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Several Identities for Inverse-Conjugate Compositions

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Abstract In this paper, we first present several identities related to the inverse-conjugate compositions having parts of size ≤ 3, the compositions into parts equal to 1 or 2, the compositions into odd parts and the compositions into parts greater than 1. In addition, we provide a bijective proof of a relation for inverse-conjugate compositions having parts of size $\leq k$.

Keywords inverse-conjugate compositions; identity; Fibonacci number; Tribonacci number; bijective proof

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1. Introduction

A composition of a positive integer n is a representation of n as a sequence of positive integers called parts which sum to n. For example, the compositions of 4 are: (4), $(3,1)$, $(1,3)$, $(2,2)$, $(2,1,1)$, $(1,2,1)$, $(1,1,2)$, $(1,1,1,1)$. It is known that there are 2^{n-1} unrestricted compositions of n. MacMahon [1] devised a graphical representation of a composition, called a zig-zag graph, which resembles the partition Ferrers graph except that the first dot of each part is aligned with the last part of its predecessor. For example, the zig-zag graph of the composition $(6,3,1,2,2)$ is shown in Figure 1.

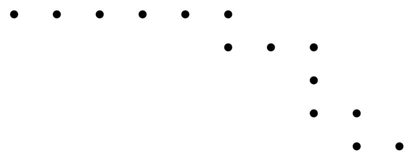

Figure 1 zig-zag graph

The conjugate of a composition is obtained by reading its graph by columns from left to right. Figure 1 gives the conjugate of the composition $(6,3,1,2,2)$ as $(1,1,1,1,1,2,1,3,2,1)$.

Let C denote a composition of n. A k-composition is a composition with k parts, i.e., $C=(c_1,c_2,\ldots,c_k)$. The conjugate of C is denoted by C' and the inverse of C is the reversal composition $\overline{C}=(c_k,c_{k-1},\ldots,c_1)$. C is called inverse-conjugate if $C'=\overline{C}$. For example, $(2,1,3,1)$ is an inverse-conjugate composition of 7.

In 1975, Hoggatt-Bicknell [2] studied ordinary compositions with parts $\leq k$, and obtained the following result [3, p.72]

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Theorem 1.1 ([2]) *Let $C_k(N)$ be the number of compositions of a positive integer N using only the parts $1, 2, \ldots, k$. Then*

$$C_k(N) = F_{N+1}^{(k)}, \tag{1.1}$$

where $F_r^{(n)}$ is the n-step Fibonacci number.

The n-step Fibonacci numbers $F_r^{(n)}$ (see [3]) extend the ordinary Fibonacci numbers.

Definition 1.2 ([3]) *The n-step Fibonacci numbers are defined for any positive integer n by*

$$F_r^{(n)} = \sum_{i=1}^{n} F_{r-i}^{(n)}, \quad r > 2, \tag{1.2}$$

with $F_r^{(n)} = 0$ for $r \leq 0$, $F_1^{(n)} = F_2^{(n)} = 1$.

Note that the case $n = 1$ gives the sequence of ones, $F_r^{(1)}$: $1, 1, 1, \ldots$ while the case $n = 2$ gives the Fibonacci numbers, that is ($F_r^{(2)} = F_r$): $F_1 = F_2 = 1$, $F_r = F_{r-1} + F_{r-2}$, $r > 2$.

Inverse-conjugate compositions have been studied by some researchers [1, 4–6]. It is known that these compositions are defined for only odd weights, and that there are 2^{n-1} inverse-conjugate compositions of $2n - 1$.

Recently Guo-Munagi [7] considered inverse-conjugate compositions with parts of size not exceeding a fixed integer $k > 0$, and obtained their enumeration properties as well as connections with other types of compositions, as summarised in the following three theorems:

Theorem 1.3 ([7]) *Let $IC_k(N)$ denote the number of inverse-conjugate compositions of N into parts of size $\leq k$. Then*

$$IC_k(2n-1) = \sum_{j=1}^{k-1} IC_k(2(n-j)-1), \quad n > k \tag{1.3}$$

with $IC_k(2t-1) = 2^{t-1}$, $t = 1, 2, \ldots, k$.

Theorem 1.4 ([7]) *Let $IC_k(N)$ denote the number of inverse-conjugate compositions of N into parts of size $\leq k$. Then*

$$IC_k(2n-1) = 2F_n^{(k-1)}, \quad n \geq k-1, \tag{1.4}$$

where $F_r^{(n)}$ is the n-generalized Fibonacci number.

Theorem 1.5 ([7]) *Let $C_k(n)$ be the number of compositions of a positive integer n using only the parts $1, 2, \ldots, k$. Then*

$$IC_{k+1}(2n-1) = 2C_k(n-1), \quad n > 1. \tag{1.5}$$

In Section 2, we first present several identities related to the inverse-conjugate compositions having parts of size ≤ 3, the compositions into parts equal to 1 or 2, the compositions into odd parts and the compositions into parts greater than 1. And bijective proofs are shown. In addition, we provide a bijective proof of Theorem 1.5.

2. Main results

We first cite the following terminologies and lemmas from [4] that will be used in the proofs later.

Let $A = (a_1, a_2, \ldots, a_i)$ and $B = (b_1, b_2, \ldots, b_j)$ be compositions. The concatenation of the parts of A and B is defined as $A|B = (a_1, a_2, \ldots, a_i, b_1, b_2, \ldots, b_j)$. In particular for a nonnegative integer c, we have $A|(c) = (A, c)$ and $(c)|A = (c, A)$. The join of A and B with notation $A \uplus B := (a_1, a_2, \ldots, a_i + b_1, b_2, \ldots, b_j)$.

Lemma 2.1 ([4]) *An inverse-conjugate composition C (or its inverse) has the form:*

$$C = (1^{b_r-1}, b_1, 1^{b_{r-1}-2}, b_2, 1^{b_{r-2}-2}, b_3, \ldots, b_{r-1}, 1^{b_1-2}, b_r), b_i \geq 2.$$

Lemma 2.2 ([4]) *If $C = (c_1, \ldots, c_k)$ is an inverse-conjugate composition of $n = 2k-1 > 1$, or its inverse, then there is an index j such that $c_1 + \cdots + c_j = k-1$ and $c_{j+1} + \cdots + c_k = k$ with $c_{j+1} > 1$.*

Moreover,

$$\overline{(c_1, \ldots, c_j)} = (c_{j+1} - 1, c_{j+2}, \ldots, c_k)'. \tag{2.1}$$

Thus C can be written in the form

$$C = A|(1) \uplus B \ \text{ such that } \ B' = \overline{A}, \tag{2.2}$$

where A and B are generally different compositions of $k-1$.

To begin with we present the following results for the inverse-conjugate compositions of odd numbers into parts not exceeding 3 according to Theorems 1.3–1.5.

Theorem 2.3 *Let $IC_3(N)$ denote the number of inverse-conjugate compositions of N into parts of size ≤ 3. Then*

$$IC_3(2n+1) = IC_3(2n-1) + IC_3(2n-3), \quad n > 2, \tag{2.3}$$

with $IC_3(1) = 1$, $IC_3(3) = 2$, $IC_3(5) = 4$.

Theorem 2.4 *Let $IC_3(N)$ denote the number of inverse-conjugate compositions of N into parts of size ≤ 3. Then*

$$IC_3(2n-1) = 2F_n, \quad n \geq 2, \tag{2.4}$$

where F_n is the Fibonacci numbers.

Theorem 2.5 *Let $C_2(n)$ be the number of compositions of a positive integer n using only the parts $1, 2$. Then*

$$IC_3(2n+1) = 2C_2(n), \quad n \geq 1. \tag{2.5}$$

From Theorems 2.4 and 2.5, we also observed that the number of compositions of n into parts of size 1 or 2 is F_{n+1}. And Theorem 2.5 presents an identity between the number of inverse-conjugate compositions having parts of size ≤ 3 and the number of compositions into parts equal to 1 or 2. In this paper, we provide a bijective proof of Theorem 2.5.

Because an inverse-conjugate composition is always paired with its inverse, we give bijective proofs for only inverse-conjugate compositions having parts ≤ 3 in which the first part is 1. The proofs for compositions with the first part > 1 are similar.

Proof For an inverse-conjugate composition $C = (c_1, c_2, \dots, c_n)$ of $2n+1$ having parts of size ≤ 3, and the first part is 1. From Lemma 2.2 we know that there is an index j such that $c_1 + c_2 + \cdots + c_j = n$ and $c_{j+1} + \cdots + c_n = n+1$ with $c_{j+1} > 1$, or $c_1 + c_2 + \cdots + c_j = n+1$ and $c_{j+1} + \cdots + c_n = n$ with $c_j > 1$. And from Lemma 2.1, we know that the number of 1's to the right of the part 3 is at most 1 in an inverse-conjugate composition. So we consider the following two cases.

Case 1 When $c_1 + c_2 + \cdots + c_j = n$, using C we first obtain a composition $B = (c_1, c_2, \dots, c_j)$. Next, for each part of size 3 in composition B, we do the following operation: if 3 is followed by a 1, we replace 3 by "2, 1"; otherwise replace 3 by "1, 1, 1". In this way, we obtain a composition of n with parts of size ≤ 2 and the first part is 1. But this case does not include compositions with three parts being "1, 1, 1" on the left end.

Case 2 When $c_1 + c_2 + \cdots + c_j = n+1$ with $c_j > 1$, we first get a composition $A = (1, c_{j+1}, \dots, c_n)'$, where the last part of A is 1 because of $c_n > 1$, and the first part of A is > 1. Using A we obtain a composition D of n by deleting the last part 1 of A. Similarly, for composition D, we replace 3 by "2, 1" if the part on the right side of it is 1, otherwise replace 3 by "1, 1, 1". In this way, we get a composition of n having parts of size ≤ 2 and the first part is > 1. And this case includes compositions having three parts are "1, 1, 1" on the left end.

Conversely, for a composition K into parts of size ≤ 2 of n, we consider the following three cases.

Case a When there are at most two 1's on the left end of K, we first do the following operation: replace "2, 1" with 3 if there are parts "2, 1, 1", or replace "1, 1, 1" with 3 if there are parts "$\underbrace{1, 1, \dots, 1}_{t \geq 3}$" from right to left in composition K. So we get a composition M. Next, we get a composition $R = M|((1)|\overline{M})'$. Thereupon the composition R is an inverse-conjugate composition of $2n+1$ with parts ≤ 3, and the first part is 1. Here the composition R satisfies $c_1+c_2+\cdots+c_j = n$ and $c_{j+1} + \cdots + c_n = n+1$ with $c_{j+1} > 1$.

For example, the composition $(1, 1, 2, 2, 1, 1, 1, 2)$ of 11 into 1's and 2's produces the inverse-conjugate composition $(1, 1, 2, 2, 3, 2, 2, 2, 1, 2, 2, 3)$ of 23 as follows:

$$(1, 1, 2, 2, 1, 1, 1, 2) \longrightarrow (1, 1, 2, 2, 3, 2) \longrightarrow (1, 1, 2, 2, 3, 2, 2, 2, 1, 2, 2, 3).$$

Case b When there are three parts "1, 1, 1" on the left end of K, we first replace "1, 1, 1" by 3, and then replace "2, 1" with 3 if there are parts "2, 1, 1", or replace "1, 1, 1" with 3 if there are parts "$\underbrace{1, 1, \dots, 1}_{t \geq 3}$" from right to left in K. In this way, we have a composition N. Next, we obtain a composition $H = N|(1)$. Using H we get a composition F by replacing the first part λ of H with $\lambda - 1$. Finally, we obtain a composition $G = \overline{F} \uplus H'$. Hence the composition G is an inverse-conjugate composition of $2n+1$ with parts ≤ 3, and the first part is 1. Here the composition G satisfies $c_1 + c_2 + \cdots + c_j = n + 1$ and $c_{j+1} + \cdots + c_n = n$ with $c_j > 1$.

Case c When the first part of K is 2, we obtain a composition P using the same steps in Case b except that the first part 2 remains the same. Hence the composition P is an inverse-conjugate composition of $2n+1$ with parts ≤ 3, and the first part is 1. Here the composition P satisfies $c_1 + c_2 + \cdots + c_j = n + 1$ and $c_{j+1} + \cdots + c_n = n$ with $c_j > 1$.

Thus we complete the proof. □

We cite an example to illustrate Theorem 2.5.

Example 2.6 Let $n = 5$. The corresponding relations between the inverse-conjugate compositions of 11 into parts of size ≤ 3 and the compositions of 5 into 1's and 2's are as follows.

$$(1,1,3,2,1,3) \longleftrightarrow (1,1,1,1,1) \longleftrightarrow (3,1,2,3,1,1),$$
$$(1,2,1,2,3,2) \longleftrightarrow (2,1,2) \longleftrightarrow (2,3,2,1,2,1),$$
$$(1,3,2,2,1,2) \longleftrightarrow (2,1,1,1) \longleftrightarrow (2,1,2,2,3,1),$$
$$(1,2,2,2,2,2) \longleftrightarrow (1,2,2) \longleftrightarrow (2,2,2,2,2,1),$$
$$(1,1,2,1,3,3) \longleftrightarrow (1,1,2,1) \longleftrightarrow (3,3,1,2,1,1),$$
$$(1,3,1,3,1,2) \longleftrightarrow (1,2,1,1) \longleftrightarrow (2,1,3,1,3,1),$$
$$(1,2,3,1,2,2) \longleftrightarrow (1,1,1,2) \longleftrightarrow (2,2,1,3,2,1),$$
$$(1,1,2,2,2,3) \longleftrightarrow (2,2,1) \longleftrightarrow (3,2,2,2,1,1).$$

It is known that the number of compositions of n into odd parts is F_n, and the number of compositions of n into parts greater than 1 is F_{n-1}. And then combined with Theorem 2.4, the following identities are also obtained.

Theorem 2.7 *Let $C_{\text{odd}}(n)$ be the number of compositions of a positive integer n into odd parts. Then*

$$IC_3(2n-1) = 2C_{\text{odd}}(n), \quad n > 1. \tag{2.6}$$

Proof For an inverse-conjugate composition C of $2n-1$ with parts ≤ 3, and the first part is 1, using proof of Theorem 2.5 we obtain a composition B of $n-1$ with parts of size $1, 2$. Next append 1 to the left end of B to obtain a composition H, then adjoin 1 and all adjacent 2's on the right of it to produce new parts from left to right in H. Hence we obtain a composition of n into odd parts.

Clearly this correspondence is one-to-one, and vice versa. We complete the proof. □

Theorem 2.8 *Let $C_{>1}(n)$ be the number of compositions of n into parts greater than 1. Then*

$$IC_3(2n-1) = 2C_{>1}(n+1), \quad n > 1. \tag{2.7}$$

Proof For an inverse-conjugate composition C of $2n-1$ with parts ≤ 3, and the first part is 1, firstly, we obtain a composition B of $n-1$ with parts of size $1, 2$ using proof of Theorem 2.5. Next a composition D is obtained by appending 1 in both the first end and the last end of B. Finally, we derive the conjugate D' of D. Since the parts of D are 1's or 2's and both ends are 1's, so the D' is a composition of $n+1$ with the parts greater than 1.

For example, the inverse-conjugate composition $(1,2,1,2,3,2)$ and its inverse composition $(2,3,2,1,2,1)$ of 11 into parts ≤ 3 produce the composition $(2,3,2)$ of 7 with parts greater than 1 as follows:

$$(1,2,1,2,3,2) \longrightarrow (1,3,2) \longrightarrow (2,1,2,1) \longrightarrow (2,1,2) \longrightarrow (1,2,1,2,1) \longrightarrow (2,3,2);$$

$$(2,3,2,1,2,1) \longrightarrow (1,2,1,2,3,2) \longrightarrow (1,3,2) \longrightarrow (2,1,2,1) \longrightarrow (2,1,2)$$
$$\longrightarrow (1,2,1,2,1) \longrightarrow (2,3,2).$$

Obviously, this correspondence is one-to-one, and vice versa. We complete the proof. □

We cite an example to illustrate Theorem 2.8.

Example 2.9 Let $n=6$. The corresponding relations between the inverse-conjugate compositions of 11 into parts of size ≤ 3 and the compositions of 7 into parts greater than 1 are as follows.

$$(1,1,3,2,1,3) \longleftrightarrow (7) \longleftrightarrow (3,1,2,3,1,1),$$
$$(1,2,1,2,3,2) \longleftrightarrow (2,3,2) \longleftrightarrow (2,3,2,1,2,1),$$
$$(1,3,2,2,1,2) \longleftrightarrow (2,5) \longleftrightarrow (2,1,2,2,3,1),$$
$$(1,2,2,2,2,2) \longleftrightarrow (3,2,2) \longleftrightarrow (2,2,2,2,2,1),$$
$$(1,1,2,1,3,3) \longleftrightarrow (4,3) \longleftrightarrow (3,3,1,2,1,1),$$
$$(1,3,1,3,1,2) \longleftrightarrow (3,4) \longleftrightarrow (2,1,3,1,3,1),$$
$$(1,2,3,1,2,2) \longleftrightarrow (5,2) \longleftrightarrow (2,2,1,3,2,1),$$
$$(1,1,2,2,2,3) \longleftrightarrow (2,2,3) \longleftrightarrow (3,2,2,2,1,1).$$

3. A bijective proof of Theorem 1.5

Theorem 1.5 is the generalization of Theorem 2.5, so it has an important theoretical meaning to provide a bijective proof of Theorem 1.5. Although the proof is similar to that of Theorem 2.5, we still give a bijective proof of Theorem 1.5 in this section.

Proof For an inverse-conjugate composition $C=(c_1,c_2,\ldots,c_n)$ of $2n-1$ with parts of size $\leq k$, and the first part is 1. From Lemma 2.2 we know that there is an index j such that $c_1+c_2+\cdots+c_j=n-1$ and $c_{j+1}+\cdots+c_n=n$ with $c_{j+1}>1$, or $c_1+c_2+\cdots+c_j=n$ and $c_{j+1}+\cdots+c_n=n-1$ with $c_j>1$. Using Lemma 2.1 we know that the number of 1's on the right of k is at most $k-2$ in an inverse-conjugate composition. Thus we consider the following two cases.

Case 1 When $c_1+c_2+\cdots+c_j=n-1$, we first obtain a composition $B=(c_1,c_2,\ldots,c_j)$. Next, for B we do the following transform: If there are no 1's on the right of the part k, we replace k by "$\underbrace{1,1,\ldots,1}_{k}$". If k is followed by d 1's, we replace k with "$d+1,\underbrace{1,1,\ldots,1}_{k-d-1}$", where $1\leq d\leq k-2$. In this way, we obtain a composition of $n-1$ into parts of size $\leq k-1$ and the first part is 1. But this case does not include the compositions with k parts being "$1,1,\ldots,1$" on the left end.

Case 2 When $c_1 + c_2 + \cdots + c_j = n$ with $c_j > 1$, we first obtain a composition $A = (1, c_{j+1}, \ldots, c_n)'$, where the last part of A is 1 because of $c_n > 1$, and the first part of A is > 1. Next, a composition D of $n-1$ is got by deleting the last part 1 of A. Similarly, we replace k by "$d+1, \underbrace{1, 1, \ldots, 1}_{k-d-1}$" when there are d 1's on the right of the part k, where $0 \leq d \leq k-2$. In this way, we obtain a composition of $n-1$ with parts of size $\leq k-1$ and the first part is > 1. And this case includes compositions with k parts being "$\underbrace{1, 1, \ldots, 1}_{k}$" on the left end.

Conversely, for a composition S with parts of size $\leq k-1$ of $n-1$, we consider the following three cases.

Case a When the first part of S is 1 and there are not k parts "$\underbrace{1, \ldots, 1}_{k}$" on the left end, we do the following operation: replace "$l, \underbrace{1, 1, \ldots, 1}_{k-l}$" with k if there are parts "$l, \underbrace{1, 1, \ldots, 1}_{k-1}$", where $2 \leq l \leq k-1$, or replace "$\underbrace{1, 1, \ldots, 1}_{k}$" with k if there are t parts $\underbrace{1, 1, \ldots, 1}_{t \geq k}$ from right to left in composition S. So we obtain a composition T. Next, we have a composition $R = T\big|((1)\big|\overline{T})'$. Thereupon the composition R is an inverse-conjugate composition of $2n-1$ with parts $\leq k$, and the first part is 1. Here the composition R satisfies $c_1 + c_2 + \cdots + c_j = n-1$ and $c_{j+1} + \cdots + c_n = n$ with $c_{j+1} > 1$.

Case b When there are k parts "$\underbrace{1, \ldots, 1}_{k}$" on the left end of S, we first replace "$\underbrace{1, \ldots, 1}_{k}$" by k, and then replace "$l, \underbrace{1, 1, \ldots, 1}_{k-l}$" with k if there are parts "$l, \underbrace{1, 1, \ldots, 1}_{k-1}$", where $2 \leq l \leq k-1$, or replacing "$\underbrace{1, 1, \ldots, 1}_{k}$" with k if there are t parts "$\underbrace{1, 1, \ldots, 1}_{t \geq k}$" from right to left in S. In this way, we have a composition U. Next, we have a composition $H = U\big|(1)$, and then we obtain a composition F by replacing the first part λ of H with $\lambda - 1$. Finally, we obtain a composition $G = \overline{F} \uplus H'$. Hence the composition G is an inverse-conjugate composition of $2n-1$ with parts $\leq k$, and the first part is 1. Here the composition G satisfies satisfies $c_1 + c_2 + \cdots + c_j = n$ and $c_{j+1} + \cdots + c_n = n-1$ with $c_j > 1$.

Case c When the first part of S is h, where, $1 < h < k$, we obtain a composition P using the same steps in Case 2 except that the first part h remains the same. Hence the composition P is an inverse-conjugate composition of $2n-1$ with parts $\leq k$, and the first part is 1. Here the composition P satisfies $c_1 + c_2 + \cdots + c_j = n$ and $c_{j+1} + \cdots + c_n = n-1$ with $c_j > 1$.

We complete the proof. □

In particular, we give the following interesting relations for the inverse-conjugate compositions into parts of size ≤ 4.

Corollary 3.1 *Let $IC_4(N)$ denote the number of inverse-conjugate compositions of N into* *parts of size* ≤ 4. *Then*

$$IC_4(2n+1) = IC_4(2n-1) + IC_4(2n-3) + IC_4(2n-5), \quad n > 3, \tag{3.1}$$

with $IC_4(1) = 1$, $IC_4(3) = 2$ $IC_4(5) = 4$, $IC_4(7) = 8$.

Corollary 3.2 *Let* $IC_4(N)$ *denote the number of inverse-conjugate compositions of* N *into parts of size* ≤ 4. *Then*

$$IC_4(2n+1) = 2C_3(n), \quad n \geq 1. \tag{3.2}$$

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