_i)| \\ &\ge |f_{i_k}^{(m_k)}(\xi_{i_k})| - \sum_{i=1}^{n_{m_{k-1}}} \|f_i^{(m_k)}\|_i > 2\varepsilon_0 - \varepsilon_0 = \varepsilon_0. \end{aligned}$$

This contradicts the assumption of $f^{(m)} \overset{sw^*}{\to} \theta$, so T is a continuous linear operator from $((l^0(\{X_i\}))^*, sw^*)$ onto $c_{00}(\{X_i^*\})$.

Let us verify the discontinuity of T^{-1} now. For each natural number i, take an $f_i \in X_i^*$ with $\|f_i\|_i = 1$. Then the sequence

$$F^{(m)} = (\frac{1}{m} f_1, \dots, \overset{m\text{th}}{\frac{1}{m} f_m}, \theta, \dots) \in c_{00}(\{X_i^*\}), \quad m = 1, 2, \dots,$$

satisfies

$$\|F^{(m)}\|_\infty = \sup_{1 \le i \le m} \|\frac{1}{m} f_i\|_i = \frac{1}{m} \to 0, \quad m \to \infty,$$

i.e., $F^{(m)} \to \theta$ in $c_{00}(\{X_i^*\})$. For each i, by $\|f_i\|_i = 1$, there is a $\xi_i \in X_i$ such that $|f_i(\xi_i)| > \frac{1}{2}$. Then for element

$$x = \big(\frac{f_1(\xi_1)}{|f_1(\xi_1)|}\xi_1, \frac{f_2(\xi_2)}{|f_2(\xi_2)|}\xi_2, \dots, \frac{f_i(\xi_i)}{|f_i(\xi_i)|}\xi_i, \dots\big) \in l^0(\{X_i\}),$$

by equality (2.6) we have

$$T^{-1}F^{(m)}(x) = \sum_{i=1}^{m} \frac{1}{m}|f_i(\xi_i)| > \frac{1}{2},$$

i.e., $T^{-1}F^{(m)} \overset{sw^*}{\not\to} \theta\ (m \to \infty)$. This proves the discontinuity of T^{-1}. □

The inequality (3.3) means that the sequences (n_{m_k}) and (i_k) of natural numbers are fastened to each other tightly, so the above construction methods could be called zipper methods.

Theorem 3.2 *Let $(X_i, \|\cdot\|_i)$ be a sequence of finite dimensional normed spaces. Then the mapping $T : ((l^0(\{X_i\}))^*, sw^*) \to c^0_{00}(\{X_i^*\})$ defined by (2.5) is continuous, but its inverse T^{-1} is not.*

Proof Suppose $f^{(m)} \in (l^0(\{X_i\}))^*$ is a non-zero sequence satisfying $f^{(m)} \overset{sw^*}{\to} \theta$. Then by Theorem 3.1 its image

$$F^{(m)} = Tf^{(m)} = (f_1^{(m)}, \dots, f_{n_m}^{(m)}, \theta, \dots), \text{where } f_{n_m}^{(m)} \neq \theta$$

satisfies $\lim_{m\to\infty} \|F^{(m)}\|_\infty = 0$. By the relation between the F-norm $\|\cdot\|_0$ and the norm $\|\cdot\|_\infty$ we have $\lim_{m\to\infty} \|F^{(m)}\|_0 = 0$, so $Tf^{(m)} \to \theta$ holds in $c^0_{00}(\{X_i^*\})$, or T is continuous now.

Similar to the second part of proof of Theorem 3.1, the sequence

$$F^{(m)} = (\frac{1}{m}f_1, \dots, \overset{m\text{th}}{\frac{1}{m}f_m}, \theta, \dots), \quad m = 1, 2, \dots,$$

also converges to θ in $c^0_{00}(\{X_i^*\})$, where $f_i \in X_i^*$ with $\|f_i\|_i = 1$. Then for $\xi_i \in X_i$ with $|f_i(\xi_i)| > \frac{1}{2}$ and the element

$$x = \big(\frac{f_1(\xi_1)}{|f_1(\xi_1)|}\xi_1, \frac{f_2(\xi_2)}{|f_2(\xi_2)|}\xi_2, \dots, \frac{f_i(\xi_i)}{|f_i(\xi_i)|}\xi_i, \dots\big) \in l^0(\{X_i\}),$$

by equality (2.6) we have

$$T^{-1}F^{(m)}(x) = \sum_{i=1}^{m} \frac{1}{m}|f_i(\xi_i)| > \frac{1}{2},$$

so $T^{-1}F^{(m)} \overset{sw^*}{\not\to} \theta\ (m \to \infty)$. This completes the proof of discontinuity of T^{-1}. □

Theorems 3.1 and 3.2 told us that the algebra isomorphism $T : (l^0(\{X_i\}))^* \to c_{00}(\{X_i^*\})$ hidden in the equations (2.1) and (2.10) is not the topological isomorphism between $((l^0(\{X_i\}))^*, sw^*)$ and $c_{00}(\{X_i^*\})$ or $c^0_{00}(\{X_i^*\})$. Now we hope to find the conditions under which the algebra isomorphism $T : (c^0_{00}(\{X_i\}))^* \to c_{00}(\{X_i^*\})$ hidden in the equation (2.10) could be lifted to the topological isomorphism between $((c^0_{00}(\{X_i\}))^*, sw^*)$ and $c_{00}(\{X_i^*\})$ or $c^0_{00}(\{X_i^*\})$.

Theorem 3.3 *Let $(X_i, \|\cdot\|_i)$ be a sequence of normed spaces. Then the mapping T : $((c^0_{00}(\{X_i\}))^*, sw^*) \to c_{00}(\{X_i^*\})$ defined by (2.5) is discontinuous, but its inverse T^{-1} is continuous.*

Proof Take $f_i \in X_i^*$ with $\|f_i\|_i = 1$ for any $i \in \mathbf{N}$. Let

$$F^{(m)} = (\theta, \dots, \theta, \overset{m\text{th}}{f_m}, \theta, \dots) \in c_{00}(\{X_i^*\}).$$

Then Theorem 2.3 shows that the sequence $f^{(m)} = T^{-1}F^{(m)} \in (c_{00}^0(\{X_i\}))^*$. For a given $x = (\xi_1, \ldots, \xi_n, \theta, \ldots) \in c_{00}^0(\{X_i\})$, we have $f^{(m)}(x) = 0$ for any $m > n$ by (2.6), so the sequence $f^{(m)}$ converges to θ in $((c_{00}^0(\{X_i\}))^*, sw^*)$. But by $\|Tf^{(m)}\|_\infty = \|F^{(m)}\|_\infty \equiv 1$ we know that its imagine $Tf^{(m)}$ does not converge to θ in $c_{00}(\{X_i^*\})$, so T is not continuous.

Assume the functional sequence $F^{(m)} = (f_1^{(m)}, f_2^{(m)}, \ldots, f_{n_m}^{(m)}, \theta, \ldots)$ converges to θ in $c_{00}(\{X_i^*\})$, i.e.,

$$\|F^{(m)}\|_\infty = \sup_{1\le i\le n_m} \|f_i^{(m)}\|_i \to 0, \quad m \to \infty.$$

Then for any given $x = (\xi_1, \xi_2, \ldots, \xi_n, \theta, \ldots) \in c_{00}^0(\{X_i\})$,

$$\begin{aligned}|T^{-1}F^{(m)}(x)| &\le \sum_{i=1}^n |f_i^{(m)}(\xi_i)| \le \sum_{i=1}^n \|f_i^{(m)}\|_i \|\xi_i\|_i \\ &\le \|F^{(m)}\|_\infty \sum_{i=1}^n \|\xi_i\|_i \to 0, \quad m \to \infty.\end{aligned}$$

Hence the sequence $T^{-1}F^{(m)}$ converges to θ in $((c_{00}^0(\{X_i\}))^*, sw^*)$, or T^{-1} is a continuous linear operator from $c_{00}(\{X_i^*\})$ to $((c_{00}^0(\{X_i\}))^*, sw^*$. $\square$

Theorem 3.4 *Let $(X_i, \|\cdot\|_i)$ be a sequence of finite dimensional normed spaces. Then the mapping $T: ((c_{00}^0(\{X_i\}))^*, sw^*) \to c_{00}^0(\{X_i^*\})$ defined by (2.5) is a topological isomorphism, or we have the topological representation*

$$((c_{00}^0(\{X_i\}))^*, sw^*) = c_{00}^0(\{X_i^*\}). \tag{3.6}$$

Proof To prove the continuity of the mapping T, suppose $f^{(m)} \in (c_{00}^0(\{X_i\}))^*$ is a non-zero sequence satisfying $f^{(m)} \stackrel{sw^*}{\to} \theta$. If its imagine

$$Tf^{(m)} = (f_1^{(m)}, f_2^{(m)}, \ldots, f_{n_m}^{(m)}, \theta, \ldots)$$

does not converge to θ in $c_{00}^0(\{X_i^*\})$, then by Proposition 1.1 there exits some coordinate sequence, without loss of generality in assuming the first coordinate sequence $(f_1^{(m)})$ that does not converge to θ in norm. By the assumption that X_1 is finite dimensional and the reason used in the proof of Theorem 3.1 we know that the norm topology and the weak star topology on X_1^* are equivalent. Thus by $\|f_1^{(m)}\|_1 \not\to 0$ there is some $\xi_1 \in X_1$ such that $f_1^{(m)}(\xi_1) \not\to 0$. Take $x = (\xi_1, \theta, \ldots) \in c_{00}^0(\{X_i\})$, then $f^{(m)}(x) = f_1^{(m)}(\xi_1) \not\to 0$, which contradicts the assumption that $f^{(m)}$ converges to θ in $((c_{00}^0(\{X_i\}))^*, sw^*)$.

To prove the continuity of the inverse mapping T^{-1}, suppose a sequence

$$F^{(m)} = (f_1^{(m)}, f_2^{(m)}, \ldots, f_{n_m}^{(m)}, \theta, \ldots) \in c_{00}^0(\{X_i^*\})$$

tends to θ, or $\|F^{(m)}\|_0 \to 0$ $(m \to \infty)$, then by Proposition 1.1 $\|f_i^{(m)}\|_i \to 0$ $(m \to \infty)$ for every $i \in \mathbf{N}$. Thus for any given $x = (\xi_1, \xi_2, \ldots, \xi_n, 0, \ldots) \in c_{00}^0(\{X_i\})$,

$$|T^{-1}F^{(m)}(x)| \le \sum_{i=1}^n |f_i^{(m)}(\xi_i)| \le \sum_{i=1}^n \|f_i^{(m)}\|_i \|\xi_i\|_i \to 0, \quad m \to \infty.$$

This shows the sequence $T^{-1}F^{(m)}$ converges to θ in $((c_{00}^0(\{X_i\}))^*, sw^*)$, i.e., the mapping T is the topological isomorphism between $((c_{00}^0(\{X_i\}))^*, sw^*)$ and $c_{00}^0(\{X_i^*\})$. □

From above four theorems we can obtain the following corollary immediately:

Corollary 3.5 *Let $(X, \|\cdot\|)$ be a finite dimensional normed space.*

(i) The mapping $T : ((l^0(X))^*, sw^*) \to c_{00}(X^*)$ is continuous, but T^{-1} is not;

(ii) The mapping $T : ((l^0(X))^*, sw^*) \to c_{00}^0(X^*)$ is continuous, but T^{-1} is not;

(iii) The mapping $T : ((c_{00}^0(X))^*, sw^*) \to c_{00}(X^*)$ is not continuous, but T^{-1} is;

(iv) The mapping $T : ((c_{00}^0(X))^*, sw^*) \to c_{00}^0(X^*)$ is a topological isomorphism, i.e., we have the topological representation

$$((c_{00}^0(X))^*, sw^*) = c_{00}^0(X^*). \tag{3.7}$$

4. The applications of the basic representation theorems

Inner product spaces and number fields with the usual topology are two typical classes of normed spaces. For these two classes of normed spaces, let us find the concrete forms of the basic representation theorems obtained in previous sections.

Theorem 4.1 *(i) Let $(X_i, \langle\cdot,\cdot\rangle_i)$ be a sequence of inner product space. Then we have the algebraic representation continued equalities*

$$(l^0(\{X_i\}))^* \overset{A}{=} (c_{00}^0(\{X_i\}))^* \overset{A}{=} c_{00}(\{X_i\}), \tag{4.1}$$

i.e., any $f \in (l^0(\{X_i\}))^*$ (or $f \in (c_{00}^0(\{X_i\}))^*$) corresponds to a unique

$$y = (\zeta_1, \zeta_1, \dots, \zeta_{n_f}, \theta, \dots) \in c_{00}(\{X_i\})$$

such that

$$f(x) = \sum_{i=1}^{n_f} \langle \xi_i, \zeta_i \rangle_i, \quad x = (\xi_i) \in l^0(\{X_i\}) \ (\text{or } x \in c_{00}^0(\{X_i\})). \tag{4.2}$$

(ii) For an inner product space $(X, \langle\cdot,\cdot\rangle)$, we have the algebraic representation continued equalities

$$(l^0(X))^* \overset{A}{=} (c^0(X))^* \overset{A}{=} (c_0^0(X))^* \overset{A}{=} (c_{00}^0(X))^* \overset{A}{=} c_{00}(X). \tag{4.3}$$

(iii) Let $(X_i, \langle\cdot,\cdot\rangle_i)$ be a sequence of finite dimensional inner product spaces. Then we have the topological representation

$$((c_{00}^0(\{X_i\}))^*, sw^*) = c_{00}^0(\{X_i\}). \tag{4.4}$$

Proof By the self-conjugate property of inner product spaces and the general form of continuous linear functionals on them [9, p.104], the conclusion (i) follows Theorems 2.1 and 2.3, (ii) follows Theorem 2.2, (iii) follows Theorem 3.4. □

Theorem 4.2 *(i) In the sense of isomorphism we have the algebraic representation continued equalities*

$$(l^0)^* \overset{A}{=} (c^0)^* \overset{A}{=} (c_0^0)^* \overset{A}{=} (c_{00}^0)^* \overset{A}{=} c_{00}, \tag{4.5}$$

i.e., any $f \in (l^0)^*$ (or $f \in (c^0)^*$, etc.) corresponds to a unique

$$y = (\zeta_1, \zeta_1, \ldots, \zeta_{n_f}, \theta, \ldots) \in c_{00}$$

such that

$$f(x) = \sum_{i=1}^{n_f} \xi_i \zeta_i, \quad x = (\xi_i) \in l^0 \text{ (or } x \in c^0, \text{ etc.)}. \tag{4.6}$$

(ii) In the sense of isomorphism we have the topological representation

$$((c^0_{00})^*, sw^*) = c^0_{00}. \tag{4.7}$$

Proof The number field $\mathbf{K}$ with the usual topology is just an inner product space under the multiplication $\langle \xi, \zeta \rangle = \xi \cdot \zeta$, so by the latter two conclusions of Theorem 4.1 we get the corresponding results of this theorem. □

The conjugate spaces of $l^0(\{X_i\})$ type spaces with weak star topology w^* have no countable θ-neighborhood basis, their topological representation should be more complicated, we will discuss it in another paper.

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