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# Some Remarks on BLUP under the General Linear Model with Linear Equality Restrictions

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**Abstract** In this paper, we give the representation of the best linear unbiased predictor (BLUP) of the new observations under $\mathscr{M}_{r_f}$. Through the representation, we give necessary and sufficient conditions that the estimators, OLSEs (ordinary least squares estimators) and BLUEs (best linear unbiased estimators), under $\mathscr{M}_f$ and $\mathscr{M}_{r_f}$, and the predictor, BLUP, under $\mathscr{M}_f$ continue to be the BLUP under $\mathscr{M}_{r_f}$, respectively.

**Keywords** general linear model; restricted linear model; BLUP; OLSE; BLUE

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## 1. Introduction

Let $\mathbb{R}^{m\times n}$ denote the set of all $m\times n$ real matrices. Given any $\mathbf{A}\in\mathbb{R}^{m\times n}$, let $\mathbf{A}'$, $\mathbf{A}^+$, $\mathbf{A}^-$, $r(\mathbf{A})$ and $\mathscr{C}(\mathbf{A})$ stand for, respectively, the transpose, the Moore-Penrose inverse, the any generalized inverse, the rank and the range (column space) of $\mathbf{A}$. Furthermore, $\mathbf{P_A}$, $\mathbf{E_A}$ and $\mathbf{F_A}$ stand for the three projectors $\mathbf{P_A}=\mathbf{AA}^+, \mathbf{E_A}=\mathbf{I}_m-\mathbf{AA}^+$ and $\mathbf{F_A}=\mathbf{I}_n-\mathbf{A}^+\mathbf{A}$.

For $\mathbf{y}\in\mathbb{R}^{n\times 1}$ and $\mathbf{y}_f\in\mathbb{R}^{l\times 1}$, which denote, respectively, an observable random vector and an unobservable random vector containing new future observations, we assume that they follow these two linear models

$$\mathbf{y}=\mathbf{X}\boldsymbol{\beta}+\boldsymbol{\varepsilon} \tag{1.1}$$

and

$$\mathbf{y}_f=\mathbf{X}_f\boldsymbol{\beta}+\boldsymbol{\varepsilon}_f,$$

where

$\mathbf{X}\in\mathbb{R}^{n\times p}$ is a known model matrix of arbitrary rank,

$\mathbf{X}_f\in\mathbb{R}^{l\times p}$ is a known model matrix related to the new observations,

$\boldsymbol{\beta}\in\mathbb{R}^{p\times 1}$ is a vector of unknown parameters,

$\boldsymbol{\varepsilon}\in\mathbb{R}^{n\times 1}$ is a random error vector,

$\boldsymbol{\varepsilon}_f\in\mathbb{R}^{l\times 1}$ is a random error vector related to new observations.

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The expectation vector and the covariance matrix of $(\mathbf{y}', \mathbf{y}_f')'$ are

$$\mathrm{E}\begin{pmatrix}\mathbf{y}\\ \mathbf{y}_f\end{pmatrix} = \begin{pmatrix}\mathbf{X}\\ \mathbf{X}_f\end{pmatrix}\boldsymbol{\beta}, \quad \mathrm{Cov}\begin{pmatrix}\mathbf{y}\\ \mathbf{y}_f\end{pmatrix} = \begin{pmatrix}\mathbf{V} & \mathbf{W}\\ \mathbf{W}' & \mathbf{V}_f\end{pmatrix} = \boldsymbol{\Sigma},$$

where $\mathbf{V}$ and $\boldsymbol{\Sigma}$ are two known positive semi-definite matrices of arbitrary rank. For simplicity, we write

$$\mathscr{M} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\} \tag{1.2}$$

to denote (1.1), and use

$$\mathscr{M}_f = \left\{\begin{pmatrix}\mathbf{y}\\ \mathbf{y}_f\end{pmatrix}, \begin{pmatrix}\mathbf{X}\boldsymbol{\beta}\\ \mathbf{X}_f\boldsymbol{\beta}\end{pmatrix}, \begin{pmatrix}\mathbf{V} & \mathbf{W}\\ \mathbf{W}' & \mathbf{V}_f\end{pmatrix}\right\}$$

to signify (1.1) with new observations. Also assume that we have additional information on the unknown parameter vector $\boldsymbol{\beta}$

$$\mathbf{A}\boldsymbol{\beta} = \mathbf{b}, \tag{1.3}$$

where $\mathbf{A} \in \mathbb{R}^{m\times p}$ is a known matrix and $\mathbf{b} \in \mathbb{R}^{m\times 1}$ is a known vector. Such information may result from different sources like past experience or long association of the experimenter with the experiment, similar kind of experiments conducted in the past, etc [1, p.224]. The model (1.1) subject to (1.3) is called a restricted linear model, which can be written in the following form

$$\mathscr{M}_r = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta} | \mathbf{A}\boldsymbol{\beta} = \mathbf{b}, \mathbf{V}\}. \tag{1.4}$$

The model (1.4) with new future observations can be written in the following form

$$\mathscr{M}_{r_f} = \left\{\begin{pmatrix}\mathbf{y}\\ \mathbf{y}_f\end{pmatrix}, \begin{pmatrix}\mathbf{X}\boldsymbol{\beta}\\ \mathbf{X}_f\boldsymbol{\beta}\end{pmatrix} \middle| \mathbf{A}\boldsymbol{\beta} = \mathbf{b}, \begin{pmatrix}\mathbf{V} & \mathbf{W}\\ \mathbf{W}' & \mathbf{V}_f\end{pmatrix}\right\}. \tag{1.5}$$

Under $\mathscr{M}$, $\mathbf{X}_f\boldsymbol{\beta}$ is said to be estimable if there exists an $\mathbf{L}$ such that $\mathrm{E}(\mathbf{Ly}) = \mathbf{X}_f\boldsymbol{\beta}$. As we all known, $\mathbf{X}_f\boldsymbol{\beta}$ is estimable under $\mathscr{M}$ if and only if $\mathscr{C}(\mathbf{X}_f') \subseteq \mathscr{C}(\mathbf{X}')$, under which the unobservable random vector $\mathbf{y}_f$ under $\mathscr{M}_f$ is also unbiasedly predictable [2]. In the remainder of this article, we suppose that $\mathscr{C}(\mathbf{X}_f') \subseteq \mathscr{C}(\mathbf{X}')$ holds. Now we recall that an unbiased linear estimator $\mathbf{Gy}$ of $\mathbf{X}_f\boldsymbol{\beta}$ is the BLUE of $\mathbf{X}_f\boldsymbol{\beta}$ under $\mathscr{M}$ if

$$\mathrm{Cov}(\mathbf{Gy}) \leq_{\mathrm{L}} \mathrm{Cov}(\mathbf{Ly}) \quad \forall\, \mathbf{L}:\ \mathbf{LX} = \mathbf{X}_f,$$

where " $\leq_{\mathrm{L}}$ " refers to the Löwner ordering, i.e.,

$$\mathbf{G}(\mathbf{X}, \mathbf{V}\mathbf{E}_{\mathbf{X}}) = (\mathbf{X}_f, \mathbf{0}), \tag{1.6}$$

(see [3]).

Moreover, a linear unbiased predictor $\mathbf{Ty}$ is the BLUP of $\mathbf{y}_f$ under $\mathscr{M}_f$ if

$$\mathrm{Cov}(\mathbf{Ty} - \mathbf{y}_f) \leq_{\mathrm{L}} \mathrm{Cov}(\mathbf{Ny} - \mathbf{y}_f) \quad \forall\, \mathbf{N}:\ \mathbf{NX} = \mathbf{X}_f,$$

i.e.,

$$\mathbf{T}(\mathbf{X}, \mathbf{V}\mathbf{E}_{\mathbf{X}}) = (\mathbf{X}_f, \mathbf{W}'\mathbf{E}_{\mathbf{X}}), \tag{1.7}$$

(see [2]).

Goldberger [4] showed, that if $\mathbf{V}$ is positive definite, then the BLUP of $\mathbf{y}_f$ under $\mathscr{M}_f$ has a form

$$\mathrm{BLUP}(\mathbf{y}_f|\mathscr{M}_f) = \mathbf{X}_f\tilde{\boldsymbol{\beta}} + \mathbf{W}'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}}),$$

where $\mathbf{X}_f\tilde{\boldsymbol{\beta}}$ and $\mathbf{X}\tilde{\boldsymbol{\beta}}$ are the BLUEs of $\mathbf{X}_f\boldsymbol{\beta}$ and $\mathbf{X}\boldsymbol{\beta}$, respectively.

The matrices $\mathbf{V}$ and $\mathbf{W}$ are often unknown and usually difficult to estimate in practice, which lead to the BLUP of $\mathbf{y}_f$ cannot be computed. Therefore, many authors considered using the BLUP of $\mathbf{y}_f$ to replace the estimators of $\mathbf{X}_f\boldsymbol{\beta}$, i.e., finding necessary and sufficient conditions for the equality of the BLUP and the estimators. On the hypothesis of $\mathscr{C}(\mathbf{X}) \subseteq \mathscr{C}(\mathbf{V})$ (see [5]) considered replacing the OLSE of $\mathbf{X}_f\boldsymbol{\beta}$ with the BLUP of $\mathbf{y}_f$, and derived an equivalent condition for the OLSE to be the BLUP. For a more general case, Baksalary [6] gave a new representation for the BLUP, as well as the equivalent conditions for the equality of the OLSE and BLUP. Elian [7] derived necessary and sufficient conditions for the BLUE to be the BLUP with the assumption that $r(\mathbf{X}) = p$ and $r(\mathbf{V}) = n$; Liu [8] considered, without any rank assumptions to the model $\mathscr{M}_f$, the equalities between the OLSE, the BLUE and the BLUP; Haslett et al. [9] revisited the equalities between the OLSE, the BLUE and the BLUP, using a different approach.

In practical applications, we usually encounter the linear restrictions (1.3) on the model (1.1). In this situation, we consider the equivalence between the estimators, OLSEs and BLUEs, under $\mathscr{M}$ and $\mathscr{M}_r$ and the BLUP under $\mathscr{M}_{r_f}$. In addition, it is interesting to give necessary and sufficient conditions for the equality of the BLUPs under $\mathscr{M}_f$ and $\mathscr{M}_{r_f}$, under which we can put aside the influence of the restrictions (1.3) on the BLUP under $\mathscr{M}_f$. For the equality of the OLSEs and BLUEs under $\mathscr{M}$ and $\mathscr{M}_r$ (see [10–17]).

## 2. Preliminaries

As we all know, a general solution to $\mathbf{A}\boldsymbol{\beta} = \mathbf{b}$ is

$$\boldsymbol{\beta} = \mathbf{A}^+\mathbf{b} + \mathbf{F}_\mathbf{A}\boldsymbol{\gamma}, \tag{2.1}$$

where $\boldsymbol{\gamma}$ is an arbitrary vector. Therefore, under the model (1.4), we have

$$\mathrm{E}(\mathbf{y} - \mathbf{X}\mathbf{A}^+\mathbf{b}) = \mathbf{X}\mathbf{F}_\mathbf{A}\boldsymbol{\gamma}.$$

Hence the model (1.4) can be reexpressed as $(\mathbf{y} - \mathbf{X}\mathbf{A}^+\mathbf{b}, \mathbf{X}\mathbf{F}_\mathbf{A}\boldsymbol{\gamma}, \mathbf{V})$, which leads to the fact that

$$\mathbf{y} - \mathbf{X}\mathbf{A}^+\mathbf{b} \in \mathscr{C}(\mathbf{X}\mathbf{F}_\mathbf{A}, \mathbf{V}) \tag{2.2}$$

holds almost surely. Tian [15] pointed out that (2.2) is equivalent to

$$\begin{pmatrix} \mathbf{y} \\ \mathbf{b} \end{pmatrix} \in \mathscr{C}\begin{pmatrix} \mathbf{X} & \mathbf{V} \\ \mathbf{A} & \mathbf{0} \end{pmatrix}.$$

Thus (2.2) also implies $\mathbf{y} \in \mathscr{C}(\mathbf{X}, \mathbf{V})$, i.e., the consistency of the model (1.4) gives rise to that of the model (1.2). It is well known that the OLSE of $\mathbf{X}_f\boldsymbol{\beta}$ under $\mathscr{M}$ has a sole form $\mathbf{X}_f\mathbf{X}^+\mathbf{y}$. So the following conclusions are readily obtained [14].

**Lemma 2.1** *Consider the model $\mathscr{M}_r$, and define $\mathbf{X}_{f\mathbf{A}} = \mathbf{X}_f\mathbf{F}_\mathbf{A}$ and $\mathbf{X}_\mathbf{A} = \mathbf{X}\mathbf{F}_\mathbf{A}$. Then:*

*(a) The* OLSE *of $\mathbf{X}_f\boldsymbol{\beta}$ under $\mathscr{M}_r$, denoted by* $\mathrm{OLSE}(\mathbf{X}_f\boldsymbol{\beta}|\mathscr{M}_r)$, *can be uniquely written as*

$$\mathbf{X}_f\mathbf{A}^+\mathbf{b} + \mathbf{X}_{f\mathbf{A}}\mathbf{X}_\mathbf{A}^+(\mathbf{y} - \mathbf{X}\mathbf{A}^+\mathbf{b}). \tag{2.3}$$

*(b) A linear statistic $\mathbf{X}_f\mathbf{A}^+\mathbf{b} + \mathbf{G}_r(\mathbf{y} - \mathbf{X}\mathbf{A}^+\mathbf{b})$ is the* BLUE *of $\mathbf{X}_f\boldsymbol{\beta}$ under $\mathscr{M}_r$, denoted by* $\mathrm{BLUE}(\mathbf{X}_f\boldsymbol{\beta}|\mathscr{M}_r)$, *if and only if the equation*

$$\mathbf{G}_r[\mathbf{X}_\mathbf{A}, \mathbf{V}\mathbf{E}_{\mathbf{X}_\mathbf{A}}] = [\mathbf{X}_{f\mathbf{A}}, \mathbf{0}] \tag{2.4}$$

*is satisfied.*

The following result characterizes the BLUP of new observations in the general linear model with linear equality restrictions.

**Lemma 2.2** *Consider the model $\mathscr{M}_{r_f}$. Then a linear statistic $\mathbf{X}_f\mathbf{A}^+\mathbf{b}+\mathbf{T}_r(\mathbf{y}-\mathbf{X}\mathbf{A}^+\mathbf{b})$, denoted by* $\mathrm{BLUP}(\mathbf{y}_f|\mathscr{M}_{r_f})$, *is the* BLUP *of $\mathbf{y}_f$ if and only if the equation*

$$\mathbf{T}_r[\mathbf{X}_\mathbf{A}, \mathbf{V}\mathbf{E}_{\mathbf{X}_\mathbf{A}}] = [\mathbf{X}_{f\mathbf{A}}, \mathbf{W}'\mathbf{E}_{\mathbf{X}_\mathbf{A}}] \tag{2.5}$$

*is satisfied.*

**Proof** Note from (2.1) that

$$\mathbf{X}_f\boldsymbol{\beta} = \mathbf{X}_f\mathbf{A}^+\mathbf{b} + \mathbf{X}_{f\mathbf{A}}\boldsymbol{\gamma}, \quad \mathbf{X}\boldsymbol{\beta} = \mathbf{X}\mathbf{A}^+\mathbf{b} + \mathbf{X}_\mathbf{A}\boldsymbol{\gamma}, \tag{2.6}$$

where $\boldsymbol{\gamma} \in \mathbb{R}^{p\times 1}$ is arbitrary. Substituting (2.6) into (1.5), we get the following unrestricted linear model

$$\mathscr{M}_{ur_f} = \left\{ \begin{pmatrix} \mathbf{z} \\ \mathbf{z}_f \end{pmatrix}, \begin{pmatrix} \mathbf{X}_\mathbf{A}\boldsymbol{\gamma} \\ \mathbf{X}_{f\mathbf{A}}\boldsymbol{\gamma} \end{pmatrix}, \begin{pmatrix} \mathbf{V} & \mathbf{W} \\ \mathbf{W}' & \mathbf{V}_f \end{pmatrix} \right\}, \tag{2.7}$$

where $\mathbf{z} = \mathbf{y} - \mathbf{X}\mathbf{A}^+\mathbf{b}$, $\mathbf{z}_f = \mathbf{y}_f - \mathbf{X}_f\mathbf{A}^+\mathbf{b}$. Applying (1.7) to (2.7) gives $\mathrm{BLUP}(\mathbf{z}_f|\mathscr{M}_{ur_f}) = \mathbf{T}_r\mathbf{z}$, i.e., $\mathrm{BLUP}(\mathbf{y}_f|\mathscr{M}_{r_f}) = \mathbf{X}_f\mathbf{A}^+\mathbf{b} + \mathbf{T}_r(\mathbf{y} - \mathbf{X}\mathbf{A}^+\mathbf{b})$. This completes the proof. $\square$

For simplification of a variety of matrix expressions involving the Moore-Penrose inverses of matrices, we need the following rank formulas for partitioned matrices given by [18].

**Lemma 2.3** *Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times k}$ and $\mathbf{C} \in \mathbb{R}^{l\times n}$. Then:*

$$r(\mathbf{A}, \mathbf{B}) = r(\mathbf{A}) + r(\mathbf{E}_\mathbf{A}\mathbf{B}) = r(\mathbf{B}) + r(\mathbf{E}_\mathbf{B}\mathbf{A}), \tag{2.8}$$

$$r\begin{pmatrix} \mathbf{A} \\ \mathbf{C} \end{pmatrix} = r(\mathbf{A}) + r(\mathbf{C}\mathbf{F}_\mathbf{A}) = r(\mathbf{C}) + r(\mathbf{A}\mathbf{F}_\mathbf{C}), \tag{2.9}$$

$$r\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{pmatrix} = r(\mathbf{B}) + r(\mathbf{C}) + r(\mathbf{E}_\mathbf{B}\mathbf{A}\mathbf{F}_\mathbf{C}), \tag{2.10}$$

$$r\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{pmatrix} \geq r(\mathbf{A}, \mathbf{B}) + r\begin{pmatrix} \mathbf{A} \\ \mathbf{C} \end{pmatrix} - r(\mathbf{A}). \tag{2.11}$$

*In particular,*

$$r(\mathbf{A},\mathbf{B}) = r(\mathbf{A}) \Leftrightarrow \mathbf{E_A B} = \mathbf{0} \Leftrightarrow \mathscr{C}(\mathbf{B}) \subseteq \mathscr{C}(\mathbf{A}), \tag{2.12}$$

$$r\begin{pmatrix}\mathbf{A}\\ \mathbf{C}\end{pmatrix} = r(\mathbf{A}) \Leftrightarrow \mathbf{CF_A} = \mathbf{0} \Leftrightarrow \mathscr{C}(\mathbf{C}') \subseteq \mathscr{C}(\mathbf{A}'). \tag{2.13}$$

**Lemma 2.4** *Let* $\mathbf{\Sigma} \in \mathbb{R}^{n\times n}$ *be nonnegative definite and* $\mathbf{X} \in \mathbb{R}^{n\times p}$*. Then*

$$\mathscr{C}\begin{pmatrix}\mathbf{\Sigma} & \mathbf{X}\\ \mathbf{X}' & \mathbf{0}\end{pmatrix} = \mathscr{C}\begin{pmatrix}\mathbf{\Sigma} & \mathbf{X} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{X}'\end{pmatrix}. \tag{2.14}$$

**Proof** Note that $\mathscr{C}(\mathbf{X}, \mathbf{\Sigma E_X}) = \mathscr{C}(\mathbf{X}, \mathbf{\Sigma})$. Then

$$\mathscr{C}\begin{pmatrix}\mathbf{\Sigma} & \mathbf{X}\\ \mathbf{X}' & \mathbf{0}\end{pmatrix} = \mathscr{C}\begin{pmatrix}\mathbf{\Sigma E_X} & \mathbf{\Sigma} & \mathbf{X}\\ \mathbf{0} & \mathbf{X}' & \mathbf{0}\end{pmatrix} = \mathscr{C}\begin{pmatrix}\mathbf{\Sigma} & \mathbf{X} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{X}'\end{pmatrix}.$$

**Remark 2.5** Using different approach, (2.14) was proved in [19].

**Lemma 2.6** *Let* $\mathbf{\Sigma} \in \mathbb{R}^{n\times n}$ *be nonnegative definite, and let* $\mathbf{A} \in \mathbb{R}^{n\times k}$ *and* $\mathbf{B} \in \mathbb{R}^{n\times l}$*. Then*

$$r\begin{pmatrix}\mathbf{\Sigma} & \mathbf{A}\\ \mathbf{B}' & \mathbf{0}\end{pmatrix} \geq r(\mathbf{\Sigma},\mathbf{A},\mathbf{B}) + r(\mathbf{A}) + r(\mathbf{B}) - r(\mathbf{A},\mathbf{B}) \geq r(\mathbf{\Sigma},\mathbf{A},\mathbf{B}). \tag{2.15}$$

$$r\begin{pmatrix}\mathbf{\Sigma} & \mathbf{A}\\ \mathbf{B}' & \mathbf{0}\end{pmatrix} \geq r(\mathbf{\Sigma},\mathbf{A}) + r(\mathbf{\Sigma},\mathbf{B}) - r(\mathbf{\Sigma}) \geq r(\mathbf{\Sigma},\mathbf{A},\mathbf{B}). \tag{2.16}$$

*Furthermore, the following statements are equivalent:*

(a) $r\begin{pmatrix}\mathbf{\Sigma} & \mathbf{A}\\ \mathbf{B}' & \mathbf{0}\end{pmatrix} = r(\mathbf{\Sigma},\mathbf{A},\mathbf{B})$.

(b) $r\begin{pmatrix}\mathbf{\Sigma} & \mathbf{A}\\ \mathbf{B}' & \mathbf{0}\end{pmatrix} = r(\mathbf{\Sigma},\mathbf{A}) + r(\mathbf{\Sigma},\mathbf{B}) - r(\mathbf{\Sigma})$, $r(\mathbf{\Sigma},\mathbf{A},\mathbf{B}) = r(\mathbf{\Sigma},\mathbf{A}) + r(\mathbf{\Sigma},\mathbf{B}) - r(\mathbf{\Sigma})$ *and* $r(\mathbf{A},\mathbf{B}) = r(\mathbf{A}) + r(\mathbf{B})$.

(c) $\mathbf{F}_{(\mathbf{\Sigma},\mathbf{A})}\begin{pmatrix}\mathbf{\Sigma} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{F}_{(\mathbf{\Sigma},\mathbf{B})} = \mathbf{0}$, $\mathscr{C}(\mathbf{E_\Sigma A}) \cap \mathscr{C}(\mathbf{E_\Sigma B}) = \{\mathbf{0}\}$ *and* $\mathscr{C}(\mathbf{A}) \cap \mathscr{C}(\mathbf{B}) = \{\mathbf{0}\}$.

**Proof** Note from (2.14) that

$$r\begin{pmatrix}\mathbf{\Sigma} & \mathbf{A} & \mathbf{B}\\ \mathbf{A}' & \mathbf{0} & \mathbf{0}\\ \mathbf{B}' & \mathbf{0} & \mathbf{0}\end{pmatrix} = r(\mathbf{\Sigma},\mathbf{A},\mathbf{B}) + r(\mathbf{A},\mathbf{B}). \tag{2.17}$$

Also notice from (2.11) and (2.14) that

$$\begin{aligned} r\begin{pmatrix}\mathbf{\Sigma} & \mathbf{A} & \mathbf{B}\\ \mathbf{B}' & \mathbf{0} & \mathbf{0}\\ \mathbf{A}' & \mathbf{0} & \mathbf{0}\end{pmatrix} &\geq r\begin{pmatrix}\mathbf{\Sigma} & \mathbf{A} & \mathbf{B}\\ \mathbf{B}' & \mathbf{0} & \mathbf{0}\end{pmatrix} + r\begin{pmatrix}\mathbf{\Sigma} & \mathbf{A}\\ \mathbf{B}' & \mathbf{0}\\ \mathbf{A}' & \mathbf{0}\end{pmatrix} - r\begin{pmatrix}\mathbf{\Sigma} & \mathbf{A}\\ \mathbf{B}' & \mathbf{0}\end{pmatrix}\\ &= 2r(\mathbf{\Sigma},\mathbf{A},\mathbf{B}) + r(\mathbf{A}) + r(\mathbf{B}) - r\begin{pmatrix}\mathbf{\Sigma} & \mathbf{A}\\ \mathbf{B}' & \mathbf{0}\end{pmatrix}, \end{aligned}$$

which combined with (2.17) leads to the first inequality in (2.15). The second inequality in (2.15) is trivial.

Now let us consider (2.16). By (2.8), we have

$$\begin{aligned} r(\mathbf{\Sigma}, \mathbf{A}, \mathbf{B}) &= r(\mathbf{\Sigma}) + r(\mathbf{E}_{\mathbf{\Sigma}}\mathbf{A}, \mathbf{E}_{\mathbf{\Sigma}}\mathbf{B}) \leq r(\mathbf{\Sigma}) + r(\mathbf{E}_{\mathbf{\Sigma}}\mathbf{A}) + r(\mathbf{E}_{\mathbf{\Sigma}}\mathbf{B}) \\ &= r(\mathbf{\Sigma}, \mathbf{A}) + r(\mathbf{\Sigma}, \mathbf{B}) - r(\mathbf{\Sigma}), \end{aligned}$$

which asserts that the second inequality in (2.16) is true. As to the first inequality in (2.16), it is obvious by (2.11).

Suppose that (b) holds. In light of (2.10), we get

$$\begin{aligned} r\begin{pmatrix} \mathbf{\Sigma} & \mathbf{A} \\ \mathbf{B}' & \mathbf{0} \end{pmatrix} + r(\mathbf{\Sigma}) &= r\begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma} & \mathbf{A} \\ \mathbf{0} & \mathbf{B}' & \mathbf{0} \end{pmatrix} = r\begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} & \mathbf{\Sigma} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}' \\ \mathbf{\Sigma} & \mathbf{B} & \mathbf{0} \end{pmatrix} \\ &= r(\mathbf{\Sigma}, \mathbf{A}) + r(\mathbf{\Sigma}, \mathbf{B}) + r\left( \mathbf{E}_{(\mathbf{\Sigma}, \mathbf{A})'} \begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{F}_{\mathbf{\Sigma}, \mathbf{B}} \right), \end{aligned}$$

which, in view of $\mathbf{E}_{(\mathbf{\Sigma}, \mathbf{A})'} = \mathbf{F}_{(\mathbf{\Sigma}, \mathbf{A})}$, implies the first equality in (c). Moreover, by (2.8), obviously,

$$\begin{aligned} r(\mathbf{\Sigma}, \mathbf{A}) + r(\mathbf{\Sigma}, \mathbf{A}) - r(\mathbf{\Sigma}) &= r(\mathbf{\Sigma}) + r(\mathbf{E}_{\mathbf{\Sigma}}\mathbf{A}) + r(\mathbf{E}_{\mathbf{\Sigma}}\mathbf{B}) \\ &= r(\mathbf{\Sigma}, \mathbf{A}, \mathbf{B}) = r(\mathbf{\Sigma}) + r(\mathbf{E}_{\mathbf{\Sigma}}\mathbf{A}, \mathbf{E}_{\mathbf{\Sigma}}\mathbf{B}) \\ &= r(\mathbf{\Sigma}) + r(\mathbf{E}_{\mathbf{\Sigma}}\mathbf{A}) + r(\mathbf{E}_{\mathbf{\Sigma}}\mathbf{B}) - \dim\mathscr{C}(\mathbf{E}_{\mathbf{\Sigma}}\mathbf{A}) \cap \mathscr{C}(\mathbf{E}_{\mathbf{\Sigma}}\mathbf{B}), \\ r(\mathbf{A}, \mathbf{B}) &= r(\mathbf{A}) + r(\mathbf{B}) - \dim\mathscr{C}(\mathbf{A}) \cap \mathscr{C}(\mathbf{B}). \end{aligned}$$

Now, clearly, (b)$\Rightarrow$ (c). The reverse relation is obvious. Condition (b) is obvious alternative way to express statement (a). This completes the proof.

The following lemma is an important result to solve the equality problems about BLUP and was given in [20].

**Lemma 2.7** *Let $\mathbf{C}_1 \in \mathbb{R}^{m\times n_1}$, $\mathbf{C}_2 \in \mathbb{R}^{m\times n_2}$ and $\mathbf{D}_1 \in \mathbb{R}^{p\times n_1}$, $\mathbf{D}_2 \in \mathbb{R}^{p\times n_2}$ be given. Then the pair of matrix equations $\mathbf{X}\mathbf{C}_1 = \mathbf{D}_1$ and $\mathbf{X}\mathbf{C}_2 = \mathbf{D}_2$ have a common solution if and only if $\mathscr{C}\binom{\mathbf{D}_1'}{\mathbf{D}_2'} \subseteq \mathscr{C}\binom{\mathbf{C}_1'}{\mathbf{C}_2'}$, or equivalently,*

$$r\begin{pmatrix} \mathbf{C}_1 & \mathbf{C}_2 \\ \mathbf{D}_1 & \mathbf{D}_1 \end{pmatrix} = r(\mathbf{C}_1, \mathbf{C}_2).$$

## 3. Main results

In this section, we consider the equivalence between the estimators, OLSEs, BLUEs, under $\mathscr{M}$ and $\mathscr{M}_r$ and the BLUP under $\mathscr{M}_{r_f}$ and between the BLUPs under $\mathscr{M}_f$ and $\mathscr{M}_{r_f}$. Let us start with a useful lemma.

**Lemma 3.1** *Let $\mathbf{G}$, $\mathbf{T}$ and $\mathbf{T}_r$ be as given in (1.6),(1.7) and (2.5), respectively. Then* *(a) The pair of matrix equations (1.6) and (2.5) have a common solution if and only if*

$$r\begin{pmatrix}\mathbf{X} & \mathbf{V}\mathbf{E}_{\mathbf{X}} & \mathbf{V}\mathbf{E}_{\mathbf{X}_{\mathbf{A}}}\\ \mathbf{X}_f & \mathbf{0} & \mathbf{W}'\mathbf{E}_{\mathbf{X}_{\mathbf{A}}}\end{pmatrix} = r(\mathbf{X},\mathbf{V}).$$

*(b) The pair of matrix equations (1.7) and (2.5) have a common solution if and only if*

$$r\begin{pmatrix}\mathbf{X} & \mathbf{V}\mathbf{E}_{\mathbf{X}_{\mathbf{A}}}\\ \mathbf{X}_f & \mathbf{W}'\mathbf{E}_{\mathbf{X}_{\mathbf{A}}}\end{pmatrix} = r(\mathbf{X},\mathbf{V}).$$

**Proof** By Lemma 2.7, we get that the pair of matrix equations (1.6) and (2.5) have a common solution if and only if

$$r\begin{pmatrix}\mathbf{X} & \mathbf{V}\mathbf{E}_{\mathbf{X}} & \mathbf{X}_{\mathbf{A}} & \mathbf{V}\mathbf{E}_{\mathbf{X}_{\mathbf{A}}}\\ \mathbf{X}_f & \mathbf{0} & \mathbf{X}_{f\mathbf{A}} & \mathbf{W}'\mathbf{E}_{\mathbf{X}_{\mathbf{A}}}\end{pmatrix} = r(\mathbf{X},\mathbf{V}\mathbf{E}_{\mathbf{X}},\mathbf{X}_{\mathbf{A}},\mathbf{V}\mathbf{E}_{\mathbf{X}_{\mathbf{A}}}).$$

Since $\mathscr{C}(\mathbf{X},\mathbf{V}\mathbf{E}_{\mathbf{X}}) = \mathscr{C}(\mathbf{X},\mathbf{V})$, we have

$$r\begin{pmatrix}\mathbf{X} & \mathbf{V}\mathbf{E}_{\mathbf{X}} & \mathbf{V}\mathbf{E}_{\mathbf{X}_{\mathbf{A}}}\\ \mathbf{X}_f & \mathbf{0} & \mathbf{W}'\mathbf{E}_{\mathbf{X}_{\mathbf{A}}}\end{pmatrix} = r(\mathbf{X},\mathbf{V}\mathbf{E}_{\mathbf{X}_{\mathbf{A}}}).$$

Similarly, we can obtain (b). □

**Theorem 3.2** *Under the model* $\mathscr{M}_{r_f}$, $\mathrm{OLSE}(\mathbf{X}_f\boldsymbol{\beta}|\mathscr{M}) = \mathrm{BLUP}(\mathbf{y}_f|\mathscr{M}_{r_f})$ *holds almost surely if and only if any of the following equivalent statements holds:*

*(a)* $\mathscr{C}[\mathbf{V}(\mathbf{X}^+)'\mathbf{X}_f' - \mathbf{W}] \subseteq \mathscr{C}(\mathbf{X}_{\mathbf{A}})$.

*(b)* $\mathscr{C}\begin{pmatrix}\mathbf{W}\\ \mathbf{X}_f'\end{pmatrix} \subseteq \mathscr{C}\begin{pmatrix}\mathbf{V}\mathbf{X} & \mathbf{X}_{\mathbf{A}}\\ \mathbf{X}'\mathbf{X} & \mathbf{0}\end{pmatrix}$.

*(c)* $r\begin{pmatrix}\mathbf{V}\mathbf{X} & \mathbf{X}_{\mathbf{A}} & \mathbf{W}\\ \mathbf{X}'\mathbf{X} & \mathbf{0} & \mathbf{X}_f'\end{pmatrix} = r\begin{pmatrix}\mathbf{X}\\ \mathbf{A}\end{pmatrix} + r(\mathbf{X}) - r(\mathbf{A})$.

**Proof** Note that

$$\mathbf{X}_f\mathbf{X}^+\mathbf{y} = \mathbf{X}_f\mathbf{A}^+\mathbf{b} + \mathbf{X}_f\mathbf{X}^+(\mathbf{y} - \mathbf{X}\mathbf{A}^+\mathbf{b}).$$

Hence $\mathrm{OLSE}(\mathbf{X}_f\boldsymbol{\beta}|\mathscr{M}) = \mathrm{BLUP}(\mathbf{y}_f|\mathscr{M}_{r_f})$ holds almost surely if and only if

$$(\mathbf{X}_f\mathbf{X}^+ - \mathbf{T}_r)(\mathbf{y} - \mathbf{X}\mathbf{A}^+\mathbf{b}) = \mathbf{0}$$

for all $\mathbf{y} - \mathbf{X}\mathbf{A}^+\mathbf{b} \in \mathscr{C}(\mathbf{X}_{\mathbf{A}},\mathbf{V}\mathbf{E}_{\mathbf{X}_{\mathbf{A}}})$, i.e.,

$$\mathbf{X}_f\mathbf{X}^+[\mathbf{X}_{\mathbf{A}},\mathbf{V}\mathbf{E}_{\mathbf{X}_{\mathbf{A}}}] = [\mathbf{X}_{f\mathbf{A}},\mathbf{W}'\mathbf{E}_{\mathbf{X}_{\mathbf{A}}}]. \tag{3.1}$$

Since $\mathscr{C}(\mathbf{X}_f') \subseteq \mathscr{C}(\mathbf{X}')$, clearly

$$\mathbf{X}_f\mathbf{X}^+\mathbf{X}_{\mathbf{A}} = \mathbf{X}_{f\mathbf{A}}.$$

Therefore, (3.1) is equivalent to

$$\mathbf{X}_f\mathbf{X}^+\mathbf{V}\mathbf{E}_{\mathbf{X}_{\mathbf{A}}} = \mathbf{W}'\mathbf{E}_{\mathbf{X}_{\mathbf{A}}},$$

i.e., (a) holds. Suppose that (b) holds, i.e., there exist matrices $\mathbf{C}$ and $\mathbf{D}$ so that

$$\text{(i) } \mathbf{W} = \mathbf{V}\mathbf{X}\mathbf{C} + \mathbf{X}_{\mathbf{A}}\mathbf{D}, \quad \text{(ii) } \mathbf{X}_f' = \mathbf{X}'\mathbf{X}\mathbf{C}. \tag{3.2}$$

The general solution to (3.2 (ii)) is

$$\mathbf{C} = (\mathbf{X}'\mathbf{X})^{-}\mathbf{X}_f'. \tag{3.3}$$

Substituting (3.3) into (3.2 (i)) yields

$$\mathbf{W} = \mathbf{V}(\mathbf{X}^{+})'\mathbf{X}_f' + \mathbf{X}_{\mathbf{A}}\mathbf{D},$$

which is $(a)$. Conversely, recalling $\mathscr{C}(\mathbf{X}_f') \subseteq \mathscr{C}(\mathbf{X}') = \mathscr{C}(\mathbf{X}'\mathbf{X})$, i.e.,

$$\mathbf{X}_f' = \mathbf{X}'\mathbf{X}\mathbf{C}$$

for some matrix $\mathbf{C}$, (a) implies

$$\mathscr{C}[\mathbf{V}\mathbf{X}(\mathbf{X}'\mathbf{X})^{+}\mathbf{X}'\mathbf{X}\mathbf{C} - \mathbf{W}] = \mathscr{C}[\mathbf{V}\mathbf{X}\mathbf{C} - \mathbf{W}] \subseteq \mathscr{C}(\mathbf{X}_{\mathbf{A}}).$$

So we obtain (3.2). By (2.12) in Lemma 2.3, it is easy to show the equivalence between (b) and (c).

**Theorem 3.3** *Under the model $\mathscr{M}_{r_f}$, $\mathrm{OLSE}(\mathbf{X}_f\boldsymbol{\beta}|\mathscr{M}_r) = \mathrm{BLUP}(\mathbf{y}_f|\mathscr{M}_{r_f})$ holds almost surely if and only if any of the following equivalent statements holds:*

*(a)* $\mathscr{C}[\mathbf{V}(\mathbf{X}_{\mathbf{A}}^{+})'\mathbf{X}_{f\mathbf{A}}' - \mathbf{W}] \subseteq \mathscr{C}(\mathbf{X}_{\mathbf{A}})$.

*(b)* $\mathscr{C}\begin{pmatrix} \mathbf{W} \\ \mathbf{X}_{f\mathbf{A}}' \end{pmatrix} \subseteq \mathscr{C}\begin{pmatrix} \mathbf{V}\mathbf{X}_{\mathbf{A}} & \mathbf{X}_{\mathbf{A}} \\ \mathbf{X}_{\mathbf{A}}'\mathbf{X}_{\mathbf{A}} & \mathbf{0} \end{pmatrix}$.

*(c)* $r\begin{pmatrix} \mathbf{V}\mathbf{X}_{\mathbf{A}} & \mathbf{X}_{\mathbf{A}} & \mathbf{W} \\ \mathbf{X}_{\mathbf{A}}'\mathbf{X}_{\mathbf{A}} & \mathbf{0} & \mathbf{X}_{f\mathbf{A}}' \end{pmatrix} = 2r\begin{pmatrix} \mathbf{X} \\ \mathbf{A} \end{pmatrix} - 2r(\mathbf{A})$.

**Proof** From (2.3) and (2.5), $\mathrm{OLSE}(\mathbf{X}_f\boldsymbol{\beta}|\mathscr{M}_r) = \mathrm{BLUP}(\mathbf{y}_f|\mathscr{M}_{r_f})$ holds almost surely if and only if

$$(\mathbf{X}_{f\mathbf{A}}\mathbf{X}_{\mathbf{A}}^{+} - \mathbf{T}_r)(\mathbf{y} - \mathbf{X}\mathbf{A}^{+}\mathbf{b}) = \mathbf{0}$$

for all $\mathbf{y} - \mathbf{X}\mathbf{A}^{+}\mathbf{b} \in \mathscr{C}(\mathbf{X}_{\mathbf{A}}, \mathbf{V}\mathbf{E}_{\mathbf{X}_{\mathbf{A}}})$, i.e.,

$$\mathbf{X}_{f\mathbf{A}}\mathbf{X}_{\mathbf{A}}^{+}(\mathbf{X}_{\mathbf{A}}, \mathbf{V}\mathbf{E}_{\mathbf{X}_{\mathbf{A}}}) = (\mathbf{X}_{f\mathbf{A}}, \mathbf{W}'\mathbf{E}_{\mathbf{X}_{\mathbf{A}}}). \tag{3.4}$$

Noting $\mathscr{C}(\mathbf{X}_{f\mathbf{A}}' \subseteq \mathscr{C}(\mathbf{X}_{\mathbf{A}}')$, (3.4) is true if and only if

$$\mathbf{X}_{f\mathbf{A}}\mathbf{X}_{\mathbf{A}}^{+}\mathbf{V}\mathbf{E}_{\mathbf{X}_{\mathbf{A}}} = \mathbf{W}'\mathbf{E}_{\mathbf{X}_{\mathbf{A}}},$$

which is precisely (a). The equivalence between (a) and (b) is similar to that of Theorem 3.2. By (2.12) in Lemma 2.3, we can establish the equivalence between (b) and (c). □

**Theorem 3.4** *Under the model $\mathscr{M}_{r_f}$, $\mathrm{BLUE}(\mathbf{X}_f\boldsymbol{\beta}|\mathscr{M}) = \mathrm{BLUP}(\mathbf{y}_f|\mathscr{M}_{r_f})$ holds almost surely if and only if any of the following equivalent statements holds:*

*(a) The pair of matrix equations (1.6) and (2.5) have a common solution.*

*(b)* $r\begin{pmatrix} \mathbf{X} & \mathbf{V}\mathbf{E}_{\mathbf{X}} & \mathbf{V}\mathbf{E}_{\mathbf{X}_{\mathbf{A}}} \\ \mathbf{X}_f & \mathbf{0} & \mathbf{W}'\mathbf{E}_{\mathbf{X}_{\mathbf{A}}} \end{pmatrix} = r(\mathbf{X}, \mathbf{V}\mathbf{E}_{\mathbf{X}_{\mathbf{A}}})$.

*(c)* $r\begin{pmatrix} \mathbf{V} & \mathbf{X} & \mathbf{V}\mathbf{E}_{\mathbf{X}} & \mathbf{0} \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{A}' \\ \mathbf{W}' & \mathbf{X}_f & \mathbf{0} & \mathbf{0} \end{pmatrix} = r\begin{pmatrix} \mathbf{X} \\ \mathbf{A} \end{pmatrix} + r(\mathbf{V}, \mathbf{X})$.

(d) $\mathscr{C}[(\mathbf{W}', \mathbf{X}_f, \mathbf{0}, \mathbf{0})'] \subseteq \mathscr{C}\left[\begin{pmatrix} \mathbf{V} & \mathbf{X} & \mathbf{VE_X} & \mathbf{0} \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{A}' \end{pmatrix}'\right]$.

(e) $\mathscr{C}\begin{pmatrix} \mathbf{W} \\ \mathbf{X}'_f \end{pmatrix} \cap \mathscr{C}\begin{pmatrix} \mathbf{VE_X} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}' \end{pmatrix} = \{\mathbf{0}\}$ *and* $\mathbf{F}_{(\mathbf{\Sigma},\mathbf{C})}\begin{pmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{F}_{(\mathbf{\Sigma},\mathbf{D})} = \mathbf{0}$,

*where* $\mathbf{\Sigma} = \begin{pmatrix} \mathbf{V} & \mathbf{X} \\ \mathbf{X}' & \mathbf{0} \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} \mathbf{VE_X} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}' \end{pmatrix}$ *and* $\mathbf{D} = \begin{pmatrix} \mathbf{W} \\ \mathbf{X}'_f \end{pmatrix}$.

**Proof** Note that

$$\mathbf{Gy} = \mathbf{X}_f\mathbf{A}^+\mathbf{b} + \mathbf{G}(\mathbf{y} - \mathbf{XA}^+\mathbf{b}).$$

Hence $\mathrm{BLUE}_{\mathscr{M}}(\mathbf{X}_f\boldsymbol{\beta}) = \mathrm{BLUP}_{\mathscr{M}_{r_f}}(\mathbf{y}_f)$ holds almost surely if and only if

$$(\mathbf{G} - \mathbf{T}_r)(\mathbf{y} - \mathbf{XA}^+\mathbf{b}) = \mathbf{0}$$

for all $\mathbf{y} - \mathbf{XA}^+\mathbf{b} \in \mathscr{C}(\mathbf{X_A}, \mathbf{VE_{X_A}})$, i.e.,

$$(\mathbf{G} - \mathbf{T}_r)(\mathbf{X_A}, \mathbf{VE_{X_A}}) = \mathbf{0},$$

which in view of $\mathscr{C}(\mathbf{X_A}, \mathbf{VE_{X_A}}) = \mathscr{C}(\mathbf{X_A}, \mathbf{V})$ becomes

$$(\mathbf{G} - \mathbf{T}_r)(\mathbf{X_A}, \mathbf{V}) = \mathbf{0}. \tag{3.5}$$

If (1.6) and (2.5) have a common solution $\mathbf{T}_*$, i.e., (a) holds, then

$$\begin{aligned}(\mathbf{G} - \mathbf{T}_r)(\mathbf{X_A}, \mathbf{V}) &= [(\mathbf{G} - \mathbf{T}_*) + (\mathbf{T}_* - \mathbf{T}_r)](\mathbf{X_A}, \mathbf{V}) \\ &= (\mathbf{G} - \mathbf{T}_*)(\mathbf{X_A}, \mathbf{V}) \\ &= (\mathbf{G} - \mathbf{T}_*)(\mathbf{X}, \mathbf{V})\begin{pmatrix} \mathbf{F_A} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_n \end{pmatrix} \\ &= \mathbf{0}.\end{aligned}$$

Trivially (3.5) implies (a). The equivalence of (a) and (b) follows from Lemma 3.1(a). Applying Lemma 2.3 to (b) gives

$$\begin{aligned} r\begin{pmatrix} \mathbf{X} & \mathbf{VE_X} & \mathbf{VE_{X_A}} \\ \mathbf{X}_f & \mathbf{0} & \mathbf{W}'\mathbf{E_{X_A}} \end{pmatrix} &= r\begin{pmatrix} \mathbf{X} & \mathbf{VE_X} & \mathbf{V} \\ \mathbf{X}_f & \mathbf{0} & \mathbf{W}' \\ \mathbf{0} & \mathbf{0} & \mathbf{X}'_\mathbf{A} \end{pmatrix} - r(\mathbf{X_A}) \\ &= r\begin{pmatrix} \mathbf{V} & \mathbf{X} & \mathbf{VE_X} & \mathbf{0} \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{A}' \\ \mathbf{W}' & \mathbf{X}_f & \mathbf{0} & \mathbf{0} \end{pmatrix} - r(\mathbf{X_A}) - r(\mathbf{A}), \\ r(\mathbf{X}, \mathbf{VE_{X_A}}) &= r\begin{pmatrix} \mathbf{X} & \mathbf{V} \\ \mathbf{0} & \mathbf{X}'_\mathbf{A} \end{pmatrix} - r(\mathbf{X_A}) \\ &= r\begin{pmatrix} \mathbf{X} & \mathbf{V} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}' & \mathbf{A}' \end{pmatrix} - r(\mathbf{X_A}) - r(\mathbf{A}) \\ &= r(\mathbf{V}, \mathbf{X}) + r\begin{pmatrix} \mathbf{X} \\ \mathbf{A} \end{pmatrix} - r(\mathbf{X_A}) - r(\mathbf{A}).\end{aligned}$$

Substituting these two rank equalities into (b) leads to the equivalence of (b) and (c). Note from Lemma 2.4 that

$$r\begin{pmatrix} \mathbf{V} & \mathbf{X} & \mathbf{V}\mathbf{E}_{\mathbf{X}} & \mathbf{0} \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{A}' \end{pmatrix} = r(\mathbf{V}, \mathbf{X}) + r\begin{pmatrix} \mathbf{X} \\ \mathbf{A} \end{pmatrix}.$$

So the equivalence of (c) and (d) follows from (2.13) in Lemma 2.3. By Lemma 2.6, it is easy to obtain the equivalence between (c) and (e).

**Remark 3.5** It should be observed that if the future observations $\mathbf{y}_f$ are uncorrelated with the observable random vector $\mathbf{y}$, i.e., $\mathbf{W} = \mathbf{0}$, then the BLUP of $\mathbf{y}_f$ under $\mathscr{M}_{r_f}$ reduces to the BLUE of $\mathbf{X}_f\boldsymbol{\beta}$ under $\mathscr{M}_r$. In this case, the above Theorems 3.2–3.4 coincide with [14, Theorems 3.3–3.5], respectively.

**Theorem 3.6** *Under the model* $\mathscr{M}_{r_f}$, $\mathrm{BLUE}(\mathbf{X}_f\boldsymbol{\beta}|\mathscr{M}_r) = \mathrm{BLUP}(\mathbf{y}_f|\mathscr{M}_{r_f})$ *holds almost surely if and only if any of the following equivalent statements holds:*

(a) $\mathbf{W}'\mathbf{E}_{\mathbf{X}_\mathbf{A}} = \mathbf{0}$.

(b) $\mathscr{C}(\mathbf{W}) \subseteq \mathscr{C}(\mathbf{X}_\mathbf{A})$.

(c) $\mathscr{C}\begin{pmatrix} \mathbf{W} \\ \mathbf{0} \end{pmatrix} \subseteq \mathscr{C}\begin{pmatrix} \mathbf{X} \\ \mathbf{A} \end{pmatrix}$.

**Proof** From Lemma 2.1(b) and Lemma 2.2, $\mathrm{BLUE}(\mathbf{X}_f\boldsymbol{\beta}|\mathscr{M}_r) = \mathrm{BLUP}(\mathbf{y}_f|\mathscr{M}_{r_f})$ holds almost surely if and only if

$$(\mathbf{G}_r - \mathbf{T}_r)(\mathbf{y} - \mathbf{X}\mathbf{A}^+\mathbf{b}) = \mathbf{0}$$

for all $\mathbf{y} - \mathbf{X}\mathbf{A}^+\mathbf{b} \in \mathscr{C}(\mathbf{X}_\mathbf{A}, \mathbf{V}\mathbf{E}_{\mathbf{X}_\mathbf{A}})$, i.e.,

$$(\mathbf{G}_r - \mathbf{T}_r)(\mathbf{X}_\mathbf{A}, \mathbf{V}\mathbf{E}_{\mathbf{X}_\mathbf{A}}) = \mathbf{0}. \tag{3.6}$$

In light of (2.4) and (2.5), (3.6) is equivalent to

$$(\mathbf{X}_{\mathbf{A}_f}, \mathbf{0}) = (\mathbf{X}_{\mathbf{A}_f}, \mathbf{W}'\mathbf{E}_{\mathbf{X}_\mathbf{A}}),$$

i.e.,

$$\mathbf{W}'\mathbf{E}_{\mathbf{X}_\mathbf{A}} = \mathbf{0}.$$

The equivalence of (a)–(c) follows from Lemma 2.3.

Sengupta et al. [21, p.278] showed that $\mathrm{BLUE}(\mathbf{X}_f\boldsymbol{\beta}|\mathscr{M}) = \mathrm{BLUE}(\mathbf{X}_f\boldsymbol{\beta}|\mathscr{M}_r)$ holds almost surely if $\mathscr{C}(\mathbf{X}') \cap \mathscr{C}(\mathbf{A}') = \{\mathbf{0}\}$, and

$$\mathrm{BLUE}(\mathbf{X}_f\boldsymbol{\beta}|\mathscr{M}_r) = \mathbf{X}_f\hat{\boldsymbol{\beta}} - \mathrm{Cov}(\mathbf{X}_f\hat{\boldsymbol{\beta}}, \mathbf{A}\hat{\boldsymbol{\beta}})\mathrm{Cov}(\mathbf{A}\hat{\boldsymbol{\beta}})^-(\mathbf{A}\hat{\boldsymbol{\beta}} - \mathbf{b})$$

if $\mathscr{C}(\mathbf{A}') \subseteq \mathscr{C}(\mathbf{X}')$, where $\mathbf{X}_f\hat{\boldsymbol{\beta}}$ and $\mathbf{A}\hat{\boldsymbol{\beta}}$ are the BLUE of $\mathbf{X}_f\boldsymbol{\beta}$ and $\mathbf{A}\boldsymbol{\beta}$ under $\mathscr{M}$, respectively. Under $\mathscr{C}(\mathbf{X}') \subseteq \mathscr{C}(\mathbf{A}')$, we have the following.

**Corollary 3.7** *Under the model* $\mathscr{M}_{r_f}$, *if* $\mathbf{X}_\mathbf{A} = \mathbf{0}$, *i.e.,* $\mathscr{C}(\mathbf{X}') \subseteq \mathscr{C}(\mathbf{A}')$, *then the following statements are equivalent:*

(a) $\mathrm{OLSE}(\mathbf{X}_f\boldsymbol{\beta}|\mathscr{M}_r) = \mathrm{BLUP}(\mathbf{y}_f|\mathscr{M}_{r_f})$ *holds almost surely.*

(b) $\mathrm{BLUE}(\mathbf{X}_f\boldsymbol{\beta}|\mathscr{M}_r) = \mathrm{BLUP}(\mathbf{y}_f|\mathscr{M}_{r_f})$ *holds almost surely.*

(c) $\mathbf{W}=\mathbf{0}$.

The following Lemma can be found in [6,9,19].

**Lemma 3.8** *Under the model* $\mathscr{M}_f$, $\mathrm{BLUE}(\mathbf{X}_f\boldsymbol{\beta}|\mathscr{M})=\mathrm{BLUP}(\mathbf{y}_f|\mathscr{M}_f)$ *holds almost surely if and only if* $\mathscr{C}(\mathbf{W})\subseteq\mathscr{C}(\mathbf{X})$.

**Remark 3.9** In addition to (1.4), another way in the literature is to combine (1.1) and (1.3) into the following implicitly restricted model

$$\mathscr{M}_{r_i}=\left\{\begin{pmatrix}\mathbf{y}\\ \mathbf{b}\end{pmatrix},\begin{pmatrix}\mathbf{X}\boldsymbol{\beta}\\ \mathbf{A}\boldsymbol{\beta}\end{pmatrix},\begin{pmatrix}\mathbf{V}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}\end{pmatrix}\right\}. \tag{3.7}$$

In such a case, the model (3.7) with new future observations can be written in the following form

$$\mathscr{M}_{r_{f_i}}=\left\{\begin{pmatrix}\mathbf{y}\\ \mathbf{b}\\ \mathbf{y}_f\end{pmatrix},\begin{pmatrix}\mathbf{X}\boldsymbol{\beta}\\ \mathbf{A}\boldsymbol{\beta}\\ \mathbf{X}_f\boldsymbol{\beta}\end{pmatrix},\begin{pmatrix}\mathbf{V}&\mathbf{0}&\mathbf{W}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{W}'&\mathbf{0}&\mathbf{V}_f\end{pmatrix}\right\}.$$

Hence, Theorem 3.6 follows from Lemma 3.8.

**Theorem 3.10** *Under the model* $\mathscr{M}_{r_f}$, $\mathrm{BLUP}(\mathbf{y}_f|\mathscr{M}_f)=\mathrm{BLUP}(\mathbf{y}_f|\mathscr{M}_{r_f})$ *holds almost surely if and only if any of the following equivalent statements holds:*

(a) *The pair of matrix equations (1.7) and (2.5) have a common solution.*

(b) $r\begin{pmatrix}\mathbf{X}&\mathbf{V}\mathbf{E}_{\mathbf{X_A}}\\ \mathbf{X}_f&\mathbf{W}'\mathbf{E}_{\mathbf{X_A}}\end{pmatrix}=r(\mathbf{X},\mathbf{V})$.

(c) $r\begin{pmatrix}\mathbf{V}&\mathbf{X}&\mathbf{0}\\ \mathbf{X}'&\mathbf{0}&\mathbf{A}'\\ \mathbf{W}'&\mathbf{X}_f&\mathbf{0}\end{pmatrix}=r(\mathbf{V},\mathbf{X})+r\begin{pmatrix}\mathbf{X}\\ \mathbf{A}\end{pmatrix}$.

(d) $\mathscr{C}[(\mathbf{W}',\mathbf{X}_f,\mathbf{0})']\subseteq\mathscr{C}\left[\begin{pmatrix}\mathbf{V}&\mathbf{X}&\mathbf{0}\\ \mathbf{X}'&\mathbf{0}&\mathbf{A}'\end{pmatrix}'\right]$.

(e) $\mathscr{C}\begin{pmatrix}\mathbf{W}\\ \mathbf{X}_f'\end{pmatrix}\cap\mathscr{C}\begin{pmatrix}\mathbf{0}\\ \mathbf{A}'\end{pmatrix}=\{\mathbf{0}\}$ *and* $\mathbf{F}_{(\boldsymbol{\Sigma},\mathbf{C})}\begin{pmatrix}\boldsymbol{\Sigma}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}\end{pmatrix}\mathbf{F}_{(\boldsymbol{\Sigma},\mathbf{D})}=\mathbf{0}$,

*where* $\boldsymbol{\Sigma}=\begin{pmatrix}\mathbf{V}&\mathbf{X}\\ \mathbf{X}'&\mathbf{0}\end{pmatrix}$, $\mathbf{C}=\begin{pmatrix}\mathbf{V}\mathbf{E}_{\mathbf{X}}&\mathbf{0}\\ \mathbf{0}&\mathbf{A}'\end{pmatrix}$ *and* $\mathbf{D}=\begin{pmatrix}\mathbf{W}\\ \mathbf{X}_f'\end{pmatrix}$.

**Proof** Note that

$$\mathbf{T}\mathbf{y}=\mathbf{X}_f\mathbf{A}^+\mathbf{b}+\mathbf{T}(\mathbf{y}-\mathbf{X}\mathbf{A}^+\mathbf{b}).$$

Hence $\mathrm{BLUP}(\mathbf{y}_f|\mathscr{M}_f)=\mathrm{BLUP}(\mathbf{y}_f|\mathscr{M}_{r_f})$ holds almost surely if and only if

$$(\mathbf{T}-\mathbf{T}_r)(\mathbf{y}-\mathbf{X}\mathbf{A}^+\mathbf{b})=\mathbf{0}$$

for all $\mathbf{y}-\mathbf{X}\mathbf{A}^+\mathbf{b}\in\mathscr{C}(\mathbf{X_A},\mathbf{V}\mathbf{E}_{\mathbf{X_A}})$, i.e.,

$$(\mathbf{T}-\mathbf{T}_r)(\mathbf{X_A},\mathbf{V})=\mathbf{0}. \tag{3.8}$$

If (1.7) and (2.5) have a common solution $\mathbf{T}_*$, i.e., (a) holds, then

$$(\mathbf{T}-\mathbf{T}_r)(\mathbf{X}_\mathbf{A},\mathbf{V}) = [(\mathbf{T}-\mathbf{T}_*)+(\mathbf{T}_*-\mathbf{T}_r)](\mathbf{X}_\mathbf{A},\mathbf{V}) = (\mathbf{T}-\mathbf{T}_*)(\mathbf{X}_\mathbf{A},\mathbf{V})$$
$$= (\mathbf{T}-\mathbf{T}_*)(\mathbf{X},\mathbf{V})\begin{pmatrix}\mathbf{F}_\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_n\end{pmatrix} = \mathbf{0}.$$

Trivially (3.8) implies (a). The equivalence of (a) and (b) follows from Lemma 3.1(b). The equivalence between (b)–(e) is similar to that of Theorem 3.4.

**Corollary 3.11** *Under the model $\mathscr{M}_{r_f}$, if* $\mathrm{BLUE}(\mathbf{X}_f\boldsymbol{\beta}|\mathscr{M}_r) = \mathrm{BLUP}(\mathbf{y}_f|\mathscr{M}_{r_f})$ *holds almost surely, then the following statements are equivalent:*

(a) $\mathrm{BLUE}(\mathbf{X}_f\boldsymbol{\beta}|\mathscr{M}) = \mathrm{BLUP}(\mathbf{y}_f|\mathscr{M}_{r_f})$ *holds almost surely.*

(b) $\mathrm{BLUP}(\mathbf{y}_f|\mathscr{M}_f) = \mathrm{BLUP}(\mathbf{y}_f|\mathscr{M}_{r_f})$ *holds almost surely.*

## 4. Conclusion

In this paper, we obtained necessary and sufficient conditions for the equalities between some estimators and BLUP under $\mathscr{M}_{r_f}$. These results can be applied to small area estimation [9]. In addition, we also derived the conditions for the BLUP under $\mathscr{M}_f$ to be the BLUP under $\mathscr{M}_{r_f}$, under which the linear equality restrictions (1.3) have no consequences on statistical inference.

Under the model $\mathscr{M}_f$, [8, 9] showed that $\mathrm{OLSE}(\mathbf{X}_f\boldsymbol{\beta}|\mathscr{M}) = \mathrm{BLUP}(\mathbf{y}_f|\mathscr{M}_f)$ holds almost surely if and only if

$$\mathscr{C}\begin{pmatrix}\mathbf{X}_f'\\ \mathbf{W}\end{pmatrix} \subseteq \mathscr{C}\begin{pmatrix}\mathbf{X}'\mathbf{X} & \mathbf{0}\\ \mathbf{V}\mathbf{X} & \mathbf{X}\end{pmatrix}. \tag{4.1}$$

When combining (1.1) and (1.3) into the implicitly restricted model (3.7) we cannot give Theorem 3.3 by (4.1) because the OLSE of $\mathbf{X}_f\boldsymbol{\beta}$ under $\mathscr{M}_r$ do not coincide with the OLSE of $\mathbf{X}_f\boldsymbol{\beta}$ under $\mathscr{M}_{r_i}$ (see [15]).

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