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# Supereulerian Extended Digraphs

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**Abstract** A digraph $D$ is supereulerian if $D$ has a spanning eulerian subdigraph. Bang-Jensen and Thomassé conjectured that if the arc-strong connectivity $\lambda(D)$ of a digraph $D$ is not less than the independence number $\alpha(D)$, then $D$ is supereulerian. In this paper, we prove that if $D$ is an extended cycle, an extended hamiltonian digraph, an arc-locally semicomplete digraph, an extended arc-locally semicomplete digraph, an extension of two kinds of eulerian digraph, a hypo-semicomplete digraph or an extended hypo-semicomplete digraph satisfying $\lambda(D) \geq \alpha(D)$, then $D$ is supereulerian.

**Keywords** supereulerian digraph; spanning closed trail; eulerian digraph; hamiltonian digraph; arc-locally semicomplete digraph; hypo-semicomplete digraph; extended digraph

**MR(2010) Subject Classification** 05C20; 05C45

## 1. Introduction

We consider finite graphs and digraphs. Undefined terms and notations will follow [1] for graphs and [2] for digraphs. As in [1], $uv$ represents an edge joining $u$ and $v$. Throughout this paper, we use the notation $(u, v)$ to represent an arc oriented from a vertex $u$ to a vertex $v$. Unless specified, we deal with simple digraphs without loops or parallel arcs. For a digraph $D$, we often use $UG(D)$ to denote the underlying undirected graph of $D$, the graph obtained from $D$ by erasing the orientations of all arcs of $D$. We write $D \cong H$ to denote that the digraphs $D$ and $H$ are isomorphic. For an integer $n$, we define $[n] = \{1, 2, \ldots, n\}$. A walk in $D$ is an alternating sequence $W = x_1a_1x_2a_2x_3 \cdots x_{k-1}a_{k-1}x_k$ of vertices $x_i$ and arcs $a_j$ from $D$ such that $a_j = (x_j, x_{j+1})$ for every $i \in [k]$ and $j \in [k-1]$. A walk $W$ is closed if $x_1 = x_k$, and open otherwise. When the arcs of $W$ are understood from the context, we will denote $W$ by $x_1x_2 \cdots x_k$. A trail in $D$ is a walk in which all arcs are distinct. If the vertices $x_1, x_2, \ldots, x_{k-1}$ are distinct, $k \geq 2$ and $x_1 = x_k$, then $W$ is a cycle. If a trail $T$ starts at $x$ and ends at $y$, we always call it $(x, y)$-trail or $T_{[x,y]}$. Especially, a trail $T$ of $D$ is called a spanning trail if $V(T) = V(D)$. For an integer $k \geq 2$, $C_k$ ($k$-cycle) denotes the cycle on $k$ vertices. A digraph $D$ is strong if, for every pair $x, y$ of distinct vertices in $D$, there exist an $(x, y)$-walk and a $(y, x)$-walk. A digraph $D$ is connected if $UG(D)$ is connected. For a digraph $D$ and a set $B \subseteq A(D)$, the digraph $D - B$ is

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the spanning subdigraph of $D$ with arc set $A(D)-B$. We often write $D-a$ rather than $D-\{a\}$ and $D+a$ rather than $D+\{a\}$.

Following [2], for a digraph $D$ with $X, Y \subseteq V(D)$, define

$$(X, Y)_D = \{(x, y) \in A(D) \,:\, x \in X, y \in Y\}.$$

When $Y = V(D) - X$, we define

$$\partial_D^+(X) = (X, V(D) - X)_D \text{ and } \partial_D^-(X) = (V(D) - X, X)_D.$$

For a vertex $v \in V(D)$, $d_D^+(v) = |\partial_D^+(\{v\})|$ and $d_D^-(v) = |\partial_D^-(\{v\})|$ are the out-degree and the in-degree of $v$ in $D$, respectively. Let

$$N_D^+(v) = \{u \in V(D) - v : (v, u) \in A(D)\}$$

and

$$N_D^-(v) = \{u \in V(D) - v : (u, v) \in A(D)\}$$

denote the out-neighbourhood and in neighbourhood of $v$ in $D$, respectively. We call the vertices in $N_D^+(v)$, $N_D^-(v)$ the out-neighbours, in-neighbours of $v$. When the digraph $D$ is understood from the context, we often omit the subscript $D$.

In 1977, Boesch, Suffel and Tindell [3] proposed the supereulerian problem, which seeks to characterize graphs that have spanning eulerian subgraphs; and they indicated that this problem would be very difficult. Later in 1979, Pulleyblank [4] proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. Since then, there has been a lot of research in this topic about supereulerian graphs.

It is natural to study supereulerian digraphs. A digraph $D$ is hamiltonian if it contains a cycle $C$ such that $V(C) = V(D)$. A digraph $D$ is eulerian if it contains a closed trail $W$ such that $A(W) = A(D)$, or, equivalently, if $D$ is connected and for every $v \in V(D)$, $d_D^+(v) = d_D^-(v)$. A digraph $D$ is supereulerian if it contains a closed trail $W$ such that $V(W) = V(D)$, or, equivalently, if $D$ contains a spanning eulerian subdigraph. Some earlier studies were done by Gutin [5,6]. Some of the recent developments can be found in [7–12] among others.

Given a digraph $D$, an eulerian factor of $D$ is a collection of arc disjoint cycles spanning $V(D)$. For $u, v \in V(D)$, the symbol $\mu_D(x, y)$ indicates the number of arcs from $x$ to $y$, i.e., the multiplicity of the arc $(x, y)$, in $D$ (which is at most one, unless we are dealing with a multidigraph). Given a subdigraph $H$ of $D$, the contraction of $H$ (into the vertex $h$) is the digraph $D/H$ with vertex set $V(D)-V(H)$, plus a new vertex $h$ and, for all $u, v \in V(D)-V(H)$, $\mu_{D/H}(u, v) = \mu_D(u, v)$ and $\mu_{D/H}(u, h) = \Sigma_{x \in V(H)} \mu_D(u, x)$, $\mu_{D/H}(h, v) = \Sigma_{y \in V(H)} \mu_D(y, v)$.

If $A$ and $B$ are disjoint subsets of vertices of $D$, we use the notation $A \Rightarrow B$ to denote that $(a, b) \in A(D)$ for any choice of $a \in A$ and $b \in B$ and there is no arc from $B$ to $A$. An extended digraph $D[V_1, V_2, \ldots, V_n]$ of a digraph $D$ with vertices labeled, say, $1, 2, \ldots, n$ is obtained from $D$ by replacing every vertex $i$ by a set of $V_i$ independent (i.e., with no arc between them) vertices; more formally, $V(D[V_1, V_2, \ldots, V_n]) = \bigcup_{i=1}^n V_i$ and $V_i \Rightarrow V_j$ in $D[V_1, V_2, \ldots, V_n]$ if and only if

$(i,j) \in A(D)$. See Figure 1 for an example.

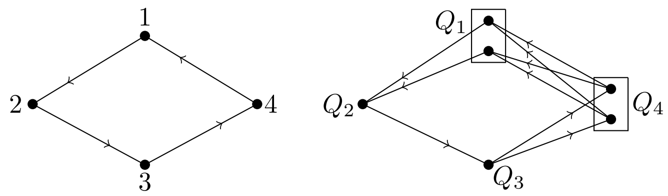



Figure 1 The digraphs $C_4$ and an extended 4-cycle $C_4[Q_1, Q_2, Q_3, Q_4]$

A digraph $T$ with vertex set $\{v_i : i \in [n]\}$ is a tournament if $\{(v_i, v_j), (v_j, v_i)\} \cap A(D) = 1$ for every $i \neq j \in [n]$. For a graph $G$, a digraph $D$ is called a biorientation of $G$ if $D$ is obtained from $G$ by replacing each edge $xy$ of $G$ by either $(x,y)$ or $(y,x)$ or the pair $(x,y)$ and $(y,x)$. Recall that a semicomplete digraph is a biorientation of a complete graph. A biorientation of a complete $p$-partite (multipartite) graph is a semicomplete $p$-partite (multipartite) digraph. Observe that semicomplete $p$-partite digraph is an extension of semicomplete digraph $D$ on $p$ vertices.

A digraph $D$ is arc-locally semicomplete if for every arc $(x,y)$ of $D$, the following two conditions hold:

(a) If $u \in N^-(x), v \in N^-(y)$ and $u \neq v$, then $u$ and $v$ are adjacent;

(b) If $u \in N^+(x), v \in N^+(y)$ and $u \neq v$, then $u$ and $v$ are adjacent.

**Theorem 1.1** ([13]) *A strong arc-locally semicomplete digraph is either semicomplete or semicomplete bipartite or an extension of a cycle.*

Next, we give the definition of hypo-semicomplete digraphs.

**Definition 1.2** *A digraph $D$ is hypo-semicomplete if for any vertex $v \in V(D)$, there is exactly one vertex which is not adjacent to $v$, and all other vertices are adjacent to $v$.*

It is easy to see that a hypo-semicomplete digraph is not semicomplete, for example, a 4-cycle is hypo-semicomplete but not semicomplete.

In [10], Bang-Jensen and Maddaloni proved the following results which will be used in our arguments.

**Theorem 1.3** ([10]) *A semicomplete multipartite digraph $D$ is supereulerian if and only if $D$ is strong and has an eulerian factor.*

**Theorem 1.4** ([10]) *Let $D$ be a digraph. If $\lambda(D) \geq \alpha(D)$, then $D$ has an eulerian factor.*

Bang-Jensen and Thomassé (2011, see [10]) proposed the following conjecture.

**Conjecture 1.5** *Let $D$ be a digraph. If $\lambda(D) \geq \alpha(D)$, then $D$ is supereulerian.*

Theorem 1.4 is an effort towards this conjecture. In [10], Conjecture 1.5 has been verified in several families of digraphs. In [10], Bang-Jensen and Maddaloni proved that if $\lambda(G) \geq \alpha(G)$ for a graph $G$, then $G$ is supereulerian.

The purpose of this research is to prove that Conjecture 1.5 holds for some extended digraphs and some families of digraphs. The main result is the following.

**Theorem 1.6** *Let $D$ be an extended cycle, an extended hamiltonian digraph, an arc-locally semicomplete digraph, an extended arc-locally semicomplete digraph, an extension of two kinds of eulerian digraph, a hypo-semicomplete digraph or an extended hypo-semicomplete digraph. If $\lambda(D) \geq \alpha(D)$, then $D$ is supereulerian.*

## 2. Main results

The main goal of this section is to prove Theorem 1.6. Throughout the rest of the paper, let $D$ be a digraph and $D'$ denote an extension of $D$. First, we study the necessary and sufficient condition involving an extended cycle to be supereulerian.

**Theorem 2.1** *An extended cycle $D'$ is supereulerian if and only if $D'$ is strong and has an eulerian factor.*

**Proof** Necessity is immediate: if there is a spanning eulerian subdigraph, then clearly $D'$ is strong and consists of the union of arc disjoint cycles spanning $V(D')$.

For sufficiency, let

$$D' = C_k[Q_1, Q_2, \ldots, Q_k]$$

be an extended cycle. The following is a procedure to produce a spanning eulerian subdigraph of $D'$, given an eulerian factor

$$\mathcal{C} = \{C^1, C^2, \ldots, C^r\}.$$

If $r = 1$, we are done, so assume that $r \geq 2$. By the definition of extended digraphs, we can conclude that the number of vertices in every cycle of $D'$ is multiples of $k$. So for every $i \in [r]$, every $C^i$ contains at least one vertex in each of $Q_1, Q_2, \ldots, Q_k$.

Form a minimal collection of vertex disjoint closed trails by merging those cycles of $\mathcal{C}$ having common vertices.

Let $\mathcal{S} = \{S_1, S_2, \ldots, S_h\}$ be the collection of vertex disjoint closed trails of $D'$ obtained after the above step is no longer applicable. Obviously $1 \leq h \leq r$. If $h = 1$ we are done, so assume that $h \geq 2$. Note that each $S_j$ of $\mathcal{S}$ has at least $k$ vertices, for every $j \in [h]$. Hence every $S_j$ contains at least one vertex in each of $Q_1, Q_2, \ldots, Q_k$.

Then we can choose $(y_j, x_j) \in A(S_j)$ such that $y_j \in V(Q_k), x_j \in V(Q_1)$. Note that

$$y_1 \neq y_2 \neq \cdots \neq y_h \quad \text{and} \quad x_1 \neq x_2 \neq \cdots \neq x_h.$$

Because if there are two vertices $y_s$ and $y_t$ in $V(Q_k)$ such that $y_s = y_t$, for some $s, t \in [h]$, then $S_s$ and $S_t$ have a common vertex and they can be merged into one closed trail, contrary to the minimality of $\mathcal{S}$. Hence $y_1 \neq y_2 \neq \cdots \neq y_h$. By the similar argument, we can get that $x_1 \neq x_2 \neq \cdots \neq x_h$.

Since every $S_j$ is a closed trail, $S_j - (y_j, x_j)$ is a $S_{j[x_j, y_j]}$ spanning trail of $S_j$. As

$$D' = C_k[Q_1, Q_2, \ldots, Q_k]$$

is an extended cycle and $Q_k \Rightarrow Q_1$, for every $m, n \in [h]$, $(y_m, x_n) \in A(D')$.

Now $(S_1 - (y_1, x_1) + (y_1, x_2)) \cup (S_2 - (y_2, x_2) + (y_2, x_3)) \cup \cdots \cup (S_h - (y_h, x_h) + (y_h, x_1))$, that is, $S_{1[x_1,y_1]} + (y_1, x_2) + S_{2[x_2,y_2]} + (y_2, x_3) + \cdots + S_{h[x_h,y_h]} + (y_h, x_1)$ is a spanning closed trail of $D'$ (see Figure 2). This completes the proof for Theorem 2.1. □

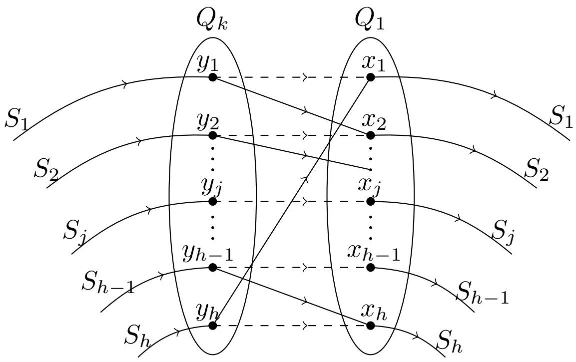



Figure 2 The spanning closed trail of $D$

By the definition of hamiltonian digraphs and Theorem 2.1, we can get the following necessary and sufficient condition.

**Corollary 2.2** *An extended hamiltonian digraph $D'$ is supereulerian if and only if $D'$ is strong and has an eulerian factor.*

Combining Theorems 1.1, 1.3 and 2.1, we can obtain the following necessary and sufficient condition immediately.

**Corollary 2.3** *Let $D$ be an arc-locally semicomplete digraph or an extension of arc-locally semicomplete digraph. Then $D$ is supereulerian if and only if $D$ is strong and has an eulerian factor.*

Next, we will present, if $D'$ is an extended eulerian digraph with an eulerian factor, then $D'$ is not necessarily supereulerian, as can be seen in the example below.

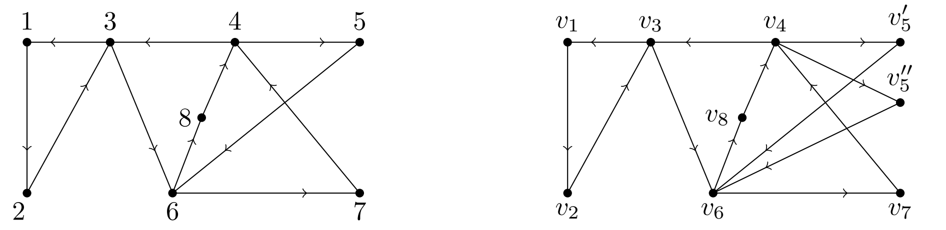
**Lemma 2.4** ([8]) *A digraph $D$ is nonsupereulerian if for some integer $m > 0$, $V(D)$ has vertex disjoint subsets $B, B_1, \ldots, B_m$ satisfying both of the following:*

*(i)* $N^-(B_i) \subseteq B$, *for* $i \in [m]$;

*(ii)* $|\partial^-(B)| \leq m - 1$.

Lemma 2.4 can be applied to find examples for digraph $D$ to be nonsupereulerian.

**Example 2.5** Let $D$ be an eulerian digraph with $V(D) = \{1, 2, \ldots, 8\}$ and let $D'$ be an extension of $D$ with $V(D') = \{v_1, v_2, v_3, v_4, v_5', v_5'', v_6, v_7, v_8\}$ (see Figure 3). Let $B, B_1, B_2, B_3$ be vertex disjoint subsets of $V(D')$ with $B = v_4$, $B_1 = \{v_1, v_2, v_3\}$, $B_2 = v_5'$ and $B_3 = v_5''$. We find that $N^-(B_i) \subseteq B$ for $i \in [3]$ and $|\partial^-(B)| = 2$. By Lemma 2.4, the digraph $D'$ is nonsupereulerian. And $D'$ has an eulerian factor $\{v_1v_2v_3v_1, v_4v_5'v_6v_8v_4, v_4v_5''v_6v_7v_4\}$.




An eulerian digraph $D$ An extended eulerian digraph $D'$

Figure 3 The digraph $D$ and $D'$

Example 2.5 indicates that if $D'$ is an extension of an eulerian digraph with an eulerian factor, then $D'$ is not necessarily supereulerian.

In the following, we present the definition of cycle decomposition and develop one result which will be used in our arguments.

A decomposition of a digraph $D$ is a family $\mathcal{F}$ of arc disjoint subdigraphs of $D$ such that $\cup_{F\in\mathcal{F}}A(F)=A(D)$. If the family $\mathcal{F}$ consists entirely of cycles, we call $\mathcal{F}$ a cycle decomposition of $D$.

**Theorem 2.6** ([1]) *A digraph $D$ admits a cycle decomposition if and only if $d_D^+(v)=d_D^-(v)$, for every $v\in V(D)$.*

Now we will discuss the supereulerian extension of two kinds of eulerian digraph.

**Theorem 2.7** *Let $D$ be an eulerian digraph satisfying for any two cycles of $D$, they have at most one common vertex. Then an extension of $D$ is supereulerian if and only if it is strong and has an eulerian factor.*

**Proof** By Theorem 2.6, $D$ has a cycle decomposition. Without loss of generality, we can let $D=\{C^1, C^2, \ldots, C^k\}$ such that $\cup_{l=1}^{k}A(C^l)=A(D)$. Let

$$C^l = x_1^l x_2^l \cdots x_{j-1}^l x_j^l x_{j+1}^l \cdots x_p^l$$

and define $S^l$ to be

$$S^l = x_{j1}^l x_{j+1}^l \cdots x_p^l x_1^l x_2^l \cdots x_{j-1}^l x_{j2}^l,$$

where $x_{j1}^l \neq x_{j2}^l$ and $l\in[k], j\in[p]$. Let $D'=D[V_1, V_2, \ldots, V_n]$ be the extension of $D$.

Necessity is immediate: if there is a spanning eulerian subdigraph, then $D'$ is strong and consists of the union of arc disjoint cycles spanning $V(D')$.

For sufficiency, suppose $D'$ is strong and has an eulerian factor $\mathcal{C}'$. Let $M$ be the multidigraph obtained from $D'$ by contracting each $V_i$ into a single vertex $v_i$, for every $i\in[n]$, it implies that $|A(M)|=|A(D')|$. Let $\mathcal{C}$ be the multidigraph obtained from $\mathcal{C}'$ by contracting each $V_i$ into a single vertex $v_i$, for every $i\in[n]$, this implies that $|A(\mathcal{C})|=|A(\mathcal{C}')|$. Obviously, $\mathcal{C}$ is a spanning sub-multidigraph of $M$. Then we can use the term $\mathcal{C}'|_M$ to denote $\mathcal{C}$.

Since for any two cycles of $D$, they have at most one common vertex. By the definition of extended digraphs, we can conclude that every cycle of $\mathcal{C}'$ in $D'$ is isomorphic to some $C^i$ or contains some $S^i$. Hence $\mathcal{C}'|_M$ is still strong. As each vertex in $V_i$ has at least one in-degree and out-degree in eulerian factor $\mathcal{C}'$, we have $d^+_{\mathcal{C}'|_M}(v_i) = d^-_{\mathcal{C}'|_M}(v_i) \geq |V_i|$. Thus, we can get from $\mathcal{C}'|_M$ a spanning closed trail of $D'$ by replacing $v_i$ by $|V_i|$ vertices spanning $V_i$, for every $i \in [n]$. This proves Theorem 2.7. $\square$

**Theorem 2.8** *Let $D$ be an eulerian digraph with two cycles which have at least two common vertices. Then an extension of $D$ is supereulerian if and only if it is strong and has an eulerian factor.*

**Proof** By Theorem 2.6, $D$ has a cycle decomposition. Without loss of generality, let $D = \{C^1, C^2\}$ such that $A(C^1) \cup A(C^2) = A(D)$. Let $D' = D[V_1, V_2, \ldots, V_n]$ be the extension of $D$.

Necessity is immediate: if there is a spanning eulerian subdigraph, then $D'$ is strong and consists of the union of arc disjoint cycles spanning $V(D')$.

For sufficiency, suppose $D'$ is strong and has an eulerian factor $\mathcal{C}'$. Let $M$ be the multidigraph obtained from $D'$ by contracting each $V_i$ into a single vertex $v_i$, for every $i \in [n]$, this implies that $|A(M)| = |A(D')|$. Let $\mathcal{C}$ be the multidigraph obtained from $\mathcal{C}'$ by contracting each $V_i$ into a single vertex $v_i$, for every $i \in [n]$, it implies that $|A(\mathcal{C})| = |A(\mathcal{C}')|$. Obviously, $\mathcal{C}$ is a spanning sub-multidigraph of $M$. Then we can use the term $\mathcal{C}'|_M$ to denote $\mathcal{C}$.

Since $D = \{C^1, C^2\}$ and by the definition of extended digraphs, $\mathcal{C}'|_M$ is still strong. As each vertex in $V_i$ has at least one in-degree and out-degree in eulerian factor $\mathcal{C}'$, we have

$$d^+_{\mathcal{C}'|_M}(v_i) = d^-_{\mathcal{C}'|_M}(v_i) \geq |V_i|.$$

Thus, we can get from $\mathcal{C}'|_M$ a spanning closed trail of $D'$ by replacing $v_i$ by $|V_i|$ vertices spanning $V_i$, for every $i \in [n]$. $\square$

In the following we will study when those kinds of conditions are necessary and sufficient for an extended hypo-semicomplete digraph to be supereulerian. We are going to use the following simple fact.

**Lemma 2.9** *Let $D' = D[V_1, V_2, \ldots, V_n]$ be an extended digraph. For any $i \in [n]$ and for any two vertex disjoint closed trails $S''_1$ and $S''_2$ of $D'$, if $V(S''_1) \cap V_i \neq \emptyset$ and $V(S''_2) \cap V_i \neq \emptyset$, then there exists a closed trail $S'$ in $D'$ such that $V(S') = V(S''_1) \cup V(S''_2)$.*

**Proof** Let $x_1 \in V(S''_1) \cap V_i$ and $x_2 \in V(S''_2) \cap V_i$. Since $S''_1$ and $S''_2$ are two vertex disjoint closed trails, we have $x_1 \neq x_2$. Let $y_1 \in N^+_{S''_1}(x_1)$ and $y_2 \in N^+_{S''_2}(x_2)$. By the definition of extended digraphs and $\{x_1, x_2\} \subseteq V_i$, we have $\{(x_1, y_2), (x_2, y_1)\} \subseteq A(D')$. Then the desired trail

$$S' := S''_1 \cup S''_2 - \{(x_1, y_1), (x_2, y_2)\} + \{(x_1, y_2), (x_2, y_1)\}. \ \square$$

**Lemma 2.10** *Suppose that $D$ is a hypo-semicomplete digraph. We have the following conclusions.*

*(i) For any three vertices $x, y, z \in V(D)$, there is an arc between $x$ and $\{y, z\}$.*

*(ii) For any $i \in [n]$ and for any two vertex disjoint closed trails $S_1$ and $S_2$ of $D$ ($|S_1| \geq 2$ and $|S_2| \geq 2$), if $(S_1, S_2)_D \neq \emptyset$ and $(S_2, S_1)_D \neq \emptyset$, then there exists a closed trail $S$ in $D$ such that $V(S) = V(S_1) \cup V(S_2)$.*

**Proof** (i) Suppose there is no arc between $x$ and $y$, by Definition 1.2, there is an arc between $x$ and $z$.

(ii) Consider the bipartite digraph $H$ with partitions $V(S_1), V(S_2)$, and arcs $(S_1, S_2)_D \cup (S_2, S_1)_D$. As $|S_1| \geq 2, |S_2| \geq 2$ and (i), we can get that every vertex of $H$ has positive degree. Thus, we consider two cases.

**Case 1** Every vertex of $H$ has positive in- and out-degree.

Then, clearly, $H$ contains a cycle $C$ that connects $S_1$ and $S_2$, so that $C \cup S_1 \cup S_2$ is the desired trail.

**Case 2** There is a vertex of $H$ with out-degree (or in-degree) equal to 0.

Let $S_1 = v_1 v_2 \cdots v_h v_1$ and $S_2 = w_1 w_2 \cdots w_k w_1$. Assume, without loss of generality, that there is a vertex of $S_1$ with no out-neighbor in $S_2$. As $(S_1, S_2)_D \neq \emptyset$, there exists another vertex of $S_1$ with out-neighbors in $S_2$. Therefore, there is an index $i \in [h]$ such that $d_H^+(v_i) = 0$ and $d_H^+(v_{i-1}) > 0$. Let $w_j \in N_H^+(v_{i-1})$, for $j \in [k]$. By (i), there exists one of the arcs $(w_j, v_i)$ and $(w_{j-1}, v_i)$. In the first case the desired trail is

$$S := \{(v_{i-1}, w_j), (w_j, v_i)\} \cup S_1 \cup S_2 - (v_{i-1}, v_i),$$

and in the second case

$$S := \{(v_{i-1}, w_j), (w_{j-1}, v_i)\} \cup S_1 \cup S_2 - (v_{i-1}, v_i) - (w_{j-1}, w_j). \quad \square$$

For any two vertex disjoint closed trails $S_1'$ and $S_2'$ of $D'$, if there are $i_{1_j}, i_{2_j}, \ldots, i_{t_j} \in [n]$ such that $V(S_j') = V_{i_{1_j}} \cup V_{i_{2_j}} \cup \cdots \cup V_{i_{t_j}}$, for $j \in [2], t \geq 2$, then we can get that Lemma 2.10 holds for extended hypo-semicomplete digraph as well. Thus, we have the following.

**Corollary 2.11** *Let $D' = D[V_1, V_2, \ldots, V_n]$ be an extended hypo-semicomplete digraph.*

*(i) For any $x, y, z \in [n]$, let $V_x, V_y, V_z$ be independent vertices of $D'$. Then there is at least one arc between $V_x$ and $\{V_y, V_z\}$.*

*(ii) For any two vertex disjoint closed trails $S_1'$ and $S_2'$ of $D'$ and for $j \in [2], t \geq 2$, if there are $i_{1_j}, i_{2_j}, \ldots, i_{t_j} \in [n]$ such that $V(S_j') = V_{i_{1_j}} \cup V_{i_{2_j}} \cup \cdots \cup V_{i_{t_j}}$, $(S_1', S_2')_{D'} \neq \emptyset$ and $(S_2', S_1')_{D'} \neq \emptyset$, then there exists a closed trail $S$ in $D'$ such that $V(S) = V(S_1') \cup V(S_2')$.*

**Theorem 2.12** *An extended hypo-semicomplete digraph $D'$ is supereulerian if and only if $D'$ is strong and has an eulerian factor.*

**Proof** Necessity is immediate: if there is a spanning eulerian subdigraph, then clearly $D'$ is strong and consists of the union of arc disjoint cycles spanning $V(D')$.

For sufficiency, let $D' = D[V_1, V_2, \ldots, V_n]$ be an extended hypo-semicomplete digraph with an eulerian factor $\mathcal{C}$. Since an eulerian factor only contains proper cycles, we have every cycle of $\mathcal{C}$

on at least two vertices. The following is a procedure to produce a spanning eulerian subdigraph of $D'$.

Form a minimal collection of vertex disjoint closed trails by merging those cycles of $\mathcal{C}$ having common vertices. For any two vertex disjoint closed trails $S''_1$ and $S''_2$ with $V(S''_1) \cap V_i \neq \emptyset$ and $V(S''_2) \cap V_i \neq \emptyset$, for any $i \in [n]$, join $S''_1$ and $S''_2$ into a closed trail $S'$ as in Lemma 2.9. After that, let $\{S'_1, S'_2, \ldots, S'_{h'}\}$ be collection of vertex disjoint closed trails of $D'$ and $\cup_{j=1}^{h'} V(S_j) = V(D)$. For any $j \in [h'], t \geq 2$, there are $i_{1_j}, i_{2_j}, \ldots, i_{t_j} \in [n]$ such that $V(S'_j) = V_{i_{1_j}} \cup V_{i_{2_j}} \cup \cdots \cup V_{i_{t_j}}$. Then, for any $l, k \in [h']$, $(S'_l, S'_k)_{D'} \neq \emptyset$ and $(S'_k, S'_l)_{D'} \neq \emptyset$, join $S'_l$ and $S'_k$ into a closed trail $S$ as in Corollary 2.11 (ii).

Let $\mathcal{S} = \{S_1, S_2, \ldots, S_h\}$ be the collection of vertex disjoint closed trails of $D'$ obtained after the above step is no longer applicable. Note that $|S_j| \geq 2$ (since every cycle of $\mathcal{C}$ is on at least two vertices), for every $j \in [h]$. If $h = 1$, we are done, so assume that $h \geq 2$. By the above step, for every $i \in [n], j \in [h], t \geq 2$, we can conclude that

$$\text{for } \forall\ S_j \in \mathcal{S}, \exists\ i_{1_j}, i_{2_j}, \ldots, i_{t_j} \in [n] \text{ such that } V(S_j) = V_{i_{1_j}} \cup V_{i_{2_j}} \cup \cdots \cup V_{i_{t_j}}, \tag{2.1}$$

and

$$\text{there is } (S_p, S_q)_{D'} \neq \emptyset \text{ or } (S_q, S_p)_{D'} \neq \emptyset \text{ in } D', \text{ for every } p \neq q \in [h]. \tag{2.2}$$

By Corollary 2.11 (i), the definition of extended digraphs and (2.1), we have

$$\begin{gathered}\text{every vertex of } S_p \text{ is at most not adjacent to one } V_{i_{t_q}} \text{ of } S_q,\\ \text{but is adjacent to all other vertices of } S_q, \text{ for every } p \neq q \in [h] \text{ and } i_{t_q} \in [n].\end{gathered} \tag{2.3}$$

Let $T$ be the digraph with vertices $v_1, \ldots, v_h$ and arcs $\{(v_p, v_q) | (S_p, S_q)_{D'} \neq \emptyset\}$, for $p, q \in [h]$. By (2.2), $T$ has no 2-cycle. By the fact that $D'$ is strong and (2.2), $T$ is a strong tournament and, thus it has a hamiltonian cycle $C$. Suppose, without loss of generality, that $C = v_1 v_2 \cdots v_h v_1$. Let $w_1 \in S_1$. Choose $w_j \in S_j \cap N^+(w_{j-1})$ for $j = 2, \ldots, h$. Note that $S_j \cap N^+(w_{j-1})$ for all $j \in [h]$, by $|S_j| \geq 2$ and (2.3), there is an arc between $w_{j-1}$ and $S_j$. As $T$ has no 2-cycle and $C = v_1 v_2 \cdots v_h v_1$, there is an arc from $w_{j-1}$ to $S_j$.

Now let $w_0$ be an out-neighbor of $w_1$ in $S_1$. Then $\{w_0, w_1\} \cap V_i = 1$, for some $i \in [n]$. By (2.3), there is an arc between $w_h$ and $\{w_0, w_1\}$. As $T$ has no 2-cycle and $C = v_1 v_2 \ldots v_h v_1$, we have that one of the arcs $(w_h, w_0)$ and $(w_h, w_1)$ is in $D'$. In the first case

$$\bigcup_{j=1}^{h} S_j \cup \bigcup_{j=1}^{h-1} (w_j, w_{j+1}) \cup (w_h, w_0) - (w_1, w_0)$$

is a spanning closed trail of $D'$. In the second case

$$\bigcup_{j=1}^{h} S_j \cup \bigcup_{j=1}^{h-1} (w_j, w_{j+1}) \cup (w_h, w_1)$$

is a spanning closed trail of $D'$. This proves Theorem 2.12. $\square$

We observe that the result of extended hypo-semicomplete digraph holds for hypo-semicomplete digraph as well. Thus, we have the following.

**Corollary 2.13** *A hypo-semicomplete digraph $D$ is supereulerian if and only if $D$ is strong and has an eulerian factor.*

Combining all the results of this section and Theorem 1.4, we complete the proof for Theorem 1.6.

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