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Carleson Type Measures Supported on $(-1,1)$ and Hankel Matrices

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Abstract In this paper, we establish a connection between Carleson type measures supported on $(-1,1)$ and certain Hankel matrices. The connection is given by the study of Hankel matrices acting on Dirichlet type spaces.

Keywords Carleson type measures; Hankel matrices; Dirichlet type spaces

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1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Denote by $H(\mathbb{D})$ the space of functions analytic in $\mathbb{D}$. The Dirichlet type space $\mathcal{D}_s$, $s \in \mathbb{R}$, consists of those functions $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$ with

$$\|f\|_{\mathcal{D}_s}^2 = \sum_{n=0}^{\infty} (n+1)^{1-s} |a_n|^2 < \infty.$$

For $s > -1$, it is well known that $f \in \mathcal{D}_s$ if and only if

$$\int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^s \mathrm{d}A(z) < \infty,$$

where $dA(z)$ denotes the Lebesgue measure on $\mathbb{D}$. For $s = 0$ we obtain the classical Dirichlet space $\mathcal{D}$ and for $s = 1$ we get the Hardy space H^2. See [1–7] for more results of Dirichlet type spaces.

If a matrix satisfies that its j,k entry is a function of $j+k$, then we say that the matrix is a Hankel matrix. For $0 < p < \infty$, a finite positive Borel measure μ on $\mathbb{D}$ can yield an infinite Hankel matrix as $S_p[\mu]$ with entries

$$(S_p[\mu])_{i,j} = (i+j+1)^{p-1} \mu[i+j], \quad i,j = 0,1,2,\ldots,$$

where

$$\mu[i+j] = \int_{\mathbb{D}} z^{i+j} \mathrm{d}\mu(z).$$

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The Hankel matrix $S_p[\mu]$ acts on analytic functions by multiplication on Taylor coefficient and defines an operator

$$S_p[\mu](f)(z)=\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{\infty}(n+k+1)^{p-1}\mu[n+k]a_k\Big)z^n$$

for the analytic functions $f(z)=\sum_{n=0}^{\infty}a_nz^n$.

For $p>0$, an important tool to study function spaces is p-Carleson measures. Given an arc I of the unit circle $\mathbb{T}$, the Carleson box $S(I)$ with $|I|<1$ is given by

$$S(I)=\{r\zeta\in\mathbb{D}:1-|I|<r<1,\ \zeta\in I\},$$

where $|I|$ denotes the length of the arc I. If $|I|>1$, we set $S(I)=\mathbb{D}$. A finite positive Borel measure μ on $\mathbb{D}$ is said to be a p-Carleson measure if

$$\sup_{I\subseteq\mathbb{T}}\frac{\mu(S(I))}{|I|^p}<\infty.$$

If

$$\frac{\mu(S(I))}{|I|^p}\to 0$$

as $|I|\to 0$, we call μ the vanishing p-Carleson measure. For $p=1$, we obtain the classical Carleson measures. See [8–10] for p-Carleson measures.

In 2014, Bao and Wulan [11] established a connection among p-Carleson measures, Hankel matrices and Dirichlet type spaces as follows. In particular, the case for $p=s=1$ was obtained by Power [10] in 1980.

Theorem 1.1 ([11]) *Let $0<p<\infty$ and $0<s<2$. Suppose that μ is a finite positive Borel measure on $\mathbb{D}$ supported on $(-1,1)$.*

(1) The following conditions are equivalent.

(i) μ is a p-Carleson measure.

(ii) $\mu[n]=O(n^{-p})$.

(iii) $S_p[\mu]$ is bounded on $\mathcal{D}_s$.

(2) The following conditions are equivalent.

(i) μ is a vanishing p-Carleson measure.

(ii) $\mu[n]=o(n^{-p})$.

(iii) $S_p[\mu]$ is compact on $\mathcal{D}_s$.

Throughout the paper, we assume that K is a nonnegative function on $[0,1]$. Let μ be a finite positive Borel measure on $\mathbb{D}$. Following Smith [12], we say that μ is a K-Carleson measure if

$$\sup_{I\subseteq\mathbb{T}}\frac{\mu(S(I))}{K(|I|)}<\infty.$$

If

$$\lim_{|I|\to 0}\frac{\mu(S(I))}{K(|I|)}=0,$$

we call μ the vanishing K-Carleson measure. Clearly, if $K(t)=t^p$, $0<p<\infty$, then the K-Carleson measure gives the p-Carleson measure. We define the corresponding Hankel matrix $S_K[\mu]$ as follows.

$$(S_K[\mu])_{i,j}=\int_{\mathbb{D}}\frac{1}{(i+j+1)K(\frac{1}{i+j+1})}z^{i+j}\mathrm{d}\mu(z),\quad i,j=0,1,2,\ldots.$$

The Hankel matrix $S_K[\mu]$ induces an operator

$$S_K[\mu](f)(z)=\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{\infty}\frac{\mu[n+k]a_k}{(n+k+1)K(\frac{1}{n+k+1})}\Big)z^n$$

for $f(z)=\sum_{n=0}^{\infty}a_nz^n\in H(\mathbb{D})$.

The purpose of this paper is to establish connection among K-Carleson measure supported on $(-1,1)$, the Hankel matrix $S_K[\mu]$ and the Dirichlet type space $\mathcal{D}_s$.

2. Main results

Following Shields and Williams [13], we say that the nonnegative function K on $[0,1]$ is normal if there exist two constants $0<a\le b<\infty$ such that $K(t)/t^a$ is increasing on $(0,1]$ and $K(t)/t^b$ is decreasing on $(0,1]$. Clearly, if K is normal, then K satisfies the double condition. Namely, $K(2t)\approx K(t)$ for $0<t<1/2$. In this paper, the symbol $A\approx B$ means that $A\lesssim B\lesssim A$. We say that $A\lesssim B$ if there exists a constant C such that $A\le CB$.

Before stating and proving our main result, we need the following lemma.

Lemma 2.1 *Let K be normal and let $s<2$. Then there exist two positive constants C_1 and C_2 depending only on K and s such that*

$$C_1\sum_{n=1}^{\infty}\frac{n^{1-s}}{(K(\frac{1}{n}))^2}t^n\le\frac{(1-t^2)^{s-2}}{(K(1-t))^2}\le C_2\sum_{n=1}^{\infty}\frac{n^{1-s}}{(K(\frac{1}{n}))^2}t^n$$

for all $1/2<t<1$.

Proof For all $1/2<t<1$, we compute that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n^{1-s}}{(K(\frac{1}{n}))^2}t^n&\approx\sum_{n=1}^{\infty}\int_{\frac{1}{n+1}}^{\frac{1}{n}}\frac{t^{\frac{1}{x}}}{x^{3-s}(K(x))^2}\mathrm{d}x\\&\approx\int_0^1\frac{t^{\frac{1}{x}}}{x^{3-s}(K(x))^2}\mathrm{d}x\approx\int_1^{\infty}\frac{y^{1-s}t^y}{(K(\frac{1}{y}))^2}\mathrm{d}y\\&\approx\int_{-\ln t}^{\infty}\frac{x^{1-s}e^{-x}}{(\ln\frac{1}{t})^{2-s}(K(\frac{1}{x}\ln\frac{1}{t}))^2}\mathrm{d}x.\end{aligned}$$

Note that K is normal. Then there exist two constants $0<a\le b<\infty$ such that $K(t)/t^a$ is increasing on $(0,1]$ and $K(t)/t^b$ is decreasing on $(0,1]$. Then if $-\ln t<x\le 1$, then $\ln\frac{1}{t}\le\frac{1}{x}\ln\frac{1}{t}$ and hence

$$\frac{K(\ln\frac{1}{t})}{K(\frac{1}{x}\ln\frac{1}{t})}=x^a\frac{K(\ln\frac{1}{t})/(\ln\frac{1}{t})^a}{K(\frac{1}{x}\ln\frac{1}{t})/(\frac{1}{x}\ln\frac{1}{t})^a}\le x^a.$$

Similarly, if $1 \le x < \infty$, then

$$\frac{K(\ln \frac{1}{t})}{K(\frac{1}{x}\ln \frac{1}{t})} \le x^b.$$

Note that $s < 2$ and $\ln \frac{1}{t} \approx (1-t)$ for all $1/2 < t < 1$. These together with the above estimates give

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n^{1-s}}{(K(\frac{1}{n}))^2} t^n &\approx \int_{-\ln t}^{\infty} \frac{x^{1-s}e^{-x}}{(\ln \frac{1}{t})^{2-s}(K(\frac{1}{x}\ln \frac{1}{t}))^2}\mathrm{d}x \\
&\lesssim \frac{(1-t)^{s-2}}{(K(1-t))^2}\Big(\int_0^{\infty} x^{1+2a-s}e^{-x}\mathrm{d}x + \int_0^{\infty} x^{1+2b-s}e^{-x}\mathrm{d}x\Big) \\
&\approx \frac{(1-t)^{s-2}}{(K(1-t))^2}(\Gamma(2+2a-s) + \Gamma(2+2b-s)) \\
&\lesssim \frac{(1-t)^{s-2}}{(K(1-t))^2},
\end{aligned}$$

where $\Gamma(.)$ is the Gamma function.

On the other hand, since $K(t)/t^a$ is increasing and $a > 0$, K is also an increasing function. This gives that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n^{1-s}}{(K(\frac{1}{n}))^2} t^n &\gtrsim \int_3^{\infty} \frac{x^{1-s}e^{-x}}{(\ln \frac{1}{t})^{2-s}(K(\frac{1}{x}\ln \frac{1}{t}))^2}\mathrm{d}x \\
&\gtrsim \frac{(1-t)^{s-2}}{(K(1-t))^2}\int_3^{\infty} x^{1-s}e^{-x}\mathrm{d}x \\
&\approx \frac{(1-t)^{s-2}}{(K(1-t))^2}.
\end{aligned}$$

The proof is completed. □

The following theorem is the main result of this paper which generalizes Theorem 1.1.

Theorem 2.2 *Let $0 < s < 2$ and let K be normal. Suppose that μ is a finite positive Borel measure on $\mathbb{D}$ supported on $(-1, 1)$.*

(1) The following conditions are equivalent.

(i) μ is a K-Carleson measure.

(ii) $\mu[n] = O(K(\frac{1}{n}))$.

(iii) $S_K[\mu]$ is bounded on $\mathcal{D}_s$.

(2) The following conditions are equivalent.

(i) μ is a vanishing K-Carleson measure.

(ii) $\mu[n] = o(K(\frac{1}{n}))$.

(iii) $S_K[\mu]$ is compact on $\mathcal{D}_s$.

Proof We give the proof of (1) as follows.

(i)⇒ (ii). Since μ is a K-Carleson measure supported on $(-1, 1)$, we see that

$$\mu((t, 1)) \lesssim K(1-t), \quad 0 < t < 1,$$

and

$$\mu((-1, -t)) \lesssim K(1-t), \quad 0 < t < 1.$$

Consequently,

$$\begin{aligned}|\mu[n]| \leq& \int_{-1}^{1}|t|^n \mathrm{d}\mu(t) = n\int_0^1 t^{n-1}\mu\{x\in(-1,1):|x|>t\}\mathrm{d}t\\ =&n\int_0^1 t^{n-1}[\mu((t,1))+\mu((-1,-t))]\mathrm{d}t\\ \lesssim& n\int_0^1 t^{n-1}K(1-t)\mathrm{d}t.\end{aligned}$$

Note that K is normal. Namely there exist two constants $0<a\leq b<\infty$ such that $K(t)/t^a$ is increasing on $(0,1]$ and $K(t)/t^b$ is decreasing on $(0,1]$. Then

$$\begin{aligned}&\int_0^1 t^{n-1}K(1-t)\mathrm{d}t = \int_0^1 (1-t)^{n-1}K(t)\mathrm{d}t\\ &= \int_0^{\frac{1}{n}}(1-t)^{n-1}K(t)\mathrm{d}t + \int_{\frac{1}{n}}^1 (1-t)^{n-1}K(t)\mathrm{d}t\\ &\leq n^aK(\frac{1}{n})\int_0^1(1-t)^{n-1}t^a\mathrm{d}t + n^bK(\frac{1}{n})\int_0^1(1-t)^{n-1}t^b\mathrm{d}t\\ &\approx \frac{1}{n}K(\frac{1}{n}).\end{aligned}$$

Thus $\mu[n]=O(K(\frac{1}{n}))$.

(ii)$\Rightarrow$ (iii). Let $0<s<2$ and let $f(z)=\sum_{n=0}^{\infty}a_nz^n\in\mathcal{D}_s$. Since $\mu[n]=O(K(\frac{1}{n}))$, we deduce that

$$\begin{aligned}\|S_K[\mu](f)\|_{\mathcal{D}_s}^2 =& \sum_{n=0}^{\infty}(n+1)^{1-s}\Big|\sum_{k=0}^{\infty}\frac{\mu[n+k]a_k}{(n+k+1)K(\frac{1}{n+k+1})}\Big|^2\\ \lesssim& \sum_{n=0}^{\infty}(n+1)^{1-s}\Big(\sum_{k=0}^{\infty}\frac{|a_k|}{n+k+1}\Big)^2 \lesssim \|f\|_{\mathcal{D}_s}^2,\end{aligned}$$

where the last inequality is from [11]. Thus $S_K[\mu]$ is bounded on $\mathcal{D}_s$.

(iii)$\Rightarrow$ (i). It suffices to consider $1/2<t<1$. Set

$$f_t(z)=(1-t^2)^{1-\frac{s}{2}}\sum_{n=0}^{\infty}((-t)^n+t^n)z^n.$$

Then

$$\|f_t\|_{\mathcal{D}_s}^2 = 4(1-t^2)^{2-s}\sum_{n=0}^{\infty}(2n+1)^{1-s}t^{4n}\approx 1.$$

Therefore, we see that

$$\begin{aligned}&\|S_K[\mu]f_t\|_{\mathcal{D}_s}^2\\ &\approx \sum_{n=0}^{\infty}(n+1)^{1-s}\Big(\sum_{k=0}^{\infty}\frac{\mu[n+2k](1-t^2)^{1-\frac{s}{2}}t^{2k}}{(n+2k+1)K(\frac{1}{n+2k+1})}\Big)^2\\ &\gtrsim (1-t^2)^{2-s}\sum_{n=0}^{\infty}(2n+1)^{1-s}\Big(\sum_{k=0}^{\infty}\frac{\mu[2n+2k]t^{2k}}{(2n+2k+1)K(\frac{1}{2n+2k+1})}\Big)^2\\ &\gtrsim (1-t^2)^{2-s}\sum_{n=0}^{\infty}(2n+1)^{1-s}\Big(\sum_{k=0}^{\infty}\frac{t^{2k}\int_t^1 x^{2n+2k}\mathrm{d}\mu(x)}{(2n+2k+1)K(\frac{1}{2n+2k+1})}\Big)^2\end{aligned}$$

$$\gtrsim (1-t^2)^{2-s}\sum_{n=0}^{\infty}(2n+1)^{1-s}\Big(\sum_{k=0}^{\infty}\frac{t^{2n+4k}\mu((t,1))}{(2n+2k+1)K(\frac{1}{2n+2k+1})}\Big)^2$$
$$\gtrsim (1-t^2)^{2-s}\sum_{n=0}^{\infty}(2n+1)^{1-s}\Big(\sum_{k=0}^{n}\frac{t^{2n+4k}\mu((t,1))}{(2n+1)K(\frac{1}{2n+1})}\Big)^2.$$

Note that $S_K[\mu]$ is bounded on $\mathcal{D}_s$. Combining this with Lemma 2.1, one gets that

$$\begin{aligned}1 &\gtrsim \|S_K[\mu]f_t\|^2_{\mathcal{D}_s}\\ &\gtrsim (1-t^2)^{2-s}\sum_{n=0}^{\infty}(2n+1)^{1-s}\Big(\sum_{k=0}^{n}\frac{t^{2n+4k}\mu((t,1))}{(2n+1)K(\frac{1}{2n+1})}\Big)^2\\ &\gtrsim (1-t^2)^{2-s}\sum_{n=0}^{\infty}\frac{(n+1)^{1-s}}{(K(\frac{1}{n+1}))^2}t^{12n}(\mu((t,1)))^2\\ &\approx \frac{(\mu((t,1)))^2}{(K(1-t))^2}.\end{aligned}$$

Hence,

$$\mu((t,1)) \lesssim K(1-t).$$

A similar computation gives

$$\mu((-1,-t)) \lesssim K(1-t).$$

Thus μ is a K-Carleson measure.

Next we give the proof of (2) as follows.

(i)⇒ (ii) is similar to the corresponding proof in part (1) with a few changes.

(ii)⇒(iii). Let $f(z)=\sum_{n=0}^{\infty}a_nz^n\in\mathcal{D}_s$ for $0<s<2$. Set

$$S_K^{(m)}[\mu](f)(z)=\sum_{n=0}^{m}\Big(\sum_{k=0}^{\infty}\frac{1}{(n+k+1)K(\frac{1}{n+k+1})}\mu[n+k]a_k\Big)z^n.$$

Then $S_K^{(m)}[\mu]$ is a finite rank operator. Thus $S_K^{(m)}[\mu]$ is compact on $\mathcal{D}_s$. If $\mu[n]=o(K(\frac{1}{n}))$, then for any $\epsilon>0$, there exists a positive constant N satisfying $|\mu[n]|<\epsilon K(\frac{1}{n})$ for $n>N$. Since

$$\|(S_K[\mu]-S_K^{(m)}[\mu])(f)\|^2_{\mathcal{D}_s}=\sum_{n=m+1}^{\infty}(n+1)^{1-s}\Big|\sum_{k=0}^{\infty}\frac{\mu[n+k]a_k}{(n+k+1)K(\frac{1}{n+k+1})}\Big|^2,$$

for $m>N$, we have

$$\|(S_K[\mu]-S_K^{(m)}[\mu])(f)\|^2_{\mathcal{D}_s}\lesssim \epsilon^2\sum_{n=m+1}^{\infty}(n+1)^{1-s}\Big(\sum_{k=0}^{\infty}\frac{|a_k|}{n+k+1}\Big)^2.$$

The following inequality appeared in [11].

$$\sum_{n=0}^{\infty}(n+1)^{1-s}\Big(\sum_{k=0}^{\infty}\frac{|a_k|}{n+k+1}\Big)^2\lesssim \|f\|^2_{\mathcal{D}_s}.$$

These yield

$$\|(S_K[\mu]-S_K^{(m)}[\mu])(f)\|^2_{\mathcal{D}_s}\lesssim \epsilon^2\|f\|^2_{\mathcal{D}_s}.$$

In other words,

$$\|S_K[\mu]-S_K^{(m)}[\mu]\|\lesssim \epsilon$$

holds for $m > N$. Thus, $S_K[\mu]$ is compact on $\mathcal{D}_s$.

(iii)$\Rightarrow$(i). For $0 < t < 1$, let

$$f_t(z) = (1-t^2)^{1-\frac{s}{2}} \sum_{n=0}^{\infty} [(-t)^n + t^n] z^n.$$

Then

$$\|f_t\|_{\mathcal{D}_s}^2 = 4(1-t^2)^{2-s} \sum_{n=0}^{\infty} (2n+1)^{1-s} t^{4n} \approx 1$$

and $\lim_{t\to 1} f_t(z) = 0$ for any $z \in \mathbb{D}$. Bear in mind that all Hilbert spaces are reflexive. Then f_t is convergent weakly to zero in $\mathcal{D}_s$ as $t \to 1$. Since $S_K[\mu]$ is compact on $\mathcal{D}_s$, one gets that

$$\lim_{t\to 1} \|S_K[\mu] f_t\|_{\mathcal{D}_s} = 0.$$

Checking the corresponding proof in part (1), we know that

$$\mu((t,1)) \lesssim \|S_p[\mu] f_t\|_{\mathcal{D}_s} K(1-t).$$

Consequently,

$$\lim_{t\to 1} \frac{\mu((t,1))}{K(1-t)} = 0.$$

Similarly,

$$\lim_{t\to 1} \frac{\mu((-1,-t))}{K(1-t)} = 0.$$

The proof of Theorem 2.2 is completed. □

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References

[1] Guanlong BAO, Zengjian LOU, Ruishen QIAN, et al. *On multipliers of Dirichlet type spaces.* Complex Anal. Oper. Theory, 2015, **9**(8): 1701–1732.

[2] Guanlong BAO, Jun YANG. *The Libera operator on Dirichlet spaces.* Bull. Iranian Math. Soc., 2015, **41**(6): 1511–1517.

[3] L. BROWN, A. SHIELDS. *Cyclic vectors in the Dirichlet space.* Trans. Amer. Math. Soc., 1984, **285**(1): 269–303.

[4] E. DIAMANTOPOULOS. *Operators induced by Hankel matrices on Dirichlet spaces.* Analysis (Munich), 2004, **24**(4): 345–360.

[5] Songxiao LI. *Some new characterizations of Dirichlet type spaces on the unit ball of* $\mathbb{C}^n$. J. Math. Anal. Appl., 2006, **324**(2): 1073–1083.

[6] Songxiao LI. *Generalized Hilbert operator on the Dirichlet-type space.* Appl. Math. Comput., 2009, **214**(1): 304–309.

[7] Ruishen QIAN, Yecheng SHI. *Inner function in Dirichlet type spaces.* J. Math. Anal. Appl., 2015, **421**(2): 1844–1854.

[8] R. AULASKARI, D. STEGENGA, Jie XIAO. *Some subclasses of BMOA and their characterization in terms of Carleson measures.* Rocky Mountain J. Math., 1996, **26**(2): 485–506.

[9] J. GARNETT. *Bounded Analytic Functions.* Academic Press, New York, 1981.

[10] S. POWER. *Vanishing carleson measures.* Bull. London Math. Soc., 1980, **12**(3): 207–210.

[11] Guanlong BAO, Hasi WULAN. *Hankel matrices acting on Dirichlet spaces.* J. Math. Anal. Appl., 2014, **409**(1): 228–235.

[12] W. SMITH. BMO(ρ) *and Carleson measures.* Trans. Amer. Math. Soc., 1985, **287**(1): 107–126.

[13] A. SHIELDS, D. WILLIAMS. *Bounded projections, duality, and multipliers in spaces of analytic functions.* Trans. Amer. Math. Soc., 1971, **162**: 287–302.