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Gronwll-Bellman Type Nonlinear Sums-Difference Inequalities and Applications in Difference Equations

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Abstract In this paper, we establish some general sums-difference inequalities with two variables. The inequalities involve finite sum and every term contains the unknown function of the composite function with the power of p_i. In the end, we study boundedness of the solution of the difference equations as applications.

Keywords sum-difference inequality; power; monotonicity; boundary value problem; boundedness

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1. Introduction

Integral inequalities provide a very useful and important device in the study of many qualitative as well as quantitative properties of solutions of differential equations. Various generalizations of Gronwall-Bellman type inequality [1, 2] and their applications have attracted great interests of many mathematicians [3–11]. Some recent works can be found, e.g., in [12–14] and some references therein. Agarwal et al. [15] investigated the inequality

$$u(t) \le a(t) + \sum_{i=1}^{n} \int_{b_i(t_0)}^{b_i(t)} g_i(t,s) w_i(u(s))\mathrm{d}s, \quad t_0 \le t < t_1.$$

Chen et al. [16] studied the following retarded integral inequality

$$\begin{aligned}\psi(u(x,y)) \le & c + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} g(s,t)u(s,t)\mathrm{d}t\mathrm{d}s + \\ & \int_{\gamma(x_0)}^{\gamma(x)} \int_{\delta(y_0)}^{\delta(y)} f(s,t)u(s,t)\varphi(u(s,t))\mathrm{d}t\mathrm{d}s,\end{aligned}$$

where c is a constant. Wang et al. [17] investigated the inequality

$$\begin{aligned}\psi(u(x,y)) \le & a(x,y) + \sum_{i=1}^{n} \Big\{ \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} u^q(s,t) g_i(x,y,s,t)\mathrm{d}s\mathrm{d}t + \\ & \int_{\delta_i(x_0)}^{\delta_i(x)} \int_{\gamma_i(y_0)}^{\gamma_i(y)} u^q(s,t) f_i(x,y,s,t)\varphi_i(u(s,t))\mathrm{d}s\mathrm{d}t \Big\}.\end{aligned}$$

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Zhou et al. [18] studied the following retarded integral inequality

$$u(t) \leq a(t) + \sum_{i=1}^{n} \Big\{ \int_{b_i(t_0)}^{b_i(t)} f_i(t,s)\phi_i(u(s))\mathrm{d}s \Big\}^{p_i},$$

where $p_i \geq 1, a, b_i, f_i, \phi_i, u$ are nonnegative continuous functions for $i = 1, 2, \ldots, n$.

With the progress of the theory of difference equations, more attentions are paid to some discrete versions of Gronwall type inequalities (e.g., [19, 20] for some early works). Some recent works can be found, e.g., in [21–26] and some references therein. Cheung [27] discussed the inequality

$$u^p(m,n) \leq c + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)u(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)u(s,t)\varphi(u(s,t)),$$

where $c \geq 0$, and a, b are nonnegative real-valued functions in $\mathbb{Z}_+^2$, and φ is a continuous nondecreasing function with $\varphi(r) > 0$, for $r > 0$. Ma and Cheung [28] studied the inequality

$$\psi(u(m,n)) \leq a(m,n) + c(m,n)\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} \psi'(u(s,t))[d(s,t)w(u(s,t)) + e(s,t)].$$

Wang et al. [29] investigated the inequality

$$\psi(u(m,n)) \leq c(m,n) + \sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_i(m,n,s,t)\varphi_i(u(s,t)).$$

Zheng et al. [30] studied the inequality

$$\begin{aligned} u^p(m,n) \leq & c(m,n) + \sum_{i=1}^{l_1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\Big[b_i(s,t,m,n)u^{q_i}(s,t) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c_i(\xi,\eta,m,n)u^{r_i}(\xi,\eta)\Big] + \\ & \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\Big[d_i(s,t,m,n)u^{h_i}(s,t) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e_i(\xi,\eta,m,n)u^{j_i}(\xi,\eta)\Big]. \end{aligned}$$

Feng et al. [31] discussed the inequalities including four sums

$$\begin{aligned} u^p(m,n) \leq & c(m,n) + \sum_{s=m_0}^{m-1} w(s,n)u^p(m,n) \\ & \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\Big[b(s,t,m,n)u^{q}(s,t) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} c(\xi,\eta,m,n)u^{r}(\xi,\eta)\Big] + \\ & \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\Big[d(s,t,m,n)u^{h}(s,t) + \sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t} e(\xi,\eta,m,n)u^{j}(\xi,\eta)\Big]. \end{aligned}$$

In this paper, we establish some new more general form of sums-difference inequalities, give the upper bound estimation and apply the obtained results to the boundedness of the solution of the difference equations.

2. Main result

Throughout this paper, $\mathbb{R}$ denotes the set of all real numbers. Let $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{N}_0 := \{0, 1, \ldots\}$. $m_1, n_1 \in \mathbb{N}_0 \cup \infty$ are given numbers, $I := [0, m_1) \cap \mathbb{N}_0$ and $J := [0, n_1) \cap \mathbb{N}_0$ are two fixed lattices of integer points in $\mathbb{R}$, $\Lambda := I \times J \subset \mathbb{N}_0^2$. For any $(s, t) \in \Lambda$, let $\Lambda_{(s,t)}$ denote the sublattice $[0, s) \times [0, t) \cap \Lambda$ of Λ. For functions $w(m), z(m, n), m, n \in \mathbb{N}_0$, let $\Delta w(m) := w(m + 1) - w(m)$ and $\Delta_1 z(m, n) := z(m+1, n) - z(m, n)$. Obviously, the linear difference equation $\Delta x(m) = b(m)$ with the initial condition $x(0) = 0$ has the solution $\sum_{s=0}^{m-1} b(s)$. For convenience, in the sequel we define that $\sum_{s=0}^{0-1} b(s) = 0$.

Consider

$$\psi(u(m, n)) \leq c(m, n) + \sum_{i=1}^{k} \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{j=0}^{s-1} \sum_{l=0}^{t-1} h_i(s, t, j, l) u^q \varphi_i^{p_i}(u(j, l)),$$

and suppose that

(H_1) ψ is a strictly increasing continuous function on $\mathbb{R}_+$, $\psi(u) > 0$ for all $u > 0$;

(H_2) All φ_i $(i = 1, 2, \ldots, k)$ are continuous functions on $\mathbb{R}_+$ and positive on $(0, \infty)$;

(H_3) $c(m, n) > 0$ on $I \times J$, and $c(m, n)$ is nondecreasing in each variable;

(H_4) $p_i > 1, q > 0$ are constants;

(H_5) All h_i $(i = 1, 2, \ldots, k)$ are nonnegative functions on $\Lambda \times \Lambda$.

We technically consider a sequence of functions $w_i(s)$, which can be calculated recursively by

$$\begin{cases} w_1(s) := \max_{\tau \in [0,s]} \varphi_1(\tau), \\ w_{i+1}(s) := \max_{\tau \in [0,s]} \{\frac{\varphi_{i+1}(\tau)}{w_i(\tau)}\} w_i(s), \quad i = 1, 2, \ldots, k - 1. \end{cases} \tag{2.1}$$

We define the functions:

$$\Psi(u) := \int_0^u \frac{\mathrm{d}s}{(\psi^{-1}(s))^q}, \quad u > 0, \tag{2.2}$$

$$W_i(u) := \int_1^u \frac{\mathrm{d}s}{w_i^{p_i}(\psi^{-1}(\Psi^{-1}(s)))}, \quad i = 1, 2, \ldots, k, \ u > 0. \tag{2.3}$$

Obviously, both Ψ and W_i are strictly increasing and continuous functions. Let Ψ^{-1}, W_i^{-1} denote Ψ, W_i inverse function, respectively. Then both Ψ^{-1} and W_i^{-1} are also continuous and increasing functions. Furthermore, let

$$\tilde{h}_i(m, n, s, t) := \max_{(\tau,\xi) \in [0,m] \times [0,n]} h_i(m, n, s, t), \tag{2.4}$$

$$\tilde{f}_i(m, n, s, t) := \max_{(\tau,\xi) \in [0,m] \times [0,n]} f_i(m, n, s, t),$$

which are nondecreasing in m and n for each fixed s and t and satisfies

$$\tilde{h}_i(m, n, s, t) \geq h_i(m, n, s, t) \geq 0, \quad \text{for all } i = 1, 2, \ldots, k.$$

Lemma 2.1 *Suppose w is continuous and positive functions on $\mathbb{R}_+$, f is nonnegative function on $\Lambda \times \Lambda$, u is a nonnegative function on Λ, then we can obtain*

$$\sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{j=0}^{s-1} \sum_{l=0}^{t-1} f(s, t, j, l) w(u(j, l)) = \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} w(u(s, t)) \sum_{j=s+1}^{m-1} \sum_{l=t+1}^{n-1} f(j, l, s, t).$$

Proof We use mathematical induction with respect to m and n. If $m=n=2$, we obtain

$$\sum_{s=0}^{1}\sum_{t=0}^{1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1} f(s,t,j,l)w(u(j,l)) = f(1,1,0,0)w(u(0,0)),$$

$$\sum_{s=0}^{1}\sum_{t=0}^{1} w(u(s,t)) \sum_{j=s+1}^{1}\sum_{l=t+1}^{1} f(j,l,s,t) = w(u(0,0))f(1,1,0,0).$$

Thus

$$\sum_{s=0}^{1}\sum_{t=0}^{1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1} f(s,t,j,l)w(u(j,l)) = \sum_{s=0}^{1}\sum_{t=0}^{1} w(u(s,t)) \sum_{j=s+1}^{1}\sum_{l=t+1}^{1} f(j,l,s,t).$$

It means that the lemma is true for $m=n=2$. Suppose that the lemma is true for $m=m_1, n=n_1$, that is

$$\sum_{s=0}^{m_1-1}\sum_{t=0}^{n_1-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1} f(s,t,j,l)w(u(j,l)) = \sum_{s=0}^{m_1-1}\sum_{t=0}^{n_1-1} w(u(s,t)) \sum_{j=s+1}^{m_1-1}\sum_{l=t+1}^{n_1-1} f(j,l,s,t).$$

Consider $m=m_1+1, n=n_1+1$, then we have

$$\begin{aligned}
&\sum_{s=0}^{m_1}\sum_{t=0}^{n_1} w(u(s,t)) \sum_{j=s+1}^{m_1}\sum_{l=t+1}^{n_1} f(j,l,s,t)\\
&= \sum_{s=0}^{m_1-1}\sum_{t=0}^{n_1-1} w(u(s,t)) \sum_{j=s+1}^{m_1}\sum_{l=t+1}^{n_1} f(j,l,s,t)\\
&= \sum_{s=0}^{m_1-1}\sum_{t=0}^{n_1-1} w(u(s,t)) \sum_{j=s+1}^{m_1-1}\sum_{l=t+1}^{n_1-1} f(j,l,s,t) + \sum_{s=0}^{m_1-1}\sum_{t=0}^{n_1-1} w(u(s,t))f(m_1,n_1,s,t)\\
&= \sum_{s=0}^{m_1-1}\sum_{t=0}^{n_1-1} w(u(s,t)) \sum_{j=s+1}^{m_1-1}\sum_{l=t+1}^{n_1-1} f(j,l,s,t) + \sum_{j=0}^{m_1-1}\sum_{l=0}^{n_1-1} f(m_1,n_1,j,l)w(u(j,l))\\
&= \sum_{s=0}^{m_1}\sum_{t=0}^{n_1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1} f(s,t,j,l)w(u(j,l)).
\end{aligned}$$

Using the inductive assumption, thus

$$\sum_{s=0}^{m_1}\sum_{t=0}^{n_1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1} f(s,t,j,l)w(u(j,l)) = \sum_{s=0}^{m_1}\sum_{t=0}^{n_1} w(u(s,t)) \sum_{j=s+1}^{m_1}\sum_{l=t+1}^{n_1} f(j,l,s,t).$$

It implies that it is true for $m=m_1+1, n=n_1+1$. Therefore, it is true for any natural number $m\ge 2, n\ge 2$. $\Box$

Theorem 2.2 *Suppose that* (H_1-H_5) *hold and* $u(m,n)$ *is a nonnegative function on* Λ *satisfying*

$$\psi(u(m,n)) \le c(m,n) + \sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1} h_i(s,t,j,l)u^q\varphi_i^{p_i}(u(j,l)). \tag{2.5}$$

Then

$$u(m,n) \le \psi^{-1}\Big\{\Psi^{-1}\big[W_k^{-1}\big(W_k(E_k(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_k(m,n,s,t)\big)\big]\Big\}, \tag{2.6}$$

for $(m,n)\in\Lambda_{(M_1,N_1)}$, *where*

$$E_1(m,n):=\Psi(c(m,n)),$$

$$E_i(m,n):=W_{i-1}^{-1}\Big(W_{i-1}(E_{i-1}(m,n))+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_{i-1}(m,n,s,t)\Big),\quad i=2,3,\ldots,k,$$

and $(M_1,N_1)\in\Lambda$ *is arbitrarily given on the boundary of the lattice*

$$\mathcal{R}:=\Big\{(m,n)\in\Lambda: W_i(E_i(m,n))+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_i(m,n,s,t)\le\int_1^{\infty}\frac{\mathrm{d}s}{w_i(\psi^{-1}(\Psi^{-1}(s)))},$$
$$W_i^{-1}\Big(W_i(E_i(m,n))+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_i(m,n,s,t)\Big)\le\int_1^{\infty}\frac{\mathrm{d}s}{(\psi^{-1}(s))^q},\ i=1,2,\ldots,k\Big\}.$$

Proof We monotonize some given functions φ_i in the sums. The sequence $w_i(s)$ defined by $\varphi_i(s)$ in (2.1) are nondecreasing and nonnegative functions and satisfy $w_i^{p_i}(s)\ge\varphi_i^{p_i}(s)$, $i=1,2,\ldots,k$. Moreover, the ratio $w_{i+1}^{p_i}(s)/w_i^{p_i}(s)$ are also nondecreasing, $i=1,2,\ldots,k$. By (2.4), (2.5), from (2.1), we have

$$\psi(u(m,n))\le c(m,n)+\sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{h}_i(s,t,j,l)u^q(s,t)w_i^{p_i}(u(j,l)).\tag{2.7}$$

By H_3, from (2.7), we have

$$\psi(u(m,n))\le c(M,N)+\sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{h}_i(s,t,j,l)u^q(s,t)w_i^{p_i}(u(j,l)),\tag{2.8}$$

for all $(m,n)\in\Lambda_{(M,\,N)}$, where $0\le M\le M_1$ and $0\le N\le N_1$ are chosen arbitrarily. Let $z(m,n)$ denote the function on the right-hand side of (2.8), which is a nonnegative and nondecreasing function on $\Lambda_{(M,\,N)}$ and $z(0,n)=C(M,N)$. Then we obtain the equivalent form of (2.8)

$$u(m,n)\le\psi^{-1}(z(m,n)),\quad\forall(m,n)\in\Lambda_{(M,\,N)}.\tag{2.9}$$

Since w_i is nondecreasing and satisfies $w_i(u)>0$, for $u>0$. By the definition of z and (2.9), from (2.8), we have

$$\begin{aligned}\Delta_1 z(m,n)=&\sum_{i=1}^{k}\sum_{t=0}^{n-1}\sum_{j=0}^{m-1}\sum_{l=0}^{t-1}\tilde{h}_i(m,t,j,l)u^q(m,t)(w_i(u(m,l)))^{p_i}\\ \le&\sum_{i=1}^{k}\sum_{t=0}^{n-1}\sum_{j=0}^{m-1}\sum_{l=0}^{t-1}\tilde{h}_i(m,t,j,l)(\psi^{-1}(z(m,t)))^q(w_i(\psi^{-1}(z(m,l))))^{p_i}.\end{aligned}\tag{2.10}$$

Using the monotonicity of ψ^{-1} and z, from (2.10), we have

$$\Delta_1 z(m,n)\le(\psi^{-1}(z(m,n)))^q\sum_{i=1}^{k}\sum_{t=0}^{n-1}\sum_{j=0}^{m-1}\sum_{l=0}^{t-1}\tilde{h}_i(m,t,j,l)(w_i(\psi^{-1}(z(m,l))))^{p_i}.\tag{2.11}$$

That is

$$\frac{\Delta_1 z(m,n)}{(\psi^{-1}(z(m,n)))^q}\le\sum_{i=1}^{k}\sum_{t=0}^{n-1}\sum_{j=0}^{m-1}\sum_{l=0}^{t-1}\tilde{h}_i(m,t,j,l)(w_i(\psi^{-1}(z(m,l))))^{p_i}.\tag{2.12}$$

By the mean-value theorem for integrals, for arbitrarily given $(m,n),(m+1,n)\in\Lambda_{(M,N)}$, in the open interval $(z(m,n),z(m+1,n))$, there exists ξ, which satisfies

$$\begin{aligned}\Psi(z(m+1,n))-\Psi(z(m,n))=\int_{z(m,n)}^{z(m+1,n)}\frac{\mathrm{d}s}{(\psi^{-1}(s))^q}&=\frac{\Delta_1 z(m,n)}{(\psi^{-1}(\xi))^q}\\ &\le\frac{\Delta_1 z(m,n)}{(\psi^{-1}(z(m,n)))^q},\end{aligned}\tag{2.13}$$

where we use the definition of Ψ in (2.2). From (2.12) and (2.13), we obtain

$$\Psi(z(m+1,n))\le\Psi(z(m,n))+\sum_{i=1}^{k}\sum_{t=0}^{n-1}\sum_{j=0}^{m-1}\sum_{l=0}^{t-1}\tilde{h}_i(m,t,j,l)(w_i(\psi^{-1}(z(m,l))))^{p_i}.\tag{2.14}$$

Keep n fixed and substitute m with s in (2.14). Then, taking the sums on both sides of (2.14) over $s=0,1,\ldots,m-1$, we have

$$\begin{aligned}\Psi(z(m,n))&\le\Psi(z(0,n))+\sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{h}_i(s,t,j,l)(w_i(\psi^{-1}(z(j,l))))^{p_i}\\ &\le\Psi(c(M,N))+\sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{h}_i(s,t,j,l)(w_i(\psi^{-1}(z(j,l))))^{p_i}\\ &=C_k(M,N)+\sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{h}_i(s,t,j,l)(w_i(\psi^{-1}(z(j,l))))^{p_i},\end{aligned}\tag{2.15}$$

where

$$C_k(M,N)=\Psi(c(M,N)).\tag{2.16}$$

Let

$$v(m,n)=\Psi(z(m,n)).\tag{2.17}$$

From (2.15), we have

$$v(m,n)\le C_k(M,N)+\sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{h}_i(s,t,j,l)(w_i(\psi^{-1}(\Psi^{-1}(v(j,l)))))^{p_i},\tag{2.18}$$

for all $(m,n)\in\Lambda_{(M,N)}$. Using Lemma 2.1, (2.18) can be written as

$$v(m,n)\le C_k(M,N)+\sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_i(m,n,s,t)(w_i(\psi^{-1}(\Psi^{-1}(v(s,t)))))^{p_i},\tag{2.19}$$

where $\tilde{g}_i(m,n,s,t)=\sum_{j=s+1}^{m-1}\sum_{l=t+1}^{n-1}\tilde{h}_i(j,l,s,t)$. Obviously, $\tilde{g}_i(m,n,s,t)$, $i=1,2,\ldots,k$ are nondecreasing in m and n for each fixed s and t and $\tilde{g}_i(m,n,s,t)\ge 0$. Then from (2.19), we have

$$v(m,n)\le C_k(M,N)+\sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_i(M,N,s,t)w_i^{p_i}(\psi^{-1}(\Psi^{-1}(v(s,t)))),\tag{2.20}$$

for all $(m,n)\in\Lambda_{(M,N)}$.

From (2.20), we can conclude that

$$v(m,n) \le W_k^{-1}\Big(W_k(E_k(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_k(M,N,s,t)\Big), \tag{2.21}$$

for all $(m,n) \in \Lambda_{(M,\,N)}$, where

$$E_i(M,N) := W_{i-1}^{-1}\Big(W_{i-1}(E_{i-1}(M,N)) + \sum_{s=0}^{M-1}\sum_{t=0}^{N-1}\tilde{g}_{i-1}(M,N,s,t)\Big), \quad i = 2,\dots,k, \tag{2.22}$$

$$E_1(M,N) := C_1(M,N).$$

For $k=1$, let $z_1(m,n)$ denote the function on the right-hand side of (2.20), which is a nonnegative and nondecreasing function on $\Lambda_{(M,\,N)}$, $z_1(0,n) = C_1(M,N)$ and $v(m,n) \le z_1(m,n)$. Then we get

$$\begin{aligned}\Delta_1 z_1(m,n) &= \sum_{t=0}^{n-1}\tilde{g}_1(M,N,s,t)(w_1(\psi^{-1}(\Psi^{-1}(v(s,t)))))^{p_1}\\ &\le \sum_{t=0}^{n-1}\tilde{g}_1(M,N,s,t)(w_1(\psi^{-1}(\Psi^{-1}(z_1(s,t)))))^{p_1},\end{aligned} \tag{2.23}$$

for all $(m,n) \in \Lambda_{(M,\,N)}$. From (2.23), we have

$$\frac{\Delta_1 z_1(m,n)}{w_1^{p_1}(\psi^{-1}(\Psi^{-1}(z_1(m,n))))} \le \sum_{t=0}^{n-1}\tilde{g}_1(M,N,m,t). \tag{2.24}$$

By the mean-value theorem for integrals, there exists ξ in the open interval $(z_1(m,n),\ z_1(m+1,n))$, for arbitrarily given $(m,n), (m+1,n) \in \Lambda_{(M,N)}$, such that

$$\begin{aligned}W_1(z_1(m+1,n)) - W_1(z_1(m,n)) &= \int_{z_1(m,n)}^{z_1(m+1,n)} \frac{\mathrm{d}s}{w_1^{p_1}(\psi^{-1}(\Psi^{-1}(s)))}\\ = \frac{\Delta_1 z_1(m,n)}{w_1^{p_1}(\psi^{-1}(\Psi^{-1}(\xi)))} &\le \frac{\Delta_1 z_1(m,n)}{w_1^{p_1}(\psi^{-1}(\Psi^{-1}(z_1(m,n))))}.\end{aligned} \tag{2.25}$$

From (2.24) and (2.25), we have

$$W_1(z_1(m+1,n)) \le W_1(z_1(m,n)) + \sum_{t=0}^{n-1}\tilde{g}_1(M,N,m,t). \tag{2.26}$$

Keep n fixed and substitute m with s in (2.26). Then, taking the sums on both sides of (2.26) over $s = 0,1,\dots,m-1$, we have

$$\begin{aligned}W_1(z_1(m,n)) &\le W_1(z_1(0,n)) + \sum_{s=m_0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_1(M,N,s,t)\\ &= W_1(C_1(M,N)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_1(M,N,s,t),\end{aligned} \tag{2.27}$$

for all $(m,n) \in \Lambda_{(M,\,N)}$. Using $v(m,n) \le z_1(m,n)$, from (2.27), we get

$$v(m,n) \le z_1(m,n) \le W_1^{-1}\Big(W_1(C_1(M,N)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_1(M,N,s,t)\Big), \tag{2.28}$$

for all $(m,n) \in \Lambda_{(M,\,N)}$. This proves that (2.21) is true for $k=1$.

Next, we make the inductive assumption that (2.21) is true for $k=l$, then

$$v(m,n) \le W_l^{-1}\Big(W_l(E_l(M,N)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_l(M,N,s,t)\Big), \tag{2.29}$$

for all $(m,n) \in \Lambda_{(M,N)}$, where

$$E_1(M,N) := C_l(M,N),$$
$$E_i(M,N) := W_{i-1}^{-1}\Big(W_{i-1}(E_{i-1}(M,N)) + \sum_{s=0}^{M-1}\sum_{t=0}^{N-1}\tilde{g}_{i-1}(M,N,s,t)\Big),\ i=2,3,\dots,l.$$

Now we consider

$$v(m,n) \le C_{l+1}(M,N) + \sum_{i=1}^{l+1}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_i(M,N,s,t)w_i^{p_i}(\psi^{-1}(\Psi^{-1}(v(s,t)))), \tag{2.30}$$

for all $(m,n) \in \Lambda_{(M,N)}$. Let $z_2(m,n)$ denote the nonnegative and nondecreasing function of the right-hand of (2.30). Then $z_2(0,n) = C_{l+1}(M,N)$ and $v(m,n) \le z_2(m,n)$.

Let

$$\phi_{i+1}(u) := \frac{w_{i+1}(u)}{w_1^{p_1/p_{i+1}}(u)}, \quad i=1,2,\dots,l. \tag{2.31}$$

By (2.1), we conclude that ϕ_i, $i=1,2,\dots,l+1$ are nondecreasing functions.

From (2.30), we have

$$\begin{aligned}
&\frac{\Delta_1 z_2(m,n)}{w_1^{p_1}(\psi^{-1}(\Psi^{-1}(z_2(m,n))))}\\
&\quad= \frac{\sum_{i=1}^{l+1}\sum_{t=0}^{n-1}\tilde{g}_i(M,N,m,t)w_i^{p_i}(\psi^{-1}(\Psi^{-1}(v(m,t))))}{w_1^{p_1}(\psi^{-1}(\Psi^{-1}(z_2(m,n))))}\\
&\quad\le \frac{\sum_{i=1}^{l+1}\sum_{t=0}^{n-1}\tilde{g}_i(M,N,m,t)w_i^{p_i}(\psi^{-1}(\Psi^{-1}(z_2(m,t))))}{w_1^{p_1}(\psi^{-1}(\Psi^{-1}(z_2(m,n))))}\\
&\quad\le \sum_{t=0}^{n-1}\tilde{g}_1(M,N,m,t) + \sum_{i=2}^{l+1}\sum_{t=0}^{n-1}\tilde{g}_i(M,N,m,t)\phi_i^{p_i}(\psi^{-1}(\Psi^{-1}(z_2(m,t))))\\
&\quad= \sum_{t=0}^{n-1}\tilde{g}_1(M,N,m,t) + \sum_{i=1}^{l}\sum_{t=0}^{n-1}\tilde{g}_{i+1}(M,N,m,t)\phi_{i+1}^{p_{i+1}}(\psi^{-1}(\Psi^{-1}(z_2(m,t)))).
\end{aligned} \tag{2.32}$$

By the mean-value theorem for integrals, there exists ξ in the open interval $(z_2(m,n), z_2(m+1,n))$, for arbitrarily given $(m,n),(m+1,n) \in \Lambda_{(M,\,N)}$, then, we obtain

$$\begin{aligned}
&W_1(z_2(m+1,n)) - W_1(z_2(m,n)) = \int_{z_2(m,n)}^{z_2(m+1,n)}\frac{\mathrm{d}s}{w_1^{p_1}(\psi^{-1}(\Psi^{-1}(s)))}\\
&\quad= \frac{\Delta_1 z_2(m,n)}{w_1^{p_1}(\psi^{-1}(\Psi^{-1}(\xi)))} \le \frac{\Delta_1 z_2(m,n)}{w_1^{p_1}(\psi^{-1}(\Psi^{-1}(z_2(m,n))))}.
\end{aligned} \tag{2.33}$$

From (2.32) and (2.33), we get

$$
\begin{aligned}
&W_1(z_2(m+1,n)) - W_1(z_2(m,n)) \\
&\quad \le \sum_{t=0}^{n-1} \tilde{g}_1(M,N,m,t) + \sum_{i=1}^{l}\sum_{t=0}^{n-1} \tilde{g}_{i+1}(M,N,m,t)\phi_{i+1}^{p_{i+1}}(\psi^{-1}(\Psi^{-1}(z_2(m,t)))). \qquad (2.34)
\end{aligned}
$$

Substitute m with s in (2.34) and keep n fixed, then taking the sum on both sides of (2.34) over $s = 0, 1, \ldots, m-1$, we have

$$
\begin{aligned}
W_1(z_2(m,n)) \le & W_1(C_{l+1}(M,N)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} \tilde{g}_1(M,N,s,t) + \\
& \sum_{i=1}^{l}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1} \tilde{g}_{i+1}(M,N,s,t)\phi_{i+1}^{p_{i+1}}(\psi^{-1}(\Psi^{-1}(z_2(s,t)))), \qquad (2.35)
\end{aligned}
$$

for all $(m,n) \in \Lambda_{(M,\,N)}$.

Let

$$\theta(m,n)) := W_1(z_2(m,n)), \qquad (2.36)$$

$$\rho_1(M,N) := W_1(C_{l+1}(M,N)) + \sum_{s=0}^{M-1}\sum_{t=0}^{N-1} \tilde{g}_1(M,N,s,t). \qquad (2.37)$$

Using (2.36) and (2.37), from (2.35) we have, for $\forall (m,n) \in \Lambda_{(M,\,N)}$,

$$\theta(m,n)) \le \rho_1(M,N) + \sum_{i=1}^{l}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1} \tilde{g}_{i+1}(M,N,s,t)\phi_{i+1}^{p_{i+1}}[\psi^{-1}(\Psi^{-1}(W_1^{-1}(\theta(s,t))))]. \qquad (2.38)$$

It has the same form as (2.20). We are ready to use the inductive assumption for (2.38). Let $\delta(s) := \psi^{-1}(\Psi^{-1}(W_1^{-1}(s)))$. Since $\psi^{-1}, \Psi^{-1}, W_1^{-1}, \phi_i$ are continuous, nondecreasing and positive on $(0,\infty)$, each $\phi_i(\delta(s))$ is continuous and nondecreasing on $(0,\infty)$. Moreover

$$\frac{\phi_{i+1}^{p_{i+1}}(\delta(s))}{\phi_i^{p_i}(\delta(s))} = \frac{w_{i+1}^{p_{i+1}}(\delta(s))}{w_i^{p_i}(\delta(s))} = \max_{\tau\in[0,\,\delta(s)]}\Big\{\frac{\varphi_{i+1}(\tau)}{w_i(\tau)}\Big\}, \quad i = 2,\ldots,l,$$

which is also continuous and nondecreasing on $[0,\infty)$ and positive on $(0,\infty)$. Therefore, by the inductive assumption in (2.29), from (2.38), we have

$$\theta(m,n) \le \Phi_{l+1}^{-1}\Big(\Phi_{l+1}(\rho_l(M,N)) + \sum_{s=0}^{m-1}\sum_{n=0}^{n-1} \tilde{g}_{l+1}(M,N,s,t)\Big), \qquad (2.39)$$

for all $(m,n) \in \Lambda_{(M,\,N)}$, where

$$\Phi_{i+1}(u) := \int_0^u \frac{\mathrm{d}s}{\phi_{i+1}^{p_{i+1}}(\psi^{-1}(\Psi^{-1}(W_1^{-1}(s))))}, \quad u > 0,\ i = 1,2,\ldots,l, \qquad (2.40)$$

$$\rho_i(M,N) := \Phi_{i-1}^{-1}\Big(\Phi_{i-1}(\rho_{i-1}(M,N)) + \sum_{s=0}^{M-1}\sum_{n=0}^{N-1} g_i(M,N,s,t)\Big), \quad i = 2,3,\ldots,l. \qquad (2.41)$$

Note that

$$\Phi_{i+1}(u)=\int_0^u \frac{w_1^{p_1}(\psi^{-1}(\Psi_p^{-1}(W_1^{-1}(s))))\mathrm{d}s}{w_{i+1}^{p_{i+1}}(\psi^{-1}(\Psi_p^{-1}(W_1^{-1}(s))))}=\int_1^{W_1^{-1}(u)} \frac{\mathrm{d}s}{w_{i+1}^{p_{i+1}}(\psi^{-1}(\Psi_p^{-1}(s)))}$$
$$=W_{i+1}(W_1^{-1}(u)),\quad i=1,2,\dots,l. \tag{2.42}$$

Thus, from (2.36), (2.39) and (2.42), we have

$$\begin{aligned}v(m,n)&\le z_2(m,n)=W_1^{-1}(\theta(m,n))\\&\le W_1^{-1}\Big(\Phi_{l+1}^{-1}\Big(\Phi_{l+1}(\rho_l(M,N))+\sum_{s=0}^{m-1}\sum_{n=0}^{n-1}\tilde{g}_{l+1}(M,N,s,t)\Big)\Big)\\&=W_{l+1}^{-1}\Big(W_{l+1}\Big(W_1^{-1}(\rho_l(M,N))\Big)+\sum_{s=0}^{m-1}\sum_{n=0}^{n-1}\tilde{g}_{l+1}(M,N,s,t)\Big),\end{aligned}\tag{2.43}$$

for all $(m,n)\in\Lambda_{(M,N)}$. We can prove that the term of $W_1^{-1}(\rho_l(M,N))$ in (2.43) is just the same as $E_{l+1}(M,N)$ defined in (2.22). Let $\tilde{\rho}_i(M,N):=W_1^{-1}(\rho_i(M,N))$. By (2.37), we have

$$\tilde{\rho}_1(M,N)=W_1^{-1}(\rho_1(M,N))=W_1^{-1}\Big(W_1(C_{l+1}(M,N))+\sum_{s=0}^{M-1}\sum_{t=0}^{N-1}\tilde{g}_1(M,N,s,t)\Big)=E_2(M,N).$$

Then using (2.41) and (2.42), we get

$$\begin{aligned}\tilde{\rho}_i(M,N)&=W_1^{-1}\Big(\Phi_{i-1}^{-1}\Big(\Phi_{i-1}(\rho_{i-1}(M,N))+\sum_{s=0}^{M-1}\sum_{t=0}^{N-1}\tilde{g}_i(M,N,s,t)\Big)\Big)\\&=W_i^{-1}\big[W_i(W_1^{-1}(\rho_{i-1}(M,N)))+\sum_{s=0}^{M-1}\sum_{t=0}^{N-1}\tilde{g}_i(M,N,s,t)\big]\\&=W_i^{-1}\big[W_i(\tilde{\rho}_{i-1}(M,N))+\sum_{s=0}^{M-1}\sum_{t=0}^{N-1}\tilde{g}_i(M,N,s,t)\big]\\&=E_{i+1}(M,N),\quad i=2,3\dots,l.\end{aligned}\tag{2.44}$$

This proves that $W_1^{-1}(\rho_l(M,N))$ in (2.43) is just the same as $E_{l+1}(M,N)$ defined in (2.22). Hence (2.43) can be equivalently written as

$$v(m,n)\le W_{l+1}^{-1}\Big(W_{l+1}(E_{l+1}(M,N))+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_{l+1}(M,N,s,t)\Big),\quad \forall(m,n)\in\Lambda_{(M,N)}.\tag{2.45}$$

The estimation (2.21) of unknown function v in the inequality (2.18) is proved by induction. By (2.9), (2.21) and (2.45), we have

$$\begin{aligned}u(m,n)&\le\psi^{-1}(z(m,n))\le\psi^{-1}\Big(\Psi^{-1}\Big(v(m,n)\Big)\Big)\\&\le\psi^{-1}\Big(\Psi^{-1}\Big(W_k^{-1}\Big(W_k(E_k(M,N))+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_k(M,N,s,t)\Big)\Big)\Big),\end{aligned}\tag{2.46}$$

for all $(m,n)\in\Lambda_{(M,N)}$. Let $m=M,n=N$. From (2.46), we have

$$u(M,N)\le\psi^{-1}\Big(\Psi^{-1}\Big(W_k^{-1}\Big(W_k(E_k(M,N))+\sum_{s=0}^{M-1}\sum_{t=0}^{N-1}\tilde{g}_k(M,N,s,t)\Big)\Big)\Big).$$

This proves (2.6), since M and N are chosen arbitrarily.

This completes the proof of Theorem 2.2. $\square$

Corollary 2.3 *Suppose that (H_1–H_5) hold and $u(m,n)$ is a nonnegative function on Λ satisfying*

$$\psi(u(m,n)) \le c(m,n) + \sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1} h_i(s,t,j,l)\varphi_i^{p_i}(u(j,l)). \tag{2.47}$$

Then

$$u(m,n) \le \psi^{-1}\Big[W_k^{-1}\big(W_k(E_k(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_k(m,n,s,t)\big)\Big], \tag{2.48}$$

for $(m,n) \in \Lambda_{(M_1,N_1)}$, where

$$E_1(m,n) := c(m,n),$$

$$E_i(m,n) := W_{i-1}^{-1}\Big(W_{i-1}(E_{i-1}(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_{i-1}(m,n,s,t)\Big), \quad i = 2,3,\ldots,k,$$

and $(M_1,N_1) \in \Lambda$ is arbitrarily given on the boundary of the lattice

$$\mathcal{R} := \Big\{(m,n) \in \Lambda : W_i(E_i(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_i(m,n,s,t) \le \int_1^{\infty}\frac{\mathrm{d}s}{w_i(\psi^{-1}(s))},$$
$$W_i^{-1}\Big(W_i(E_i(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_i(m,n,s,t)\Big) \le \int_1^{\infty}\frac{\mathrm{d}s}{\psi^{-1}(s)},\ i = 1,2,\ldots,k\Big\}.$$

The proof of Corollary 2.3 is similar to the argument in the proof of Theorem 2.2 with appropriate modification. We omit the details here.

Remark 2.4 If $p_i = 1$ and $h_i(s,t,j,l) = h_i(m,n,s,t)$, Corollary 2.3 reduces to [29, Theorem 1].

Remark 2.5 If $k = l_1 + l_2$ and $\varphi_i(u) = u$, Corollary 2.3 reduces to [30, Theorem 1] and [31, Theorem 1].

Theorem 2.6 *Suppose that $(H_1 - H_5)$ hold and all f_i $(i = 1,2,\ldots,k)$ are nonnegative functions on $\Lambda \times \Lambda$, $p > q \ge 0$. $u(m,n)$ is a nonnegative function on Λ satisfying*

$$\begin{aligned}\psi(u(m,n)) \le & c(m,n) + \sum_{i=1}^{k}\Big(\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1} f_i(s,t,j,l)u^p(s,t) + \\ & \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1} h_i(s,t,j,l)u^q(s,t)\varphi_i^{p_i}(u(j,l))\Big).\end{aligned} \tag{2.49}$$

Then

$$u(m,n) \le \psi^{-1}\Big\{\Psi_p^{-1}\Big[W_k^{-1}\big(W_k(E_k(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_k(m,n,s,t)\big)\Big]\Big\}, \tag{2.50}$$

for $(m,n) \in \Lambda_{(M_1,N_1)}$, where

$$\Psi_p(u) = \int_0^u \frac{\mathrm{d}s}{(\psi^{-1}(s))^p}, \tag{2.51}$$

$$W_i(u) = \int_1^u \frac{\mathrm{d}s}{w_i(\psi^{-1}(\Psi_p^{-1}(s)))},$$

$$E_1(m,n) = \Psi_q(c(m,n)) + \sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{f}_i(s,t,j,l),$$

$$E_i(m,n) = W_{i-1}^{-1}\Big(W_{i-1}(E_{i-1}(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_{i-1}(m,n,s,t)\Big), \quad i = 2,3,\dots,k,$$

and $(M_1, N_1) \in \Lambda$ *is arbitrarily given on the boundary of the lattice*

$$\mathcal{R} = \Big\{(m,n) \in \Lambda : W_i(E_i(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_i(m,n,s,t) \le \int_1^{\infty}\frac{\mathrm{d}s}{w_i(\psi^{-1}(\Psi_p^{-1}(s)))},$$
$$W_i^{-1}\Big(W_i(E_i(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\tilde{g}_i(m,n,s,t)\Big) \le \int_1^{\infty}\frac{\mathrm{d}s}{\psi^{-1}(s)},\ i = 1,2,\dots,k\Big\}.$$

Proof First of all, we monotonize some given functions φ_i in the sums. Obviously, the sequence $w_i(s)$ defined by $\varphi_i(s)$ in (2.1) are nondecreasing and nonnegative functions and satisfy $w_i^{p_i}(s) \ge \varphi_i^{p_i}(s)$, $i = 1,2,\dots,k$. Moreover, the ratio $w_{i+1}^{p_i}(s)/w_i^{p_i}(s)$ are also nondecreasing, $i = 1,2,\dots,k$. By (2.49), from (2.1), we have

$$\psi(u(m,n)) \le c(m,n) + \sum_{i=1}^{k}\Big(\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{f}_i(s,t,j,l)u^p(s,t) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{h}_i(s,t,j,l)u^q(s,t)w_i^{p_i}(u(j,l))\Big). \tag{2.52}$$

By H_3, from(2.52), we have

$$\psi(u(m,n)) \le c(M,N) + \sum_{i=1}^{k}\Big(\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{f}_i(s,t,j,l)u^p(s,t) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{h}_i(s,t,j,l)u^q(s,t)w_i^{p_i}(u(j,l))\Big), \tag{2.53}$$

for all $(m,n) \in \Lambda_{(M,\,N)}$, where $0 \le M \le M_1$ and $0 \le N \le N_1$ are chosen arbitrarily. Let $z(m,n)$ denote the function on the right-hand side of (2.53), which is a nonnegative and nondecreasing function on $\Lambda_{(M,\,N)}$ and $z(0,n) = c(M,N)$. Then we obtain

$$u(m,n) \le \psi^{-1}(z(m,n)), \quad \forall (m,n) \in \Lambda_{(M,\,N)}. \tag{2.54}$$

Since w_i is nondecreasing and satisfies $w_i(u) > 0$, for $u > 0$. By the definition of z and (2.54), we have

$$\Delta_1 z(m,n) = \sum_{i=1}^{k}\sum_{t=0}^{n-1}\sum_{j=0}^{m-1}\sum_{l=0}^{t-1}\tilde{f}_i(m,t,j,l)u^p(m,t) +$$
$$\sum_{i=1}^{k}\sum_{t=0}^{n-1}\sum_{j=0}^{m-1}\sum_{l=0}^{t-1}\tilde{h}_i(m,t,j,l)u^q(m,t)w_i^{p_i}(u(j,l))$$

$$\leq \sum_{i=1}^{k}\sum_{t=0}^{n-1}\sum_{j=0}^{m-1}\sum_{l=0}^{t-1}\tilde{f}_i(m,t,j,l)(\psi^{-1}(z(m,t)))^p+$$
$$\sum_{i=1}^{k}\sum_{t=0}^{n-1}\sum_{j=0}^{m-1}\sum_{l=0}^{t-1}\tilde{h}_i(m,t,j,l)(\psi^{-1}(z(m,t)))^q w_i^{p_i}(\psi^{-1}(z(j,l))). \tag{2.55}$$

Let $\psi^{-1}(z(m,t)) > 1$. Then $(\psi^{-1}(z(m,n)))^p > (\psi^{-1}(z(m,n)))^q$. Using the monotonicity of ψ^{-1} and z, from (2.55), we have

$$\Delta_1 z(m,n) \leq (\psi^{-1}(z(m,n)))^p\Big(\sum_{i=1}^{k}\sum_{t=0}^{n-1}\sum_{j=0}^{m-1}\sum_{l=0}^{t-1}\tilde{f}_i(m,t,j,l)+$$
$$\sum_{i=1}^{k}\sum_{t=0}^{n-1}\sum_{j=0}^{m-1}\sum_{l=0}^{t-1}\tilde{h}_i(m,t,j,l)w_i^{p_i}(\psi^{-1}(z(j,l)))\Big). \tag{2.56}$$

That is

$$\frac{\Delta_1 z(m,n)}{(\psi^{-1}(z(m,n)))^p} \leq \Big(\sum_{i=1}^{k}\sum_{t=0}^{n-1}\sum_{j=0}^{m-1}\sum_{l=0}^{t-1}\tilde{f}_i(m,t,j,l)+$$
$$\sum_{i=1}^{k}\sum_{t=0}^{n-1}\sum_{j=0}^{m-1}\sum_{l=0}^{t-1}\tilde{h}_i(m,t,j,l)w_i^{p_i}(\psi^{-1}(z(j,l)))\Big). \tag{2.57}$$

On the other hand, by the mean-value theorem for integrals, for arbitrarily given $(m,n),(m+1,n) \in \Lambda_{(M,N)}$, in the open interval $(z(m,n), z(m+1,n))$, there exists ξ, which satisfies

$$\Psi_p(z(m+1,n)) - \Psi_p(z(m,n)) = \int_{z(m,n)}^{z(m+1,n)} \frac{\mathrm{d}s}{(\psi^{-1}(s))^p} = \frac{\Delta_1 z(m,n)}{(\psi^{-1}(\xi))^p}$$
$$\leq \frac{\Delta_1 z(m,n)}{(\psi^{-1}(z(m,n)))^p}. \tag{2.58}$$

We use the definition of Ψ_p in (2.51). From (2.57) and (2.58), we obtain

$$\Psi_p(z(m+1,n)) \leq \Psi_p(z(m,n)) + \Big(\sum_{i=1}^{k}\sum_{t=0}^{n-1}\sum_{j=0}^{m-1}\sum_{l=0}^{t-1}\tilde{f}_i(m,t,j,l)+$$
$$\sum_{i=1}^{k}\sum_{t=0}^{n-1}\sum_{j=0}^{m-1}\sum_{l=0}^{t-1}\tilde{h}_i(m,t,j,l)w_i^{p_i}(\psi^{-1}(z(j,l)))\Big). \tag{2.59}$$

Keep n fixed and substitute m with s in (2.59). Then, taking the sums on both sides of (2.59) over $s = 0, 1, \ldots, m-1$, we have

$$\Psi_p(z(m,n)) \leq \Psi_p(z(0,n)) + \sum_{i=1}^{k}\Big(\sum_{s=0}^{m}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{f}_i(s,t,j,l)+$$
$$\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{h}_i(s,t,j,l)w_i^{p_i}(\psi^{-1}(z(j,l)))\Big)$$
$$\leq \Psi_p(c(M,N)) + \sum_{i=1}^{k}\Big(\sum_{s=0}^{M-1}\sum_{t=0}^{N-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{f}_i(s,t,j,l)+$$

$$\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{h}_i(s,t,j,l)w_i^{p_i}(\psi^{-1}(z(j,l)))\Big)$$

$$= C_k(M,N) + \sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{h}_i(s,t,j,l)w_i^{p_i}(\psi^{-1}(z(j,l))), \tag{2.60}$$

where

$$C_k(M,N) = \Psi_p(c(M,N)) + \sum_{i=1}^{k}\sum_{s=0}^{M-1}\sum_{t=0}^{N-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{f}_i(s,t,j,l). \tag{2.61}$$

Let $v(m,n) = \Psi_p(z(m,n))$. From (2.60), we have

$$v(m,n) \le C_k(M,N) + \sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\sum_{j=0}^{s-1}\sum_{l=0}^{t-1}\tilde{h}_i(s,t,j,l)w_i^{p_i}(\psi^{-1}(\Psi_p^{-1}(v(j,l)))), \tag{2.62}$$

for all $(m,n) \in \Lambda_{(M,N)}$.

(2.62) has the same form of (2.47), from Corollary 2.3, we can obtain the estimation (2.50). This completes the proof of Theorem 2.6. □

3. Applications

In this section, we apply our results to study the boundedness of the solutions of difference equations.

Example 3.1 We consider the difference equation

$$v(m,n) = 1 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}2^{-s}\sqrt{|v(s,t)|} + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}s3^{-s}v(s,t) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\frac{s2^{-s}}{20000}e^{v(s,t)}, \tag{3.1}$$

for all $(m,n) \in \Lambda$, where Λ is defined as in Section 2. From (3.1), we have

$$|v(m,n)| \le 1 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}2^{-s}\sqrt{|v(s,t)|} + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}s3^{-s}|v(s,t)| + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\frac{s2^{-s}}{20000}e^{|v(s,t)|}.$$

Let $|v(m,n)| = u(m,n)$. We obtain

$$u(m,n) \le 1 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}2^{-s}\sqrt{u(s,t)} + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}s3^{-s}u(s,t) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\frac{s2^{-s}}{20000}e^{u(s,t)}, \tag{3.2}$$

where $c(m,n) = 1$, $f_1(m,n,s,t) = 2^{-s}$, $w_1(u) = \sqrt{u}$, $f_2(m,n,s,t) = s3^{-s}$, $w_2(u) = u$, $f_3(m,n,s,t) = \frac{s2^{-s}}{20000}$, $w_3(u) = e^u$. We can conclude that $\frac{w_3}{w_2} = \frac{e^u}{u}$ and $\frac{w_2}{w_1} = \frac{u}{\sqrt{u}}$ are non-decreasing for $u > 0$, then, we have

$$E_1(m) = \tilde{c}(m) = 1,$$

$$\tilde{f}_i(m,n,s,t) = f_i(m,n,s,t), \quad i = 1,2,3,$$

$$W_1(u) = \int_1^u \frac{\mathrm{d}s}{\sqrt{s}} = 2(\sqrt{u}-1), \quad W_1^{-1}(u) = (\frac{u}{2}+1)^2,$$

$$W_2(u) = \int_1^u \frac{\mathrm{d}s}{s} = \ln u, \quad W_2^{-1}(u) = e^u,$$

$$W_3(u) = \int_1^u \frac{\mathrm{d}s}{e^s} = e^{-1} - e^{-u}, \quad W_3^{-1}(u) = \ln \frac{1}{e^{-1} - u}. \tag{3.3}$$

From (3.3), we have

$$\begin{aligned} E_2(m,n) &= W_1^{-1}\big[W_1(E_1(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} 2^{-s}\big], \\ &= W_1^{-1}\big[2(\sqrt{E_1(m,n)} - 1) + 2 - (\frac{1}{2})^{m-1}\big], \\ &= \big(2 - (\frac{1}{2})^m\big)^2, \end{aligned}$$

and

$$\begin{aligned} E_3(m,n) &= W_2^{-1}\big[W_2(E_2(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} s3^{-s}\big], \\ &= W_2^{-1}\big[\ln E_2(m,n) + \frac{3}{4} - \frac{5}{12}(\frac{1}{3})^{m-2} - \frac{1}{2}\frac{m-2}{3^{m-1}}\big], \\ &= E_2(m,n)\exp(\frac{3}{4} - \frac{5}{12}(\frac{1}{3})^{m-2} - \frac{1}{2}\frac{m-2}{3^{m-1}}). \end{aligned}$$

Using Theorem 2.2, we obtain

$$\begin{aligned} u(m,n) &\le W_3^{-1}\big[W_3(E_3(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} \frac{s2^{-s}}{20000}\big], \\ &= W_3^{-1}\big[e^{-1} - e^{-E_3(m)} + \frac{1}{20000}\big(2 - \frac{3}{4}\frac{1}{2^{m-3}} - \frac{m-2}{2^{m-1}}\big)\big], \\ &= \ln \frac{1}{\exp(-E_3(m)) - \frac{1}{20000}\big(2 - \frac{3}{4}\frac{1}{2^{m-3}} - \frac{m-2}{2^{m-1}}\big)} \\ &= \ln \frac{1}{\exp\big(-E_2(m)\exp(\frac{3}{4} - \frac{5}{12}(\frac{1}{3})^{m-2} - \frac{1}{2}\frac{m-2}{3^{m-1}})\big) - \frac{1}{20000}\big(2 - \frac{3}{4}\frac{1}{2^{m-3}} - \frac{m-2}{2^{m-1}}\big)} \\ &= \ln \frac{1}{\exp\big(-\big(2-(\frac{1}{2})^m\big)^2\exp(\frac{3}{4} - \frac{5}{12}(\frac{1}{3})^{m-2} - \frac{1}{2}\frac{m-2}{3^{m-1}})\big) - \frac{1}{20000}\big(2 - \frac{3}{4}\frac{1}{2^{m-3}} - \frac{m-2}{2^{m-1}}\big)}. \end{aligned}$$

The above function $\ln\frac{1}{s}$ always makes sense, since $\exp(-(2-(\frac{1}{2})^m)^2\exp(\frac{3}{4}-\frac{5}{12}(\frac{1}{3})^{m-2}-\frac{1}{2}\frac{m-2}{3^{m-1}}))$ is a decreasing function, and $\frac{1}{20000}(2-\frac{3}{4}\frac{1}{2^{m-3}}-\frac{m-2}{2^{m-1}})$ is an increasing function. When $m=2, n=2$ we have

$$\begin{aligned} &\exp\big(-\big(2-(\frac{1}{2})^2\big)^2\exp(\frac{3}{4}-\frac{5}{12})\big) = \exp\big(-(\frac{7}{4})^2\exp(\frac{1}{3})\big) \approx 0.0139, \\ &\frac{1}{20000}\big(2 - \frac{3}{4}\frac{1}{2^{2-3}}\big) = 0.000025. \end{aligned}$$

When $m \to \infty, n \to \infty$, we have

$$\begin{aligned} &\lim_{m\to\infty} \exp\big(-\big(2-(\frac{1}{2})^m\big)^2\exp(\frac{3}{4} - \frac{5}{12}(\frac{1}{3})^{m-2} - \frac{1}{2}\frac{m-2}{3^{m-1}})\big) = \exp\big(-4\exp(\frac{3}{4})\big) \approx 0.00021, \\ &\lim_{m\to\infty} \frac{1}{20000}\big(2 - \frac{3}{4}\frac{1}{2^{m-3}} - \frac{m-2}{2^{m-1}}\big) = 0.0001. \end{aligned}$$

Therefore, for $\ln\frac{1}{s}$, $0 < s < 1$ always holds true. This implies that $u(m,n)$ is bounded for $(m,n) \in \mathbb{N}_0^2$.

Example 3.2 We consider the partial difference equation with the initial boundary value conditions.

$$\Delta_2\Delta_1\psi(z(m,n)) = F(m,n,\varphi_1(z(m,n)),\ldots,\varphi_k(z(m,n))), \tag{3.4}$$

$$\psi(z(m,0)) = a_1(m),\ \psi(z(0,n)) = a_2(n),\ a_1(0) = a_2(0) = 0, \tag{3.5}$$

for all $(m,n) \in \Lambda$, where $\Lambda = I \times J$ is defined as in Section 2, ψ is a continuous and strictly increasing odd function on $\mathbb{R}$, satisfying $\psi(0) = 0$ and $\psi(u) > 0$ for $u > 0$, $F : \Lambda \times \mathbb{R}^k \to \mathbb{R}$, $a_1 : I \to \mathbb{R}$ and $a_2 : J \to \mathbb{R}$, $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing continuous functions and the ratio φ_{i+1}/φ_i are also nondecreasing, $\varphi_i(u) > 0$ for $u > 0$, $i = 1, 2, \ldots, k$.

In the following corollary, we apply our result to discuss boundedness on the solution of problem (3.4).

Corollary 3.3 *Assume that $F : \Lambda \times \mathbb{R}^k \to \mathbb{R}$ is a continuous function satisfying*

$$|F(m,n,\varphi_1(u),\ldots,\varphi_k(u))| \le \sum_{i=1}^{k} g_i(M,N,m,n)|u|^q\varphi_i^{p_i}(|u|), \tag{3.6}$$

$$|a_1(m) + a_2(n)| \le a(m,n), \tag{3.7}$$

for all $(m,n) \in \Lambda$, where $p > q > 0$ is a constant, $f_i(M,N,m,n), g_i(M,N,m,n)$, $i = 1, 2, \ldots, k$, are continuous nonnegative functions and nondecreasing in M and N for each fixed m and n, $a(m,n) : \Lambda \to \mathbb{R}_+$ is nondecreasing in each variable. If $z(m,n)$ is any solution of (3.4) with the condition (3.5), then

$$|z(m,n)| \le \psi^{-1}\Big\{\Psi^{-1}\Big[\tilde{G}_k^{-1}\big(\tilde{G}_k(\tilde{H}_k(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} g_k(M,N,s,t)\big)\Big]\Big\}, \tag{3.8}$$

for all $(m,n) \in \Lambda_{(M,N)}$, where $\Psi(u)$ is defined by (2.2), and

$$\begin{aligned}
&\tilde{G}_i(u) := \int_1^u \frac{\mathrm{d}s}{\varphi_i^{p_i}(\psi^{-1}(\Psi^{-1}(s)))}, \quad u > 0,\\
&\tilde{H}_1(m,n) := \Psi(a(m,n)),\\
&\tilde{H}_i(m,n) := \tilde{G}_{i-1}^{-1}[\tilde{G}_{i-1}(\tilde{H}_{i-1}(m,n)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} g_{i-1}(M,N,s,t)],
\end{aligned}$$

Ψ_p^{-1} and $\tilde{G}_k^{-1}$ denote the inverse functions of Ψ_p and $\tilde{G}$, respectively.

Proof The solution $z(m,n)$ of (3.4) satisfies the following equivalent difference equation

$$\psi(z(m,n)) = a_1(m) + a_2(n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} F(s,t,\varphi_1(z(s,t)),\ldots,\varphi_k(z(s,t))). \tag{3.9}$$

By (3.6), (3.7) and (3.9), we obtain

$$\begin{aligned}
|\psi(z(m,n))| \le& |a_1(m) + a_2(n)| + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} |F(s,t,\varphi_1(z(s,t)),\ldots,\varphi_k(z(s,t)))|\\
\le& a(m,n) + \sum_{i=1}^{k}\sum_{s=0}^{m-1}\sum_{t=0}^{n-1} |z(s,t)|^q g_i(M,N,s,t)\big]\varphi_i^{p_i}(|z(s,t)|).
\end{aligned} \tag{3.10}$$

Since $|\psi(z(m,n))| = \psi(|z(m,n)|)$, (3.10) has the same form of (2.5). Applying Theorem 2.2 to inequality (3.10), we obtain the estimation of $z(m,n)$ as given in (3.8).

If there exists a constant $M > 0$,

$$\tilde{H}_i(m,n) < M, \quad \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} g_i(M,N,s,t) < M, \quad i = 1,2,\ldots,k, \tag{3.11}$$

for all $(m,n) \in \Lambda_{(M,N)}$, then every solution $z(m,n)$ of (3.4) is bounded on $\Lambda_{(M,N)}$. $\square$

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References

[1] T. H. GRONWALL. *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations*. Ann. of Math. (2), 1919, **20**(4): 292–296.

[2] R. BELLMAN. *The stability of solutions of linear differential equations*. Duke Math. J., 1943, **10**(4): 643–647.

[3] R. P. AGARWAL, Y. H. KIM, S. K. SEN. *New retarded integral inequalities with applications*. J. Inequl. Appl., 2008, Art. ID 908784, 15 pages.

[4] D. BAINOV, P. SIMEONOV. *Integral Inequalities and Applications*. Kluwer Academic, Dordrecht, 1992.

[5] I. A. BIHARI. *A generalization of a lemma of Bellman and its application to uniqueness problem of differential equation*. Acta Math. Acad. Sci. Hungar., 1956, **7**(1): 81–94.

[6] W. S. CHEUNG. *Some new nonlinear inequalities and applications to boundary value problems*. Nonlinear Anal., 2006, **64**(9): 2112–2128.

[7] S. K. CHOI, Shengfu DENG, N. P. KOO, et al. *Nonlinear integral inequalities of Bihari-type without class H*. Math. Inequal. Appl., 2005, **8**(4): 643–654.

[8] S. S. DRAGOMIR, Y. H. KIM. *Some integral inequalities for functions of two variables*. Electron. J. Differential Equations, 2003, **10**: 1–13.

[9] Fangcui JIANG, Fangwei MENG. *Explicit bounds on some new nonlinear integral inequality with delay*. J. Comput. Appl. Math., 2007, **205**(1): 479–486.

[10] Y. H. KIM. *Gronwall, Bellman and Pachpatte type integral inequalities with applications*. Nonlinear Anal., 2009, **71**(12): e2641–e2656.

[11] O. LIPOVAN. *Integral inequalities for retarded Volterra equations*. J. Math. Anal. Appl., 2006, **322**(1): 349–358.

[12] Qinghua MA, Enhao YANG. *Some new Gronwall-Bellman-Bihari type integral inequalities with delay*. Period. Math. Hungar., 2002, **44**(2): 225–238.

[13] B. G. PACHPATTE. *Inequalities for Differential and Integral Equations*. Academic Press, New York, 1998.

[14] Weinian ZHANG, Shengfu DENG. *Projected Gronwall-Bellman's inequality for integrable functions*. Math. Comput. Modelling, 2001, **34**(3-4): 393–402.

[15] R. P. AGARWAL, Shengfu DENG, Weinian ZHANG. *Generalization of a retarded Gronwall-like inequality and its applications*. Appl. Math. Comput., 2005, **165**(3): 599–612.

[16] C. J. CHEN, W. S. CHEUNG, Dandan ZHAO. *Gronwall-Bellman-Type integral inequalities and applications to BVPs*. J. Inequal. Appl. 2009, Art. ID 258569, 15 pages.

[17] Wusheng WANG, Zizun LI, Yong LI, et al. *Nonlinear retarded integral inequalities with two variables and applications*. J. Inequal. Appl. 2010, Art. ID 240790, 21 pages.

[18] Jun ZHOU, Jun SHEN, Weinian ZHANG. *A powered Gronwall-type inequality and applications to stochastic differential equations*. Discrete Contin. Dyn. Syst., 2016, **36**(12): 7207–7234.

[19] R. P. AGARWAL. *Difference Equations and Inequalities*. Marcel Dekker, New York, 1992.

[20] B. G. PACHPATTE, S. G. DEO. *Stability of discrete time systems with retarded argument*. Utilitas Math., 1973, **4**: 15–33.

[21] B. G. PACHPATTE. *A note on some discrete inequalities*. Tamsui Oxford J. Math. Sci., 2005, **21**(2): 183–190.

[22] M. PINTO. *Null solutions of difference systems under vanishing perturbation*. J. Difference Equ. Appl., 2003, **9**(1): 1–13.

[23] Wusheng WANG. *A generalized sum-difference inequality and applications to partial difference equations*. Adv. Difference Equ., 2008, Art. ID 695495, 12 pages.

[24] Wusheng WANG, Zizun LI, W. S. CHEUNG. *Some new nonlinear retarded sum-difference inequalities with applications*. Adv. Difference Equ. 2011, 2011:41, 11 pages.

[25] Linfeng ZHOU, Weinian ZHANG. *A projected discrete Gronwall's inequality with sub-exponential growth*. J. Difference Equ. Appl., 2010, **16**(8): 931–943.

[26] Bin ZHENG, Bosheng FU. *Some Volterra-Fredholm type nonlinear discrete inequalities involving four iterated infinite sums*. Adv. Difference Equ. 2012, 2012: 228, 18 pages.

[27] W. S. CHEUNG, Jingli REN. *Discrete non-linear inequalities and applications to boundary value problems*. J. Math. Anal. Appl., 2006, **319**(2): 708–724.

[28] Qinghua MA, W. S. CHEUNG. *Some new nonlinear difference inequalities and their applications*. J. Comput. Appl. Math., 2007, **202**(2): 339–351.

[29] Wusheng WANG, Xiaoliang ZHOU. *An extension to nonlinear sum-difference inequality and applications*. Adv. Difference Equ. 2009, Art. ID 486895, 17 pages.

[30] Bin ZHENG, Qinghua FENG. *Some new Volterra-Fredholm-type discrete inequalities and their applications in the theory of difference equations*. Abstr. Appl. Anal. 2011, Art. ID 584951, 24 pages.

[31] Qinghua FENG, Fanwei MENG, Bosheng FU. *Some new generalized Volterra-Fredholm type finite difference inequalities involving four iterated sums*. Appl. Math. Comput., 2013, **219**(15): 8247–8258.