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The d-Shadowing Property on Nonuniformly Expanding Maps

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Abstract In this paper, it is proved that a nonuniformly expanding map f having the $\underline{d}$-shadowing property or $\overline{d}$-shadowing property is topologically transitive.

Keywords nonuniformly expanding map; d-shadowing property; topological transitivity

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1. Introduction

Throughout this paper, denote $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$. A dynamical system is a pair (M, f), where M is a compact metric space with a metric d and $f : M \longrightarrow M$ is a continuous surjection. The orbit of $x \in M$ is the set $\{f^n(x) : n \in \mathbb{N}\}$ which is denoted by $\mathrm{Orb}(x, f)$. We say that f is topologically transitive if for any non-empty open subsets U and V of M, there exists a natural number k such that $f^k(U) \cap V \neq \emptyset$.

The pseudo-orbit shadowing is one of the most important tools to explore dynamical properties of discrete dynamical systems, which is close to the stability of dynamical system. It originated from the works of Anosov [1] and Bowen [2]. For any $\delta > 0$, a sequence $\{x_i\}_{i=0}^{n} \subset M$ is said to be a δ-pseudo-orbit of f if $d(f(x_i), x_{i+1})) < \delta$ for any $0 \leq i < n$. A sequence $\{x_i\}_{i=0}^{\infty} \subset M$ is called to be ε-shadowed by a point $z \in X$ if $d(f^i(z), x_i) < \varepsilon$ for each $i \in \mathbb{Z}^+$. From different understandings on pseudo-orbit and shadowing way, several generalized definitions of shadowing have been developed (for example, average shadowing property [3], limit shadowing property [4], asymptotic average shadowing property [5], ergodic shadowing property [6], $\underline{d}$-shadowing property [7], $\overline{d}$-shadowing property [7] and $\mathcal{F}$-shadowing property [8], and so on). In this paper, we prove that every nonuniformly expanding map with $\underline{d}$-shadowing property is topologically transitive. Firstly, we present some basic definitions of shadowing.

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2. Preliminaries

In this section, some basic concepts and notations are introduced.

Definition 2.1 ([3]) *For $\delta > 0$, a sequence $\{x_i\}_{i=0}^{\infty} \subset M$ is called a δ-average pesudo-orbit of f if there exists $N > 0$ such that for any $n > N$ and any $k \in \mathbb{Z}^+$,*

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

A dynamical system (M, f) has the average shadowing property (abbreviated ASP) if for any $\varepsilon > 0$, there exists $\delta > 0$ such that every δ-average pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ is ε-shadowed in average by some point $z \in M$, that is

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), x_i) < \varepsilon.$$

Definition 2.2 ([5]) *A sequence $\{x_i\}_{i=0}^{\infty} \subset M$ is called an asymptotic average pesudo-orbit of f if*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) = 0.$$

A dynamical system (M, f) has the asymptotic average shadowing property (abbreviated AASP) if every asymptotic average pesudo-orbit $\{x_i\}_{i=0}^{\infty} \subset M$ is asymptotic shadowed in average by a point $z \in M$, i.e.,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), x_i) = 0.$$

In 2010, Fakhari and Ghane [6] introduced the concept of δ-ergodic pseudo-orbit.

Definition 2.3 ([6]) *For $\delta > 0$, a sequence $\{x_i\}_{i=0}^{\infty} \subset M$ is called a δ-ergodic pesudo-orbit of f if*

$$\lim_{n\to\infty} \frac{1}{n}|\{0 \le i < n : d(f(x_i), x_{i+1}) < \delta\}| = 1.$$

$\{x_i\}_{i=0}^{\infty} \subset M$ is δ-ergodic shadowed by a point $z \in M$ if

$$\lim_{n\to\infty} \frac{1}{n}|\{0 \le i < n : d(f^i(z), x_i) < \delta\}| = 1.$$

Definition 2.4 ([7]) *Let (M, f) be a dynamical system. If for any $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-ergodic pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ is ε-shadowed by a point $z \in M$ in such a way that either*

(i) $\liminf_{n\to\infty} \frac{1}{n}|\{0 \le i < n : d(f^i(z), x_i) < \varepsilon\}| > 0$, *or*

(ii) $\limsup_{n\to\infty} \frac{1}{n}|\{0 \le i < n : d(f^i(z), x_i) < \varepsilon\}| > \frac{1}{2}$,

then we say that f has the $\underline{d}$-shadowing property in case (i) and $\overline{d}$-shadowing property in case (ii).

For any $A \subset \mathbb{Z}^+$, the upper density of A is defined by

$$\overline{d}(A) := \limsup_{n\to\infty} \frac{1}{n}|A \cap \{0, 1, \ldots, n-1\}|. \tag{1}$$

Replacing lim sup with lim inf in (1) gives the definition of $\underline{d}(A)$, the lower density of A. If there exists a number $d(A)$ such that $\overline{d}(A) = \underline{d}(A) = d(A)$ then we say that the set A has density $d(A)$. Fix any $\alpha \in [0, 1)$ and denote by $\mathscr{M}_\alpha$ (resp., $\mathscr{M}^\alpha$) the family consisting of sets $A \subset \mathbb{Z}^+$ with $\underline{d}(A) > \alpha$ (resp., $\overline{d}(A) > \alpha$).

Let $\alpha \in [0, 1)$. If for any $\varepsilon > 0$ and any δ-ergodic pseudo-orbit $\{x_i\}_{i=0}^\infty$ of f, there exists a point $z \in X$ such that

$$\{i \in \mathbb{Z}^+ : d(f^i(z), x_i) < \varepsilon\} \in \mathscr{M}^\alpha \text{ (resp., } \{i \in \mathbb{Z}^+ : d(f^i(z), x_i) < \varepsilon\} \in \mathscr{M}_\alpha),$$

then f has the $\mathscr{M}^\alpha$-shadowing property (resp., $\mathscr{M}_\alpha$-shadowing property) [9]. It is easy to see that the $\mathscr{M}^{\frac{1}{2}}$-shadowing property and $\mathscr{M}_0$-shadowing property are $\overline{d}$-shadowing and $\underline{d}$-shadowing, respectively.

Kulczycki et al. [10] explored the relation between asymptotic average shadowing and average shadowing. Wu [9] obtained the following result:

$$\text{AASP} \Longrightarrow \text{ASP} \Longleftrightarrow \mathscr{M}_\alpha\text{-shadowing} \quad (\forall \alpha \in [0, 1)) \Longrightarrow \overline{d}\text{-shadowing} + \underline{d}\text{-shadowing}. \qquad (2)$$

For the asymptotic average shadowing property, Bahabadi [12] proved the following.

Theorem 2.5 *If C^1 local diffeomorphism f is nonuniformly expanding and has the asymptotic average shadowing property, then f is transitive.*

Inspired by (2) and Theorem 2.5, it is natural to ask:

Question 2.6 *Can the hypothesis of AASP in Theorem 2.5 be reduced to ASP, $\underline{d}$-shadowing or $\overline{d}$-shadowing?*

This paper shall give a positive answer to Question 2.6 in section four.

3. Nonuniformly expanding

Let $f : M \longrightarrow M$ be a homeomorphism which is a C^1 locally diffeomorphism and M a C^∞ compact manifold with a Riemannain metric d. f is expanding if there exists a Riemannian metric $\| \cdot \|$ on TM and $\lambda > 1$ such that

$$\| Df^n(x)\nu \| \geq \lambda^n \| \nu \|, \quad \forall x \in M,\ \nu \in T_xM.$$

f is nonuniformly expanding on a set $H \subset M$ if there is a $\lambda > 0$ such that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \| Df(f^j(x))^{-1} \| < -\lambda, \quad \forall x \in H.$$

For $\sigma < 1$, we say that n is a σ-hyperbolic time if for all $1 \leq k \leq n$,

$$\prod_{j=n-k}^{n-1} \|Df(f^j(x))^{-1}\| \leq \sigma^k.$$

A σ-hyperbolic times for $x \in M$ is said to have positive frequency if there exists some $\theta > 0$ such that for any large $n \in \mathbb{N}$ there exist $l \geq \theta n$ and integers $1 \leq n_1 < n_2 < \cdots < n_l \leq n$ which are

σ-hyperbolic times for x, in fact we have

$$\frac{1}{n}|\{0 < k < n : k \text{ is a hyperbolic time for } x\}| > \theta.$$

A C^1 local diffeomorphism f is nonuniformly expanding if it is nonuniformly expanding on a set of full Lebesgue measure. The following propositions are essential.

Proposition 3.1 ([12]) *For $0 < \sigma < 1$, there exists $\delta > 0$ such that if n is a σ-hyperbolic time for x, then there exists a neighborhood V_n of x such that:*

(a) f^n maps V_n deffeomorphically on to the ball of radius δ around $f^n(x)$.

(b) For $1 \le k < n$ and $y, z \in V_n$, $d(f^{n-k}(y), f^{n-k}(z)) \le \sigma^{\frac{k}{2}} d(f^n(y), f^n(z))$.

Proposition 3.2 ([12]) *If f is a nonuniformly expanding map, then there are $0 < \sigma < 1$ and $\theta > 0$ such that the frequency of σ-hyperbolic times for a set H of full Lebesgue measure is greater than θ.*

4. Main results

Lemma 4.1 ([7]) *$B \subset \mathbb{Z}^+$ has positive lower density if and only if for any sequence A with $\overline{d}(A) = 1$, $A \cap B \neq \emptyset$. In particular, $\overline{d}(A \cap B) > 0$.*

Theorem 4.2 *If C^1 local diffeomorphism f is nonuniformly expanding and has the $\underline{d}$-shadowing property, then f is topologically transitive.*

Proof Assume that f is a nonuniformly expanding map and let $H \subset M$ be a set of full measure. Given any two fixed open subsets U and V of M, then $H \cap U \neq \emptyset$ and $H \cap V \neq \emptyset$. Take $x \in H \cap U$ and $y \in H \cap V$ and choose $\varepsilon > 0$ such that $N_\varepsilon(x) := \{z \in M : d(z, x) < \varepsilon\} \subset U$ and $N_\varepsilon(y) \subset V$. There exists $\zeta > 0$ such that if $d(p, q) < \zeta$ then $d(f^{-1}(p), f^{-1}(q)) < \varepsilon$.

Let D denote the diameter of M. It follows from Proposition 3.2 that there exist $0 < \sigma < \frac{\zeta}{D}$ and $0 < \theta < 1$ such that the frequency of σ-hyperbolic times for the set H is greater than θ. Let n_x and n_y be σ-hyperbolic time for x and y, respectively. Proposition 3.1 implies that there exist $0 < \delta < \frac{\varepsilon}{2}$, neighborhood V_{n_x} of x and V_{n_y} of y such that f^{n_u} maps V_{n_u} diffeomorphically on to the ball of radius δ around $f^n(u)$ for $u \in \{x, y\}$.

Take $a_0 = A_0 = 0, a_1 = A_1 = 2$, and $a_n = 2^{a_0 + a_1 + \cdots + a_{n-1}}$, $A_n = a_0 + a_1 + \cdots + a_n$ for $n \ge 2$. Choose $A = [0, A_1] \cup [A_2, A_3] \cup \cdots \cup [A_{2n}, A_{2n+1}] \cup \cdots$, $B = [A_1, A_2] \cup [A_3, A_4] \cup \cdots \cup [A_{2n+1}, A_{2n+2}] \cup \cdots$, and $C = \{0, A_1, A_2, \cdots\}$. It can be verified that $d(C) = 0$, $\overline{d}(A) = \overline{d}(B) = 1$ and $\underline{d}(A) = \underline{d}(B) = 0$. Define a sequence $\{x_i\}_{i=0}^{\infty}$ by

$$x_i = f^{i - A_{2n}}(x), \quad A_{2n} \le i < A_{2n+1};$$

$$x_i = f^{i - A_{2n+1}}(y), \quad A_{2n+1} \le i < A_{2n+2}.$$

It is easy to see that $\{x_i\}_{i=0}^{\infty}$ is an ergodic pseudo-orbit of f. The $\underline{d}$-shadowing property of f implies that there exists a point $z \in X$ such that

$$\liminf_{n \to \infty} \frac{1}{n}|\{0 \le i < n : d(f^i(z), x_i) < \delta\}| > 0. \tag{3}$$

Let $E = \{i \in \mathbb{Z}^+ : d(f^i(z), x_i) < \delta\}$. Applying Lemma 4.1 implies that $\overline{d}(E \cap A) > 0$ and $\overline{d}(E \cap B) > 0$. This implies that there exist infinitely many σ-hyperbolic times n_x such that corresponding to every n_x there exists a positive integer m_x such that $d(f^{m_x}(z), f^{n_x}(x)) < \delta$ and that there exist infinitely many σ-hyperbolic times n_y such that corresponding to every n_y there exists a positive integer m_y such that $d(f^{m_y}(z), f^{n_y}(y)) < \delta$.

Since f^{n_u} maps V_{n_u} diffeomorphically on to the ball of radius δ around f^{n_u} for $u \in \{x, y\}$, applying Proposition 3.1 implies that for any $p, q \in V_{n_u}$,

$$d(f(p), f(q)) \leq \sigma^{\frac{n-1}{2}} d(f^n(p), f^n(q)) < \sigma^{\frac{n-1}{2}} D < \zeta.$$

Then $d(p, q) < \varepsilon$. This implies that $V_{n_x} \subset U$ and $V_{n_y} \subset V$. Therefore, for any $u \in \{x, y\}$, $f^{m_u}(z) \in f^{n_u}(V_u)$. This means that f is topologically transitive. □

Similarly to the proof of Theorem 4.2, for the $\overline{d}$-shadowing property, we have the following result.

Theorem 4.3 *If C^1 local diffeomorphism f is nonuniformly expanding and has the $\overline{d}$-shadowing property, then f is topologically transitive.*

Corollary 4.4 *If C^1 local diffeomorphism f is nonuniformly expanding and has the average shadowing property, then f is topologically transitive.*

Corollary 4.5 *If C^1 local diffeomorphism f is nonuniformly expanding and has the $\mathscr{M}_\alpha$-shadowing property for some $\alpha \in [0, 1)$, then f is topologically transitive.*

Corollary 4.6 *If C^1 local diffeomorphism f is nonuniformly expanding and has the $\mathscr{M}^\alpha$-shadowing property for some $\alpha \in [\frac{1}{2}, 1)$, then f is topologically transitive.*

Theorem 4.7 *If C^1 local diffeomorphism f is nonuniformly expanding and has the average shadowing property, then f is weakly mixing.*

Proof Since f is uniformly expanding and has the average shadowing property, it can be verified that $f \times f$ is uniformly expanding and has the average shadowing property. Applying Corollary 4.4, $f \times f$ is transitive. Then f is weakly mixing. □

In general, it is difficult to investigate the asymptotic average shadowing property, average shadowing property, $\underline{d}$-shadowing property and $\overline{d}$-shadowing property of f. However, the transitivity of f is easily verified. This is the motivation of this paper. If f is nonuniformly expanding and not transitive, then f does not have asymptotic average shadowing, average shadowing, $\underline{d}$-shadowing or $\overline{d}$-shadowing.

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