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Faber Polynomial Coefficient Estimates on a Subclass of Bi-Univalent Functions

Xiaoyuan WANG[1,*], Zhiren WANG[1], Li YIN[2]
1. *College of Science, Yanshan University, Hebei* 066004*, P. R. China;*
2. *Department of Mathematics, Binzhou University, Shandong* 256603*, P. R. China*

Abstract The main objective of this paper is to derive the upper bounds for the coefficients of functions in a subclass of analytic and bi-univalent functions associated with Faber polynomials. The consequences presented here point out and correct the errors of some earlier results.

Keywords analytic functions; univalent functions; bi-univalent functions; Faber polynomials

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U}=\{z: z\in\mathbb{C} \ \text{ and } \ |z|<1\}.$$

We also denote $\mathcal{S}$ by the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$.

For $f(z)$ and $F(z)$ analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $F(z)$, written $f\prec F$, if there exists a Schwarz function

$$\mu(z)=\sum_{n=1}^{\infty}c_nz^n,$$

with the conditions $\mu(0)=0$ and $|\mu(z)|<1$ in $\mathbb{U}$, such that $f(z)=F(\mu(z))$. Further, let $\mathcal{P}$ be the class of functions

$$\varphi(z)=1+\sum_{n=1}^{\infty}\xi_nz^n,$$

which are analytic in $\mathbb{U}$ and satisfy the condition $\Re(\varphi(z))>0$ in $\mathbb{U}$. By the Carathéodory Lemma [1], we know that $|\xi_n|\le 2$.

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*Corresponding author
E-mail address: mewangxiaoyuan@163.com (Xiaoyuan WANG); wangzhiren528@sina.com (Zhiren WANG); yinli_79@163.com (Li YIN)

The Koebe one-quarter theorem [1] states that the image of $\mathbb{U}$ under every function f from $\mathcal{S}$ contains a disk of radius 1/4. Thus, every univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z, \quad z \in \mathbb{U},$$

$$f^{-1}(f(\omega)) = \omega(|\omega| < r_0(f); r_0(f) \geq \frac{1}{4}),$$

where

$$f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots .$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Denote by Σ the class of bi-univalent functions defined in the unit disc $\mathbb{U}$.

Suppose throughout the paper that $0 < q < p \leq 1$. Chakrabarti and Jagannathan [2] introduced the following operator.

Definition 1.1 *The (p,q)-derivative of the function f give by (1.1) is defined as*

$$D_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p-q)z}, \quad z \neq 0$$

and $D_{p,q}f(0) = f'(0)$, provided that $f'(0)$ exists.

From Definition 1.1, we have

$$D_{p,q}f(z) = 1 + \sum_{n=2}^{\infty}[n]_{p,q}\, a_n z^{n-1},$$

where $[n]_{p,q}$ denotes the (p,q)-number or twin-basic number

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}.$$

If $p = 1$, the (p, q)-derivative reduces to the Jackson q-derivative given by [3]

$$(D_q f)(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad z \neq 0.$$

Making use of the Faber polynomial expansion of function $f \in \mathcal{A}$ with the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as follows [4] (see also [5])

$$g(\omega) = f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty}\frac{1}{n}K_{n-1}^{-n}(a_2, a_3, \ldots)\omega^n, \tag{1.2}$$

where

$$\begin{aligned} K_{n-1}^{-n} =& \frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!}a_2^{n-3}a_3 + \frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4 + \\ & \frac{(-n)!}{(2(-n+2))!(n-5)!}a_2^{n-5}[a_5 + (-n+2)a_3^2] + \\ & \frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}[a_6 + (-2n+5)a_3a_4] + \sum_{j\geq 7}a_2^{n-j}V_j, \end{aligned}$$

such that V_j $(7 \leq j \leq n)$ is a homogeneous polynomial in the variables $a_2, a_3, \ldots, a_n$ (see [6]).

In particular, the first four terms of K_{n-1}^{-n} are given by

$$K_1^{-2} = -2a_2, \quad K_2^{-3} = 3(2a_2^2 - a_3), \quad K_3^{-4} = -4(5a_2^3 - 5a_2a_3 + a_4),$$
$$K_4^{-5} = 5(14a_2^2 - 21a_2^2a_3 + 6a_2a_4 + 3a_3^2 - a_5).$$

In general, for any $\beta \in \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$, an expansion of K_{n-1}^{β} is given by [4]

$$K_{n-1}^{\beta} = \beta a_n + \frac{\beta(\beta-1)}{2}E_{n-1}^2 + \frac{\beta!}{(\beta-3)!3!}E_{n-1}^3 + \cdots + \frac{\beta!}{(\beta-n+1)!(n-1)!}E_{n-1}^{n-1}, \tag{1.3}$$

where $E_{n-1}^{\beta} = E_{n-1}^{\beta}(a_2, a_3, \ldots)$. In view of [7], we see that

$$E_{n-1}^{m}(a_2, a_3, \ldots, a_n) = \sum_{n=2}^{\infty} \frac{m!}{j_1! \cdots j_{n-1}!} a_2^{j_1} \cdots a_n^{j_{n-1}}$$

and the sum is taken over all non-negative integers $j_1, \ldots, j_{n-1}$ satisfying

$$\begin{cases} j_1 + j_2 + \cdots + j_{n-1} = m, \\ j_1 + 2j_2 + \cdots + (n-1)j_{n-1} = n-1. \end{cases} \tag{1.4}$$

It is clear that $E_{n-1}^{n-1}(a_2, \ldots, a_n) = a_2^{n-1}$; while $a_1 = 1$, and the sum is taken over all non-negative integers $j_1, \ldots, j_n$ satisfying

$$\begin{cases} j_1 + j_2 + \cdots + j_n = m, \\ j_1 + 2j_2 + \cdots + nj_n = n. \end{cases} \tag{1.5}$$

Evidently, the first and the last polynomials are

$$E_n^1(a_1, a_2, \ldots, a_n) = a_n \ \text{ and } \ E_n^n(a_1, a_2, \ldots, a_n) = a_1^n.$$

In a recent paper, Altınkaya and Yalçın [8] introduced a new subclass of analytic and bi-univalent functions defined by using the (p,q)-derivative operator.

Definition 1.2 *A function $f(z)$ given by (1.1) is said to be in the class $D_\Sigma(p,q;\lambda,\varphi)$ $(\lambda \geq 1)$ if the following subordination conditions hold*

$$f \in \Sigma \text{ and } (1-\lambda)\frac{f(z)}{z} + \lambda(D_{p,q}f)(z) \prec \varphi(z), \quad z \in \mathbb{U}; \lambda \geq 1, \tag{1.6}$$

$$g \in \Sigma \text{ and } (1-\lambda)\frac{g(\omega)}{\omega} + \lambda(D_{p,q}g)(\omega) \prec \varphi(\omega), \quad \omega \in \mathbb{U}; \lambda \geq 1, \tag{1.7}$$

where $g(\omega) = f^{-1}(\omega)$.

The following result was obtained in [8]:

Theorem 1.3 *Let $f \in D_\Sigma(p,\, q; \lambda, \varphi)$. If $a_m = 0$ $(2 \leq m \leq n-1)$, then*

$$|a_n| \leq \frac{2}{|1 + ([n]_{p,q} - 1)\lambda|}, \quad n \geq 4.$$

We here provide a simple counterexample to clarify that the above assertion is not correct. For the Schwarz functions $\mu(z) = \sum_{n=1}^{\infty} c_n z^n$ and $\nu(z) = \sum_{n=1}^{\infty} d_n \omega^n$. From (2.9) and (2.10) in [8], we see that

$$[1 + ([n]_{p,q} - 1)\lambda]a_n = \sum_{k=1}^{n-1} \xi_k E_{n-1}^{k}(c_1, c_2, \ldots, c_{n-1}), \quad n \geq 2, \tag{1.8}$$

$$[1+([n]_{p,q}-1)\lambda]b_n=\sum_{k=1}^{n-1}\xi_k E_{n-1}^k(d_1,d_2,\ldots,d_{n-1}),\quad n\geq 2. \tag{1.9}$$

For $n=4$ in (1.8) and (1.9), we find that

$$[1+([4]_{p,q}-1)\lambda]a_4=\xi_1c_3+2\xi_2c_1c_2+\xi_3c_1^3, \tag{1.10}$$

$$[1+([4]_{p,q}-1)\lambda]b_4=\xi_1d_3+2\xi_2d_1d_2+\xi_3d_1^3, \tag{1.11}$$

which contradicts the original assertion [8]

$$[1+([4]_{p,q}-1)\lambda]a_4=\xi_1c_3, \tag{1.12}$$

$$-[1+([4]_{p,q}-1)\lambda]a_4=\xi_1d_3, \tag{1.13}$$

since one cannot ensure that $c_1=0$, $d_1=0$.

Note from Definition 1.2 that

$$\lim_{p\to 1-}D_\Sigma(p,q;\lambda,\varphi)=\left\{f:f\in\Sigma \text{ and } \left\{\begin{array}{l}\lim\limits_{p\to 1-}[(1-\lambda)\dfrac{f(z)}{z}+\lambda(D_{p,\,q}f)(z)]\\ \lim\limits_{p\to 1-}[(1-\lambda)\dfrac{g(\omega)}{\omega}+\lambda(D_{p,\,q}g)(\omega)]\end{array}\right\}\right\}=D_\Sigma(q;\lambda,\varphi).$$

Motivated by Wang and Bulut [9], we shall derive the upper bounds for the coefficients of functions belong to the class $D_\Sigma(p,q;\lambda,\varphi)$ ($\lambda\geq 1$) of analytic and bi-univalent functions. The useful consequences presented here point out and correct the errors of the main results in Altınkaya and Yalçın [8].

2. Main result

We now give the following result.

Theorem 2.1 *Let $f\in D_\Sigma(p,\,q;\lambda,\varphi)$. If $a_m=0$ $(2\leq m\leq n-1)$, then*

$$|a_n|\leq\min\Big\{\frac{2\sum_{k=1}^{n-1}|E_{n-1}^k(c_1,c_2,\ldots,c_{n-1})|}{|1+([n]_{p,q}-1)\lambda|},\frac{2\sum_{k=1}^{n-1}|E_{n-1}^k(d_1,d_2,\ldots,d_{n-1})|}{|1+([n]_{p,q}-1)\lambda|}\Big\},\quad n\geq 4,$$

where $\mu(z)=\sum_{n=1}^\infty c_nz^n$ and $\nu(\omega)=\sum_{n=1}^\infty d_n\omega^n$ are two Schwarz functions.

Proof For f given by (1.1), we have

$$(1-\lambda)\frac{f(z)}{z}+\lambda(D_{p,\,q}f)(z)=1+\sum_{n=2}^\infty[1+([n]_{p,\,q}-1)\lambda]a_nz^{n-1}, \tag{2.1}$$

$$\begin{aligned}(1-\lambda)\frac{g(\omega)}{\omega}+\lambda(D_{p,\,q}g)(\omega)&=1+\sum_{n=2}^\infty[1+([n]_{p,\,q}-1)\lambda]\frac{1}{n}K_{n-1}^{-n}(a_2,a_3,\ldots,a_n)\omega^{n-1}\\&=1+\sum_{n=2}^\infty[1+([n]_{p,\,q}-1)\lambda]b_n\omega^{n-1}.\end{aligned} \tag{2.2}$$

On the other hand, the inequalities (1.6) and (1.7) imply that there exist two Schwarz functions $\mu(z)=\sum_{n=1}^\infty c_nz^n$ and $\nu(\omega)=\sum_{n=1}^\infty d_n\omega^n$, such that

$$(1-\lambda)\frac{f(z)}{z}+\lambda(D_{p,\,q}f)(z)=\varphi(\mu(z)), \tag{2.3}$$

$$(1-\lambda)\frac{g(\omega)}{\omega}+\lambda(D_{p,\,q}g)(\omega)=\varphi(\nu(\omega)), \tag{2.4}$$

where

$$\varphi(\mu(z))=1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\xi_k E_n^k(c_1,c_2,\ldots,c_n)z^n, \tag{2.5}$$

$$\varphi(\nu(\omega))=1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\xi_k E_n^k(d_1,d_2,\ldots,d_n)\omega^n. \tag{2.6}$$

Thus, from (2.1), (2.3) and (2.5), we deduce that

$$[1+([n]_{p,q}-1)\lambda]a_n=\sum_{k=1}^{n-1}\xi_k E_{n-1}^k(c_1,c_2,\ldots,c_{n-1}),\quad n\geq 2. \tag{2.7}$$

Similarly, combining (2.2), (2.4) and (2.6), we obtain

$$[1+([n]_{p,q}-1)\lambda]b_n=\sum_{k=1}^{n-1}\xi_k E_{n-1}^k(d_1,d_2,\ldots,d_{n-1}),\quad n\geq 2. \tag{2.8}$$

Since $a_m=0$ $(2\leq m\leq n-1)$, we derive $b_n=-a_n$ and hence

$$\begin{cases} \big[1+([n]_{p,q}-1)\lambda\big]a_n=\displaystyle\sum_{k=1}^{n-1}\xi_k E_{n-1}^k(c_1,c_2,\ldots,c_{n-1}), \\ -\big[1+([n]_{p,q}-1)\lambda\big]a_n=\displaystyle\sum_{k=1}^{n-1}\xi_k E_{n-1}^k(d_1,d_2,\ldots,d_{n-1}). \end{cases} \tag{2.9}$$

By noting that $|\xi_k|\leq 2$, from (2.9), we get the desired assertion of Theorem 2.1. □

By letting $p\to 1$ in Theorem 2.1, we get the following corollary.

Corollary 2.2 *Let $f\in D_\Sigma(q;\lambda,\varphi)$. If $a_m=0$ $(2\leq m\leq n-1)$, then*

$$|a_n|\leq\min\Big\{\frac{2(1-q)\sum_{k=1}^{n-1}|E_{n-1}^k(c_1,c_2,\ldots,c_{n-1})|}{|1-q+(q-q^n)\lambda|}, \frac{2(1-q)\sum_{k=1}^{n-1}|E_{n-1}^k(d_1,d_2,\ldots,d_{n-1})|}{|1-q+(q-q^n)\lambda|}\Big\},\quad n\geq 4. \tag{2.10}$$

In order to correct the error of [8, Theorem 2.2], we add the additional conditions $c_k=d_k=0$ $(1\leq k\leq n-2)$ in Theorem 2.1.

Corollary 2.3 *Let $f\in D_\Sigma(p,\,q;\lambda,\varphi)$. If $a_m=0$ $(2\leq m\leq n-1)$ and $c_k=d_k=0$ $(1\leq k\leq n-2)$, then*

$$|a_n|\leq\frac{2}{|1+([n]_{p,q}-1)\lambda|},\quad n\geq 4,$$

where $\mu(z)=\sum_{n=1}^{\infty}c_nz^n$ and $\nu(\omega)=\sum_{n=1}^{\infty}d_n\omega^n$ are two Schwarz functions.

Remark 2.4 Estimating for the initial coefficients a_2 and a_3, we can see Part 3 of Altınkaya and Yalçın [8].

Remark 2.5 Recently, a series of papers involve the Faber polynomial coefficient estimates for bi-univalent functions. Similar errors also exist in the following papers as pointed out above: [10, Theorem 1], [11, Theorem 7], [12, Theorem 2.1], [13, Theorem 1], [14, Theorems 2.2, 2.4, 2.6 and 2.8].

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