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Fractional Brownian Bridge Measures and Their Integration by Parts Formula

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Abstract In this paper, we focus on the characterization for fractional Brownian bridge measures. We give the integration by parts formula for such measures by Bismut's method and their pull back formula. Conversely, we prove that such measures can be determined through their integration by parts formula.

Keywords fractional Brownian bridge measures; integration by parts formula; characterization

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1. Introduction

A fractional Brownian bridge is a kind of Gaussian bridge. Similarly to a Brownian bridge, a fractional Brownian bridge has its anticipative and non-anticipative representations which are studied in [1]. In this paper, we aim to characterize a fractional Brownian bridge measure through its integration by parts formula.

Since integration by parts formulas for measures are important in stochastic analysis, a lot of interesting work has been done on these fields. The integration by parts formula was investigated for Wiener measures on the path space in [2–4]. For Brownian bridge measures on the loop space, [5] gave the integration by parts formula on loop group, in which the vector field is C^1; [6, 7] established the integration by parts formula for such measures over Riemannian manifold with Levi-Civita connection. For fractional Wiener measures, [8, 9] gave its integration by parts formula under different integrals. [10] established the integration by parts formula for fractional Ornstein-Uhlenbeck measures.

Conversely, it is significant to consider the characterizations of measures through their integration by parts formulas. It is proved that Gaussian measures can be characterized through their integration by parts formula. [11] showed that the integration by parts formula can characterize abstract Wiener measures. [12] proved that Wiener measures can be characterized by their integration by parts formula on the path space. [10] gave the characterization for fractional Ornstein-Uhlenbeck measures.

2. Preliminaries

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For a continuous Gaussian process G starting at 0 and such that $\mathbb{E}(G_t) = 0, t \in [0,1]$, its associated bridge process is defined by

$$X_t = G_t - tG_1, \quad 0 \le t \le 1.$$

If the Gaussian process G is a fractional Brownian motion, X is called a fractional Brownian bridge.

We set the loop space on $\mathbb{R}^n$ as follows

$$\Omega = \{\omega \in C([0,1];\mathbb{R}^n) \mid \omega(0) = \omega(1) = 0\}.$$

Let $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ be a filtered probability space, where P is fractional Brownian bridge measure such that coordinate process $(X_t)_{0\le t\le 1} = (\omega_t)_{0\le t\le 1}$ is a fractional Brownian bridge, $(\mathscr{F}_t)_{0\le t\le 1}$ is the P-completed natural filtration of $(X_t)_{0\le t\le 1}$, and $\mathscr{F} = \mathscr{F}_1$ is the P-completion of the Borel σ-algebra of Ω. By [1], fractional Brownian bridge $(X_t)_{0\le t\le 1}$ satisfies the following integral equation

$$X_t = B_t^H - \int_0^t \Big(X_s + \int_0^s \Psi(s,u)\mathrm{d}X_u\Big)\frac{k(1,s)k(t,s)}{\int_s^1 k(1,u)^2\mathrm{d}u}\mathrm{d}s, \quad \text{a.s.} \tag{2.1}$$

where B^H is a fractional Brownian motion with $H > \frac{1}{2}$ and

$$\begin{aligned} k(t,s) &= c_H s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}\mathrm{d}u, \\ \Psi(t,s) &= \frac{\sin(\pi(H+\frac{1}{2}))}{\pi}s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}\int_t^1 \frac{u^{H+\frac{1}{2}}(u-t)^{H+\frac{1}{2}}}{u-s}\mathrm{d}u, \end{aligned} \tag{2.2}$$

in which $c_H = \sqrt{\frac{H(2H-1)}{B(2-2H,H-\frac{1}{2})}}$. By [1, Proposition 18], the non-anticipative representation of the fractional Brownian bridge is

$$X_t = B_t^H - \int_0^t \varphi(t,s)\mathrm{d}B_s^H, \quad \text{a.s.} \tag{2.3}$$

where

$$\varphi(t,s) = \int_s^t \Big\{\int_s^u \frac{(1+\Psi(v,s))k(1,v)^2}{(\int_v^1 k(1,w)\mathrm{d}w)^2}\mathrm{d}v - \frac{1+\Psi(u,s)}{\int_u^1 k(1,v)^2\mathrm{d}v}\Big\}k(1,u)k(t,u)\mathrm{d}u.$$

We set

$$L^2(\Omega;P) = \{F \mid F:\Omega\to\mathbb{R}, \|F\|_2 := (\mathbb{E}_P|F|^2)^{\frac{1}{2}} < \infty\}.$$

By [8], the isomorphism operator $K: L^2(\Omega;P) \to I_{0+}^{H+\frac{1}{2}}(L^2(\Omega;P))$ is defined by

$$(Kh)_t = \int_0^t k(t,s)h_s\mathrm{d}s,$$

where $h \in L^2(\Omega;P)$ and $I_{0+}^{H+\frac{1}{2}}(L^2(\Omega;P))$ is $(H+\frac{1}{2})$-Hölder left fractional Riemann-Liouville integral operator. The inverse operator of K is denoted by K^{-1}. By [8], the Cameron-Martin vector field on Ω is

$$\mathcal{H}_0 = \{Kh \mid h \text{ is adapted process}, h \in L^2(\Omega;P) \text{ and } (Kh)_1 = 0\},$$

with scalar product

$$\langle Kh, Kg\rangle_{\mathcal{H}_0} = \langle h, g\rangle_{L^2(\Omega;P)} = \mathbb{E}_P\Big[\int_0^1 \langle h_t, g_t\rangle \mathrm{d}t\Big].$$

The directional derivative of F along Kh is

$$D_hF(\omega) = \lim_{\delta\to 0}\frac{1}{\delta}(F(\omega+\delta(Kh)) - F(\omega)),$$

if the limit exists in $L^2(\Omega, P)$. Denote by $\mathcal{F}C^\infty(\Omega)$ the set of all the smooth cylindrical functions on Ω, i.e.,

$$\mathcal{F}C^\infty(\Omega) = \{F \mid F(\omega) = f(\omega_{t_1}, ..., \omega_{t_n}), 0 < t_1 \le t_2 \le \cdots \le t_n \le 1, f \in C^\infty(\mathbb{R}^n)\}.$$

For $F \in \mathcal{F}C^\infty(\Omega)$, the directional derivative of F is

$$D_hF(\omega) = \sum_{i=1}^n \langle \nabla^i F, (Kh)_{t_i}\rangle_{\mathbb{R}^n},$$

where $\nabla^i F = \nabla^i f(\omega_{t_1}, \ldots, \omega_{t_n})$ is the gradient with respect to the i-th variable of f. The gradient $DF : \Omega \to \mathcal{H}_0$ is determined by $\langle DF, Kh\rangle_{\mathcal{H}_0} = D_hF$. The domain of D is denoted by $\mathrm{Dom}(D)$.

3. Integration by parts formula

By Bismut's idea [13], we need to construct a $\mathbb{R}^n$-valued process β such that for any $r \in (-\epsilon, \epsilon)$, the following integral equation

$$X_t(r) = B_t^H(r) - \int_0^t \Big(X_s(r) + \int_0^s \Psi(s,u)\mathrm{d}X_u(r)\Big)\frac{k(1,s)k(t,s)}{\int_s^1 k(1,u)^2\mathrm{d}u}\mathrm{d}s, \tag{3.1}$$

has solution $(X_t(r))_{0\le t\le 1}$ satisfying

$$\begin{aligned}&(X_t(r))_{0\le t\le 1} \in \Omega \text{ for any } r,\\ &\frac{\mathrm{d}}{\mathrm{d}r}X_t(r)\big|_{r=0} \text{ exists and } \frac{\mathrm{d}}{\mathrm{d}r}X_t(r)\big|_{r=0} = (Kh)_t,\end{aligned} \tag{3.2}$$

where $B_t^H(r)$ is defined by

$$B_t^H(r) = \int_0^t k(t,s)\mathrm{d}B_s(r) = \int_0^t k(t,s)\mathrm{d}\Big(B_s + r\int_0^s (K^{-1}\beta.)_u\mathrm{d}u\Big), \tag{3.3}$$

in which B is a $\mathbb{R}^n$-valued Brownian motion. Note that $(X_t(0))_{0\le t\le 1} = (X_t)_{0\le t\le 1}$ and $B_t(0)_{0\le t\le 1} = (B_t)_{0\le t\le 1}$. The following lemma gives the expression of β such that the solution $(X_t(r))_{0\le t\le 1}$ of (3.1) satisfies (3.2).

Lemma 3.1 *If the solutions $(X_t(r))_{0\le t\le 1}$ of (3.1) satisfy (3.2), then*

$$\beta_t = (Kh)_t + \int_0^t ((Kh)_s + \int_0^s \Psi(s,u)\mathrm{d}(Kh)_u)\frac{k(1,s)k(t,s)}{\int_s^1 k(1,u)^2\mathrm{d}u}\mathrm{d}s.$$

Proof Differentiating (3.1) with respect to r at $r=0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}X_t(r)\big|_{r=0}=\frac{\mathrm{d}}{\mathrm{d}r}B_t^H(r)\big|_{r=0}-\int_0^t\Big(\frac{\mathrm{d}}{\mathrm{d}r}X_s(r)\big|_{r=0}+\int_0^s\Psi(s,u)\mathrm{d}\frac{\mathrm{d}}{\mathrm{d}r}X_u(r)\big|_{r=0}\Big)\frac{k(1,s)k(t,s)}{\int_s^1 k(1,u)^2\mathrm{d}u}\mathrm{d}s.$$

By (3.2) and (3.3), we get

$$\frac{\mathrm{d}}{\mathrm{d}r}X_t(r)\big|_{r=0}=(Kh)_t,\quad \frac{\mathrm{d}}{\mathrm{d}r}B_t^H(r)\big|_{r=0}=\beta_t,$$

which implies that

$$\beta_t=(Kh)_t+\int_0^t\Big((Kh)_s+\int_0^s\Psi(s,u)\mathrm{d}(Kh)_u\Big)\frac{k(1,s)k(t,s)}{\int_s^1 k(1,u)^2\mathrm{d}u}\mathrm{d}s. \tag{3.4}$$

This completes the proof. □

We give the integration by parts formula for fractional Brownian bridge measures.

Theorem 3.2 *For any $T\in(0,1)$, $F\in\mathrm{Dom}(D)\cap\mathcal{F}_T$ and $Kh\in\mathcal{H}_0$, the integration by parts formula for the fractional Brownian bridge measure P is*

$$\mathbb{E}_P\Big[F\int_0^T\langle(K^{-1}\beta.)_t,\mathrm{d}B_t\rangle\Big]=\mathbb{E}_P[D_hF],$$

where B is a $\mathbb{R}^n$-valued Brownian motion and

$$\beta_t=(Kh)_t+\int_0^t\Big((Kh)_s+\int_0^s\Psi(s,u)\mathrm{d}(Kh)_u\Big)\frac{k(1,s)k(t,s)}{\int_s^1 k(1,u)^2\mathrm{d}u}\mathrm{d}s.$$

Proof We set

$$\rho_t=\exp\Big\{-r\int_0^t\langle(K^{-1}\beta.)_s,\mathrm{d}B_s\rangle-\frac{r^2}{2}\int_0^t(K^{-1}\beta.)_s^2\mathrm{d}s\Big\}.$$

For $H>\frac{1}{2}$, by (3.4), we have

$$(K^{-1}\beta.)_t=h_t+((Kh)_t+\int_0^t\Psi(t,u)\mathrm{d}(Kh)_u)\frac{k(1,t)}{\int_t^1 k(1,u)^2\mathrm{d}u}.$$

It follows that

$$\begin{aligned}&\int_0^1(K^{-1}\beta.)_t^2\mathrm{d}t\\&\le 2\int_0^1 h_t^2\mathrm{d}t+4\int_0^1(Kh)_t^2\mathrm{d}\frac{1}{\int_t^1 k(1,u)^2\mathrm{d}u}+4\int_0^1\frac{(\int_0^t\Psi(t,u)\mathrm{d}(Kh)_u)^2k^2(1,t)}{(\int_t^1 k(1,u)^2\mathrm{d}u)^2}\mathrm{d}t.\end{aligned} \tag{3.5}$$

By the definition of k in (2.2), we obtain that

$$\frac{c_H}{H-\frac{1}{2}}(1-t)^{H-\frac{1}{2}}\le k(1,t)\le\frac{c_H}{H-\frac{1}{2}}t^{\frac{1}{2}-H}(1-t)^{H-\frac{1}{2}}. \tag{3.6}$$

Since Kh is H-Hölder continuous and $(Kh)_1=0$, there is a constant C_K such that

$$|(Kh)_t|\le C_K(1-t)^H\Big(\int_0^1 h_t^2\mathrm{d}t\Big)^{\frac{1}{2}}. \tag{3.7}$$

Due to

$$(Kh)_t=c_H\int_0^t\int_0^u s^{\frac{1}{2}-H}u^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}h_s\mathrm{d}s\mathrm{d}u,$$

we have

$$(Kh)'_t = c_H \int_0^t s^{\frac{1}{2}-H} t^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}} h_s \mathrm{d}s. \tag{3.8}$$

Suppose that h is a bounded adapted process. By (3.6)–(3.8), there is a constant C_1 such that

$$\begin{aligned}
&\int_0^1 (Kh)_t^2 \mathrm{d}\frac{1}{\int_t^1 k(1,u)^2\mathrm{d}u} \\
&\le \Big|\lim_{t\to 1} \frac{(Kh)_t^2}{\int_t^1 k(1,u)^2\mathrm{d}u}\Big| + \Big|\int_0^1 \frac{2(Kh)_t(Kh)'_t}{\int_t^1 k(1,u)^2\mathrm{d}u}\mathrm{d}t\Big| \\
&\le \lim_{t\to 1} \frac{C_K^2(1-t)^{2H}}{\frac{c_H^2(1-t)^{2H}}{2H(H-\frac{1}{2})^2}} + \int_0^1 \frac{2C_K(1-t)^H |c_H \int_0^t s^{\frac{1}{2}-H} t^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}}\mathrm{d}s|}{\frac{c_H^2(1-t)^{2H}}{2H(H-\frac{1}{2})^2}}\mathrm{d}t \le C_1.
\end{aligned} \tag{3.9}$$

By (2.2), there is a constant C_Ψ such that

$$\Psi(t,s) \le C_\Psi s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H} (1-t)^{H+\frac{1}{2}}. \tag{3.10}$$

By (3.8) and (3.10), it is easy to check that there is a constant C_2 such that

$$\int_0^1 \frac{(\int_0^t(\int_s^t \Psi(t,u) c_H s^{\frac{1}{2}-H} u^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}}\mathrm{d}u) h_s \mathrm{d}s)^2 k^2(1,t)}{(\int_t^1 k(1,u)^2\mathrm{d}u)^2}\mathrm{d}t \le C_2. \tag{3.11}$$

By (3.5), (3.9) and (3.11), $(\rho_t)_{0\le t\le 1}$ is a uniformly integrable martingale for any $r\in(-\epsilon,\epsilon)$ on $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ due to Novikov's criterion. Note that

$$B_s(r) = B_s + r\int_0^s (K^{-1}\beta.)_u \mathrm{d}u.$$

By Girasonv's theorem, we conclude that $(B_t(r))_{0\le t\le 1}$ is a Brownian motion for any $r\in(-\epsilon,\epsilon)$ under $\rho_1 P$. Thus by [14, Theorem 2], $(B_t^H(r))_{0\le t\le 1}$ is a fractional Brownian motion under $\rho_1 P$. Since

$$\begin{aligned}
X_t &= B_t^H - \int_0^t \Big(X_s + \int_0^s \Psi(s,u)\mathrm{d}X_u\Big)\frac{k(1,s)k(t,s)}{\int_s^1 k(1,u)^2\mathrm{d}u}\mathrm{d}s, \\
X_t(r) &= B_t^H(r) - \int_0^t \Big(X_s(r) + \int_0^s \Psi(s,u)\mathrm{d}X_u(r)\Big)\frac{k(1,s)k(t,s)}{\int_s^1 k(1,u)^2\mathrm{d}u}\mathrm{d}s,
\end{aligned}$$

we conclude that $(X_t(r))_{0\le t\le 1}$ and $(X_t)_{0\le t\le 1}$ have the same distribution under $\rho_1 P$ and P, respectively, that is, for any cylindrical function $F = f(X_{t_1},\dots,X_{t_n})$,

$$\mathbb{E}_{\rho_1 P}[f(X_{t_1}(r),\dots,X_{t_n}(r))] = \mathbb{E}_P[f(X_{t_1},\dots,X_{t_n})].$$

Differentiating the above equation with respect to r, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}\mathbb{E}_P[\rho_1 f(X_{t_1}(r),\dots,X_{t_n}(r))]\big|_{r=0} = -\mathbb{E}_P\Big[F\int_0^1 \langle (K^{-1}\beta.)_t, \mathrm{d}B_t\rangle\Big] + \mathbb{E}_P[D_h F] = 0.$$

Thus for any adapted bounded process h, we get

$$\mathbb{E}_P\Big[F\int_0^1 \langle (K^{-1}\beta.)_t, \mathrm{d}B_t\rangle\Big] = \mathbb{E}_P[D_h F]. \tag{3.12}$$

It is obvious that $K^{-1}\beta \in L^2(\Omega;\nu)$ for $h \in L^2(\Omega;\nu)$, then (3.12) holds for any $h \in L^2(\Omega;\nu)$. Moreover, since D is a closable operator, integration by parts formula (3.12) holds for any $F \in \mathrm{Dom}(D) \cap \mathcal{F}_T$. The proof is completed. □

4. Characterization of fractional Brownian bridge

Next, we show that a fractional Brownian bridge measure can be characterized through its integration by parts formula. Suppose that Y is a semi-martingale and $Y_t = \int_0^t \Gamma_s \mathrm{d}B_s + L_t$, where Γ is a $\mathbb{R}^n \times \mathbb{R}^n$-valued continuous process, B is a $\mathbb{R}^n$-valued Brownian motion and L is a $\mathbb{R}^n$-valued continuous bounded quadratic variation process.

Theorem 4.1 *Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mu)$ be a probability space. If μ is a probability measure such that*

(1) Coordinate process X satisfies

$$X_t = Y_t^H - \int_0^t \Big(X_s + \int_0^s \Psi(s,u)\mathrm{d}X_u\Big)\frac{k(1,s)k(t,s)}{\int_s^1 k(1,u)^2\mathrm{d}u}\mathrm{d}s,$$

where $Y_t^H = \int_0^t k(t,s)\mathrm{d}Y_s$;

(2) For any $T \in (0,1)$, $F \in \mathrm{Dom}(D) \cap \mathcal{F}_T$ and $Kh \in \overline{\mathcal{H}}_0 = \{Kh | h$ is adapted process, $h \in L^2(\Omega;\mu)$ and $(Kh)_1 = 0\}$, it holds that

$$\mathbb{E}_\mu\Big[F\int_0^T \langle (K^{-1}\beta_\cdot)_t, \mathrm{d}Y_t\rangle\Big] = \mathbb{E}_\mu[D_h F], \tag{4.1}$$

where

$$(K^{-1}\beta_\cdot)_t = h_t + \Big((Kh)_t + \int_0^t \Psi(t,u)\mathrm{d}(Kh)_u\Big)\frac{k(1,t)}{\int_t^1 k(1,u)^2\mathrm{d}u},$$

then μ is a fractional Brownian bridge measure.

Proof It suffices to prove that Y is a Brownian motion. We establish the proof in two steps.

(1) Let $F = 1$. By (4.1), we have

$$\mathbb{E}_\mu\Big[F\int_0^T \langle (K^{-1}\beta_\cdot)_t, \mathrm{d}Y_t\rangle\Big] = 0,$$

which implies

$$\mathbb{E}_\mu\Big[\int_0^T \langle (K^{-1}\beta_\cdot)_t, \mathrm{d}L_t\rangle\Big] = 0. \tag{4.2}$$

Considering integral equation $(K^{-1}\beta_\cdot)_t = L_t$, that is

$$(Kh)_t + \int_0^t \Big((Kh)_s + \int_0^s \Psi(s,u)\mathrm{d}(Kh)_u\Big)\frac{k(1,s)k(t,s)}{\int_s^1 k(1,u)^2\mathrm{d}u}\mathrm{d}s = (KL)_t.$$

Its solution is

$$(Kh)_t = (KL)_t - \int_0^t \varphi(t,s)\mathrm{d}(KL)_s,$$

which implies

$$h_t = L_t - \Big(K^{-1}\int_0^{\cdot} \varphi(\cdot,s)\mathrm{d}(KL)_s\Big)_t.$$

By the definition of isomorphism operator K, it holds that

$$\begin{aligned}&\int_0^t \varphi(t,s)(KL)'_s\mathrm{d}s\\ &=\int_0^t \Big\{\int_s^t \Big(\int_s^u \frac{(1+\Psi(v,s))k(1,v)^2}{(\int_v^1 k(1,w)^2\mathrm{d}w)^2}\mathrm{d}v-\frac{1+\Psi(u,s)}{\int_u^1 k(1,v)^2\mathrm{d}v}\Big)k(1,u)k(t,u)\mathrm{d}u\Big\}(KL)'_s\mathrm{d}s\\ &=\int_0^t k(t,u)\Big(\int_0^u \Big\{\Big(\int_s^u \frac{(1+\Psi(v,s))k(1,v)^2}{(\int_v^1 k(1,w)^2\mathrm{d}w)^2}\mathrm{d}v-\frac{1+\Psi(u,s)}{\int_u^1 k(1,v)^2\mathrm{d}v}\Big)k(1,u)(KL)'_s\Big\}\mathrm{d}s\Big)\mathrm{d}u\\ &=\Big(K\Big\{\int_0^u \Big(\int_s^u \frac{(1+\Psi(v,s))k(1,v)^2}{(\int_v^1 k(1,w)^2\mathrm{d}w)^2}\mathrm{d}v-\frac{1+\Psi(u,s)}{\int_u^1 k(1,v)^2\mathrm{d}v}\Big)k(1,u)(KL)'_s\mathrm{d}s\Big\}\Big)_t.\end{aligned}$$

Hence,

$$h_t=L_t-\int_0^t \Big(\int_s^t \frac{(1+\Psi(v,s))k(1,v)^2}{(\int_v^1 k(1,w)^2\mathrm{d}w)^2}\mathrm{d}v-\frac{1+\Psi(t,s)}{\int_t^1 k(1,v)^2\mathrm{d}v}\Big)k(1,t)(KL)'_s\mathrm{d}s. \tag{4.3}$$

Therefore, if we let h equal to (4.3), Eq. (3.2) is

$$\mathbb{E}_\mu\Big[\int_0^T \langle L_t,\mathrm{d}L_t\rangle\Big]=0,$$

which yields that $L_t=0$ for any $t\in[0,T]$. Due to the continuity of L in $[0,1]$, we obtain that $L_t=0$ for any $t\in[0,1]$.

(2) For an orthogonal basis $\{e_i: i=1,\dots,n\}$ on $\mathbb{R}^n$, let $F=\langle Y_T,e_i\rangle$. We give the derivative of $\langle Y_T,e_i\rangle$ in two ways. Consider the following equation

$$X_t(r)=Y_t^H(r)-\int_0^t \Big(X_s(r)+\int_0^s \Psi(s,u)\mathrm{d}X_u(r)\Big)\frac{k(1,s)k(t,s)}{\int_s^1 k(1,u)^2\mathrm{d}u}\mathrm{d}s,$$

where $Y_t^H(r)$ is defined by

$$Y_t^H(r)=Y_t^H+r\alpha_t,$$

in which α is a $\mathbb{R}^n$-valued adapted process. If the solution satisfies $(Kh)_t=\frac{\mathrm{d}}{\mathrm{d}r}Y_t(r)|_{r=0}$, we obtain

$$\alpha_t=\beta_t=(Kh)_t+\int_0^t \Big((Kh)_s+\int_0^s \Psi(s,u)\mathrm{d}(Kh)_u\Big)\frac{k(1,s)k(t,s)}{\int_s^1 k(1,u)^2\mathrm{d}u}\mathrm{d}s. \tag{4.4}$$

By the definition of Y^H, we have

$$Y_t^H(r)=\int_0^t k(t,s)\mathrm{d}Y_s(r)=\int_0^t k(t,s)\mathrm{d}\Big(Y_s+r\int_0^s (K^{-1}\beta_\cdot\Big)_u\mathrm{d}u),$$

which yields that

$$Y_t(r)=Y_t+r\int_0^t (K^{-1}\beta_\cdot)_s\mathrm{d}s.$$

Hence

$$D_h\langle Y_T,e_i\rangle=\frac{\mathrm{d}}{\mathrm{d}r}\langle Y_T(r),e_i\rangle\Big|_{r=0}=\int_0^T \langle (K^{-1}\beta_\cdot)_s,e_i\rangle\mathrm{d}s. \tag{4.5}$$

Let $F = \langle Y_T, e_i\rangle$. By (4.1), we have

$$\begin{aligned}\mathbb{E}_\mu[D_h\langle Y_T, e_i\rangle] =& \mathbb{E}_\mu\Big[\int_0^T \langle \Gamma_t^* e_i, \mathrm{d}B_t\rangle \int_0^1 \langle (K^{-1}\beta.)_t, \mathrm{d}Y_t\rangle\Big]\\ =& \mathbb{E}_\mu\Big[\int_0^T \langle \Gamma_t\Gamma_t^* e_i, (K^{-1}\beta.)_t\rangle \mathrm{d}t\Big]. \end{aligned} \tag{4.6}$$

Note that Γ is a $\mathbb{R}^n \times \mathbb{R}^n$-valued continuous process. Combining (4.5) and (4.6), we obtain

$$\mathbb{E}_\mu\Big[\int_0^T \langle (\Gamma_t\Gamma_t^* - I)e_i, (K^{-1}\beta.)_t\rangle \mathrm{d}t\Big] = 0. \tag{4.7}$$

By (4.4), we get

$$\begin{aligned}&\mathbb{E}_\mu\Big[\int_0^T \langle (\Gamma_t\Gamma_t^* - I)e_i, (K^{-1}\beta.)_t\rangle \mathrm{d}t\Big]\\ &= \mathbb{E}_\mu\Big[\int_0^T \langle (\Gamma_t\Gamma_t^* - I)e_i, h_t + ((Kh)_t + \int_0^t \Psi(t,u)\mathrm{d}(Kh)_u)\frac{k(1,t)}{\int_t^1 k(1,u)^2\mathrm{d}u}\rangle \mathrm{d}t\Big]\\ &= \mathbb{E}_\mu\Big[\int_0^T \langle (\Gamma_t\Gamma_t^* - I)e_i, h_t\rangle \mathrm{d}t\Big] + \mathbb{E}_\mu\Big[\int_0^T \langle P_t, (Kh)_t + \int_0^t \Psi(t,u)\mathrm{d}(Kh)_u\rangle \mathrm{d}t\Big]\\ &= \mathbb{E}_\mu\Big[\int_0^T \langle (\Gamma_t\Gamma_t^* - I)e_i, h_t\rangle \mathrm{d}t\Big] + \mathbb{E}_\mu\Big[\int_0^T \langle P_t, (Kh)_t\rangle \mathrm{d}t\Big] +\\ &\quad \mathbb{E}_\mu\Big[\int_0^T \langle P_t, \int_0^t \Psi(t,u)\mathrm{d}(Kh)_u\rangle \mathrm{d}t\Big], \end{aligned} \tag{4.8}$$

where

$$P_t = \frac{k(1,t)}{\int_t^1 k(1,u)^2\mathrm{d}u}(\Gamma_t\Gamma_t^* - I)e_i.$$

The second term of (4.8) is

$$\begin{aligned}\mathbb{E}_\mu\Big[\int_0^T \langle P_t, (Kh)_t\rangle \mathrm{d}t\Big] =& \mathbb{E}_\mu\Big[\int_0^T \langle P_t, c_H\int_0^t s^{\frac12 - H}\int_s^t u^{H-\frac12}(u-s)^{H-\frac12}\mathrm{d}u h_s \mathrm{d}s\rangle \mathrm{d}t\Big]\\ =& \mathbb{E}_\mu\Big[\int_0^T \langle c_H\int_s^T P_t s^{\frac12 - H}\int_s^t u^{H-\frac12}(u-s)^{H-\frac12}\mathrm{d}u\mathrm{d}t, h_s\rangle \mathrm{d}s\Big]\\ =& \mathbb{E}_\mu\Big[\int_0^T \langle c_H\int_t^T P_s t^{\frac12 - H}\int_t^s u^{H-\frac12}(u-t)^{H-\frac12}\mathrm{d}u\mathrm{d}s, h_t\rangle \mathrm{d}t\Big]. \end{aligned} \tag{4.9}$$

The third term of (4.8) is

$$\begin{aligned}&\mathbb{E}_\mu\Big[\int_0^T \Big\langle \int_u^T \Psi(t,u)P_t\mathrm{d}t, (Kh)_u'\Big\rangle \mathrm{d}u\Big]\\ &= \mathbb{E}_\mu\Big[\int_0^T \Big\langle \int_u^T \Psi(t,u)P_t\mathrm{d}t, c_H\int_0^u s^{\frac12 - H}u^{H-\frac12}(u-s)^{H-\frac32}h_s\mathrm{d}s\Big\rangle \mathrm{d}u\Big]\\ &= \mathbb{E}_\mu\Big[\int_0^T \Big\langle c_H\int_s^T s^{\frac12 - H}u^{H-\frac12}(u-s)^{H-\frac32}\int_u^T \Psi(t,u)P_t\mathrm{d}t\mathrm{d}u, h_s\Big\rangle \mathrm{d}s\Big]\\ &= \mathbb{E}_\mu\Big[\int_0^T \Big\langle c_H\int_t^T t^{\frac12 - H}u^{H-\frac12}(u-t)^{H-\frac32}\int_u^T \Psi(s,u)P_s\mathrm{d}s\mathrm{d}u, h_t\Big\rangle \mathrm{d}t\Big]\\ &= \mathbb{E}_\mu\Big[\int_0^T \Big\langle \int_t^T \Big(c_H P_s t^{\frac12 - H}\int_t^s u^{H-\frac12}(u-t)^{H-\frac32}\Psi(s,u)\mathrm{d}u\Big)\mathrm{d}s, h_t\Big\rangle \mathrm{d}t\Big]. \end{aligned} \tag{4.10}$$

By (4.7)–(4.10), for any h, we obtain

$$\mathbb{E}_\mu\Big[\int_0^T \Big\langle(\Gamma_t\Gamma_t^*-I)e_i+\int_t^T c_H P_s t^{\frac{1}{2}-H}\int_t^s u^{H-\frac{1}{2}}(u-t)^{H-\frac{1}{2}}(1+\Psi(s,u))\mathrm{d}u\mathrm{d}s, h_t\Big\rangle \mathrm{d}t\Big]=0.$$

Thus

$$\mathbb{E}_\mu\Big[(\Gamma_t\Gamma_t^*-I)e_i+\int_t^T c_H P_s t^{\frac{1}{2}-H}\int_t^s u^{H-\frac{1}{2}}(u-t)^{H-\frac{1}{2}}(1+\Psi(s,u))\mathrm{d}u\mathrm{d}s\big|\mathcal{F}_t\Big]=0,$$

which implies that

$$(\Gamma_t\Gamma_t^*-I)e_i+\mathbb{E}_\mu\Big[\int_t^T c_H P_s t^{\frac{1}{2}-H}\int_t^s u^{H-\frac{1}{2}}(u-t)^{H-\frac{1}{2}}(1+\Psi(s,u))\mathrm{d}u\mathrm{d}s\big|\mathcal{F}_t\Big]=0.$$

Let t tend to T. We get

$$(\Gamma_T\Gamma_T^*-I)e_i=0.$$

Since Γ is continuous in $[0,1]$, $\Gamma_T\Gamma_T=I$ for any $T\in[0,1]$. Therefore, Y is a Brownian motion. The proof is completed. $\Box$

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