Journal of Mathematical Research with Applications
Sept., 2018, Vol. 38, No. 5, pp. 509–522
DOI:10.3770/j.issn:2095-2651.2018.05.009
Http://jmre.dlut.edu.cn

A Classification of 3-dimensional Paracontact Metric Manifolds with $\varphi l = l\varphi$

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Abstract Let M^3 be a 3-dimensional paracontact metric manifold. Firstly, a classification of M^3 satisfying $\varphi Q = Q\varphi$ is given. Secondly, manifold M^3 satisfying $\varphi l = l\varphi$ and having η-parallel Ricci tensor or cyclic η-parallel Ricci tensor is studied.

Keywords paracontact metric manifold; para-Sasakian manifold; η-parallel Ricci tensor; cyclic η-parallel Ricci tensor

MR(2010) Subject Classification 53C15; 53C25; 53D10; 53D15

1. Introduction

Blair, Koufogiorgos and Sharma [1] proved that if M^3 satisfies $Q\varphi = \varphi Q$, then it is either flat, Sasakian or of constant ξ-sectional curvature $k < 1$ and of constant φ-sectional curvature $-k$. Furthermore, they proved that $Q\varphi = \varphi Q$ implies $l\varphi = \varphi l$. Perrone [2] proved that on any contact metric manifold the following conditions are equivalent:

$$\nabla_\xi h = 0,\ \nabla_\xi l = 0,\ \nabla_\xi \tau = 0,\ l\varphi = \varphi l,\ \ \tau = \mathcal{L}_\xi g. \tag{1.1}$$

Hence, the class of the 3-dimensional contact metric manifolds satisfying (1.1) generalizes the above mentioned classes in [1]. Andreou and Xenos [3] gave the study of the 3-dimensional contact metric manifolds satisfying one of (1.1) and obtained the classification theorem under the condition such as harmonic curvature, or η-parallel Ricci tensor or cyclic η-parallel Ricci tensor. In parallel with contact and complex structures in the Riemannian case, paracontact metric structures were introduced in [4] in semi-Riemannian settings, as a natural odd-dimensional counterpart to para-Hermitian structures. For a long time, the study of paracontact metric manifolds focused essentially on the special case of para-Sasakian manifolds. In 2009, Zamkovoy [5] undertook a systematic study of paracontact metric manifolds, since then, the study of paracontact metric geometry has attracted a growing number of researchers and paracontact metric manifolds have been studied under several different points of view. In particular, paracontact (κ, μ)-spaces were studied in [6]; The classification of para-Sasakian space forms was obtained in [7]; Three-dimensional homogeneous paracontact metric manifolds were classified in [8]; The geometry of H-paracontact metric manifolds were studied in [9] and so on.

Received April 10, 2018; Accepted July 17, 2018
Supported by the National Natural Science Foundation of China (Grant No. 11371076).
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Motivated by [1] and [3], the aim of the present paper is to investigate $Q\varphi=\varphi Q$ and more generally $l\varphi=\varphi l$ in 3-dimensional paracontact metric manifolds. Under this point of view, we distinguish three cases according to the type of h. This makes it interesting to study the above properties in the paracontact settings.

The paper is organized in the following way. In Section 2 we report some basic information about paracontact metric manifolds; In Section 3, we prove some properties of 3-dimensional paracontact metric manifold M^3 satisfying $Q\varphi=\varphi Q$, where we also give a classification theorem of M^3. In Section 4 we mainly discuss paracontact metric manifolds with $l\varphi=\varphi l$, and give some conditions under which $l\varphi=\varphi l$ is equivalent to $Q\varphi=\varphi Q$. In the last two sections, we studied M^3 satisfying $l\varphi=\varphi l$ and having η-parallel Ricci tensor or cyclic η-parallel Ricci tensor.

2. Preliminaries

Now, we recall some basic notions of almost paracontact manifold [6]. A $2n+1$-dimensional smooth manifold M is said to have an almost paracontact structure if it admits a (1,1)-tensor field φ, a vector field ξ and a 1-form η satisfying the following conditions:

(1) $\varphi^2=\mathrm{Id}-\eta\otimes\xi$, $\eta(\xi)=1$;

(2) the tensor field φ induces an almost paracomplex structure on each fibre of $\mathcal{D}=\ker(\eta)$, i.e., the $\pm1-$eigendistributions $\mathcal{D}^{\pm}:=\mathcal{D}_{\varphi}(\pm1)$ of φ have equal dimension n.

From the definition it follows that $\varphi(\xi)=0$, $\eta\circ\phi=0$ and $\mathrm{rank}(\varphi)=2n$. When the tensor field $\mathcal{N}_{\varphi}:=[\varphi,\varphi]-2d\eta\otimes\xi$ vanishes identically, the almost paracontact manifold is said to be normal. If an almost paracontact manifold admits a pseudo-Riemannian metric g such that

$$g(\varphi X,\varphi Y)=-g(X,Y)+\eta(X)\eta(Y) \tag{2.1}$$

for any vector fields $X,Y\in\Gamma(TM)$. Then we say that $(M^{2n+1},\phi,\xi,\eta,g)$ is an almost paracontact metric manifold. Notice that any such a pseudo-Riemannian metric is necessarily of signature $(n,n+1)$. Moreover, we can define a skew-symmetric tensor field 2-form Φ by $\Phi(X,Y)=g(X,\varphi Y)$ usually called fundamental form. For an almost paracontact metric manifold, there always exists an orthogonal basis$\{\xi,X_1,\ldots,X_n,Y_1,\ldots,Y_n\}$ such that $g(X_i,X_j)=\delta_{ij},g(Y_i,Y_j)=-\delta_{ij}$ and $Y_i=\varphi X_i$, for any $i,j\in\{1,\ldots,n\}$. Such basis is called a φ-basis.

If in addition $\Phi(X,Y)=d\eta(X,Y)$ for all vector fields X,Y on M $(M^{2n+1},\varphi,\xi,\eta,g)$ is said to be a paracontact metric manifold.

Now let $(M^{2n+1},\varphi,\xi,\eta,g)$ be a paracontact metric manifold. We denote $l=R(\cdot,\xi)\xi$ and $h=\frac{1}{2}\mathcal{L}_{\xi}\varphi$ on M^{2n+1}, where R is the Riemannian curvature tensor of g and $\mathcal{L}$ is the Lie differentiation. Thus, the two $(1,1)$-type tensor fields l and h are symmetric and satisfy

$$h\xi=0,\ l\xi=0,\ \mathrm{tr}h=0,\ \mathrm{tr}(h\varphi)=0,\ h\varphi+\varphi h=0. \tag{2.2}$$

We also have the following formulas on a paracontact metric manifold

$$\nabla_X\xi=-\varphi X+\varphi hX,\ \ \Rightarrow\nabla_{\xi}\xi=0, \tag{2.3}$$

$$\mathrm{tr}l=\mathrm{tr}h^2-2n, \tag{2.4}$$

$$\nabla_\xi h = -\varphi - \varphi l + h^2\varphi, \tag{2.5}$$

$$\nabla_\xi \varphi = 0, \tag{2.6}$$

$$\varphi l \varphi + l = 2(h^2 - \varphi^2). \tag{2.7}$$

Formulas occur in [10]. Moreover $h \equiv 0$ if and only if ξ is a killing vector and in this case M is said to be a K-paracontact manifold. A normal paracontact metric manifold is called a para-Sasakian manifold. Also in this context the para-Sasakian condition implies the K-pracontact condition and the converse holds only in dimension 3 (see [8]). Moreover, in any para-Sasakian manifold

$$R(X,Y)\xi = -(\eta(Y)X - \eta(X)Y) \tag{2.8}$$

holds, but unlike contact metric geometry, the condition (2.8) not necessarily implies that the manifold is para-Sasakian. On a 3-dimensional pseudo-Riemannian manifold, since the conformal curvature tensor vanishes identically, the curvature tensor R takes the form [9]

$$\begin{aligned} R(X,Y)Z =& g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y \\ &- \frac{r}{2} g(Y,Z)X - g(X,Z)Y, \end{aligned} \tag{2.9}$$

where r is the scalar curvature of the manifold and the Ricci operator Q is defined by

$$R(X,Y)\xi = -(\eta(Y)X - \eta(X)Y). \tag{2.10}$$

Recall that on a 3-dimensional paracontact metric manifold, we have

$$R(X,Y)\xi = -(\eta(Y)X - \eta(X)Y). \tag{2.11}$$

Given a paracontact metric (φ, ξ, η, g) and $t \neq 0$, the change of structure tensors

$$\widetilde{\eta} = t\eta, \widetilde{\xi} = \frac{1}{t}\xi, \widetilde{\varphi} = \varphi, \widetilde{g} = tg + t(t-1)\eta \otimes \eta$$

is called a D_t-homothetic deformation. And one can easily check that the new structure $\{\widetilde{\eta}, \widetilde{\xi}, \widetilde{\varphi}, \widetilde{g}\}$ is still a paracontact metric structure, the D_t-homothetic deformation destroy conditions like $R(X,Y)\xi = 0$, but they preserve the class of paracontact (k,μ)-manifold. Some remarkable subclasses of paracontact metric (k,μ)-manifolds are given. For example, in any para-Sasakian manifold, $R(X,Y)\xi = -(\eta(Y)X - \eta(X)Y)$ holds, but unlike in contact metric geometry, the converse does not hold necessarily. For more details see [6]. For those paracontact metric manifolds such that $R(X,Y)\xi = 0$ for all vector fields X, Y on M (see [5]) gave the theorem.

Theorem 2.1 ([5]) *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a paracontact manifold and suppose that $R(X,Y)\xi = 0$ for all vector fields X, Y on M. Then locally M^{2n+1} is the product of a flat $(n+1)$-dimensional manifold and n-dimensional manifold of negative constant curvature equal to -4.*

Erken and Murathan analyzed the different possibilities for the tensor field h in [9]. If h has form

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to local orthonormal φ-basis $\{e, \varphi e, \xi\}$, where $g(e, e) = -1$, then the operator h is said to be of η_1 type.

If h has form

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to a pseudo orthonormal basis $\{e_1, e_2, \xi\}$, where $g(e_1, e_1) = g(e_2, e_2) = g(e_1, \xi) = g(e_2, \xi) = 0, g(e_1, e_2) = g(\xi, \xi) = 0$, in this case h is said to be of η_2 type. If the matrix form of h has the form

$$\begin{pmatrix} 0 & -\lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to the local orthonormal φ-basis $\{e, \varphi e, \xi\}$, where $g(e, e) = -1$, then the operator h is said to be of η_3 type. And from [9, Propositions 4.3, 4.9 and 4.13] we know that on a 3-dimensional paracontact metric manifold, it holds

$$h^2 - \varphi^2 = \frac{\mathrm{tr}l}{2}\varphi^2. \tag{2.12}$$

3. On paracontact metric manifolds with $Q\varphi = \varphi Q$

In this section, we shall prove some properties of 3-dimensional paracontact metric manifolds satisfying $Q\varphi = \varphi Q$.

Lemma 3.1 *Let $(M^3, \varphi, \xi, \eta, g)$ be a paracontact metric manifold with $Q\varphi = \varphi Q$. Then the function* tr$l$ *is constant everywhere on M^3.*

Proof Since $Q\varphi = \varphi Q$, it is easy to get that $Q\xi = (\mathrm{tr}l)\xi$. By the definition of l and using (2.9), we have for any X,

$$lX = QX + (\mathrm{tr}l - \frac{r}{2})X + (\frac{r}{2} - 2\mathrm{tr}l)\eta(X)\xi. \tag{3.1}$$

Combining (3.1) with $Q\varphi = \varphi Q$, it follows that $l\varphi = \varphi l$. Using (2.7), we directly get

$$l = h^2 - \varphi^2. \tag{3.2}$$

By (2.5), we get $\nabla_\xi h = 0$ and therefore $\nabla_\xi l = 0$. We declare that $\xi(\mathrm{tr}l) = 0$. In fact, if h is of η_1 type, we choose the φ-basis $\{e, \varphi e, \xi\}$, such that $he = \lambda e$, and $g(e, e) = -1$. By (3.2), we get that $le = (\lambda^2 - 1)e$. If h is of η_3 type, we choose the φ-basis $\{e, \varphi e, \xi\}$, such that $he = \lambda\varphi e, h\varphi e = -\lambda e$, also by (3.2), we get that $le = -(\lambda^2 + 1)e$. In these two cases, $\xi(\mathrm{tr}l) = -\xi g(le, e) + \xi g(l\varphi e, \varphi e) + \xi g(l\xi, \xi) = 0$. If h is of η_2 type, we choose a pseudo orthonormal basis $\{e_1, e_2, \xi\}$, such that $he_1 = e_2, he_2 = 0$, and $\varphi e_1 = e_1, \varphi e_2 = -e_2$. By (3.2), we get that $le_1 = -e_1, le_2 = -e_2$, thus $\xi(\mathrm{tr}l) = 0$.

By (2.12) and (3.2), we obtain

$$l = \frac{\mathrm{tr}l}{2}\varphi^2 X. \tag{3.3}$$

Substituting (3.3) in (3.1), we get

$$QX = aX + b\eta(X)\xi, \tag{3.4}$$

where $a = \frac{1}{2}(r - \mathrm{tr}l)$ and $b = \frac{1}{2}(3\mathrm{tr}l - r)$. Differentiating (3.4) with respect to Y we find

$$(\nabla_Y Q)X = Y(a)X + Y(b)\eta(X)\xi + bg(\nabla_Y\xi, X)\xi + b\eta(X)\nabla_Y\xi. \tag{3.5}$$

Letting $X = Y = \xi$ and using $\xi(\mathrm{tr}l) = 0$, we get $(\nabla_\xi Q)\xi = 0$. Now we carry out discussion according to the different type of h.

If h is of η_1 type, substituting $X = Y$ by e and φe, we obtain $(\nabla_e Q)e = e(a)e$ and $(\nabla_{\varphi e}Q)\varphi e = \varphi e(a)\varphi e$ by the well known formula

$$\sum_{i=1}^{3}\varepsilon_i(\nabla_{X_i}Q)X_i = \frac{1}{2}\mathrm{grad}\, r. \tag{3.6}$$

Therefore, it follows that $\xi(r) = 0$.

If h is of η_2 type, setting $X = e_1$, $Y = e_2$ and $X = e_2$, $Y = e_1$, we get $(\nabla_{e_1}Q)e_2 = e_1(a)e_2 - b\xi$ and $(\nabla_{e_2}Q)e_1 = e_2(a)e_1 + b\xi$. Using (3.6), we get $\xi(r) = 0$.

If h is of η_3 type, resetting $X = Y = e$ and $X = Y = \varphi e$, we get $(\nabla_e Q)e = e(a)e - \lambda b\xi$ and $(\nabla_{\varphi e}Q)\varphi e = \varphi e(a)\varphi e - \lambda b\xi$. Using (3.6), we have $\xi(r) = 0$.

It is easy to get that for any vector field X, $(\nabla_\xi Q)X = \xi(a)X = 0$, and thus $\nabla_\xi Q = 0$. Using (2.9), we get $\nabla_\xi R = 0$. By the second Bianchi identity, we get

$$(\nabla_X R)(Y, \xi, Z) = (\nabla_Y R)(X, \xi, Z) = 0. \tag{3.7}$$

Substituting (3.4) into (2.9), we get

$$\begin{aligned} R(X,Y)Z =& \{cg(Y,Z) + b\eta(Y)\eta(Z)\}X - \{cg(X,Z) + b\eta(X)\eta(Z)\}Y + \\ & b\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\}\xi, \end{aligned} \tag{3.8}$$

where $c = \frac{r}{2} - \mathrm{tr}l$. Let $Z = \xi$ in (3.8). We obtain

$$R(X,Y)\xi = \frac{\mathrm{tr}l}{2}(\eta(Y)X - \eta(X)Y). \tag{3.9}$$

Differentiating (3.9), we get that

$$(\nabla_X R)(Y, \xi, \xi) = \frac{1}{2}(X\mathrm{tr}l)Y \tag{3.10}$$

for any X, Y orthogonal to ξ. Combining (3.7) with (3.10), we get that $X\mathrm{tr}l = 0$. Since $\xi\mathrm{tr}l = 0$, it follows that $\mathrm{tr}l$ is constant. Thus, we complete the proof. □

Remark 3.2 If $\mathrm{tr}l = \text{const.} = 0$, by (3.9), it follows that $R(X,Y)\xi = 0$. By Theorem 3.3 for $n = 1$ in [5], M^3 is flat.

If $\mathrm{tr}l = \text{const.} = -2$, by (2.4) for $n = 1$, we get $\mathrm{tr}h^2 = 0$. And since $\mathrm{tr}h^2 = 2\lambda^2 \geq 0$ if h is of η_1 type; $\mathrm{tr}h^2 = 2\lambda^2 < 0$ ($\lambda \neq 0$) if h is of η_3 type; $\mathrm{tr}h^2 = 0$ if h is of η_2 type. Therefore, if h is of η_1 type, then $\lambda = 0$ and M^3 is a para-Sasakian manifold, otherwise, h is of η_2 type.

Using Lemma 3.1, we can easily obtain the following proposition using a similar method to [1, Proposition 3.2].

Proposition 3.3 *Let $(M^3, \varphi, \xi, \eta, g)$ be a paracontact metric manifold. Then the following conditions are equivalent:*

(1) M^3 is η-Einstein;

(2) $Q\varphi = \varphi Q$;

(3) ξ belongs to the κ-nullity distribution, i.e., $\xi \in \mathcal{N}(\kappa)$.

To note that, differently from the contact metric case, $\xi \in \mathcal{N}(\kappa)$ is necessary but not sufficient for a paracontact metric manifold to be para-Sasakian. This fact was already pointed out in papers (see for example [6], but the first example in dimension three appeared in [9]).

By Lemma 3.1, we know that $Q\xi = \mathrm{tr}l\xi$, and by [9] we conclude that

Corollary 3.4 *Let $(M^3, \varphi, \xi, \eta, g)$ be a paracontact metric manifold with $Q\varphi = \varphi Q$. Then M^3 is H-paracontact.*

Combining with Proposition 3.3 and Corollary 3.4, we have the conclusion that 3-dimensional η-Einstein paracontact metric manifolds are H-paracontact. For a paracontact metric manifold M^3, if $\xi \in \mathcal{N}(\kappa)$, then M^3 is H-paracontact.

Using (2.3) and after direct calculations, we get the following proposition

Proposition 3.5 *Let $(M^3, \varphi, \xi, \eta, g)$ be a paracontact metric manifold. Then*

$$R(X,Y)\xi = \eta(X)(Y - hY) - \eta(Y)(X - hX) + \varphi((\nabla_X h)Y - (\nabla_Y h)X). \tag{3.11}$$

Theorem 3.6 *Let $(M^3, \varphi, \xi, \eta, g)$ be a paracontact metric manifold with $Q\varphi = \varphi Q$. Then M^3 is either flat, para-Sasakian, or h is of η_2 type or of constant ξ-sectional curvature $\kappa < 1$ and constant φ-sectional curvature $-\kappa$.*

Proof By Remark 3.2, we know that if $\mathrm{tr}l = \text{const.} = 0$, M^3 is flat; If $\mathrm{tr}l = \text{const.} = -2$, then M^3 is either para-Sasakian or h is of η_2 type.

We mainly discuss $\mathrm{tr}l = \text{const.} \neq 0, -2$. Combining (3.9) and (3.11), we obtain

$$\eta(Y)hX - \eta(X)hY - \varphi((\nabla_X h)Y - (\nabla_Y h)X) = (\kappa - 1)(\eta(Y)X - \eta(X)Y), \tag{3.12}$$

where $\kappa = \frac{\mathrm{tr}l}{2} \neq 0, -1$. Since $\mathrm{tr}l = \text{const.} \neq 0, -2$, h can be only of η_1 and η_3 types, so we only need to separate the question into two cases.

Case 1 We suppose that h is of η_1 type. We choose the local orthonormal φ-basis $\{X, \varphi X, \xi\}$, where $g(X,X) = -1, hX = \lambda X$, thus $\mathrm{tr}h^2 = 2\lambda^2$ and $\lambda = \sqrt{1+\kappa} \neq 0$, since $\kappa = \frac{\mathrm{tr}l}{2}$ is constant, then λ is also constant. Putting $Y = \varphi X$ in (3.12), we have

$$\varphi((\nabla_X h)\varphi X - (\nabla_{\varphi X} h)X) = 0, \tag{3.13}$$

which implies that

$$\varphi(-\lambda(\nabla_X \varphi X) - h\nabla_X \varphi X - \lambda \nabla_{\varphi X} X + h\nabla_{\varphi X} X) = 0. \tag{3.14}$$

Taking the inner product of (3.14) with X and recalling that $\lambda \neq 0$, we obtain $g(\nabla_{\varphi X} X, \varphi X) = 0$. What is more, $g(\nabla_{\varphi X} X, X) = 0$, and $g(\nabla_{\varphi X} X, \xi) = -(1+\lambda)$. Hence $\nabla_{\varphi X} X = -(1+\lambda)\xi$.

Similarly taking the inner product of (3.14) with φX yields $\nabla_X \varphi X = (1-\lambda)\xi$. It is easy to get that $\nabla_X X = 0,\ [X, \varphi X] = 2\xi$.

By (3.8), we have

$$R(X, \varphi X)X = -cg(X, X)\varphi X = (\frac{r}{2} - \mathrm{tr}l)\varphi X. \tag{3.15}$$

On the other hand, by direct calculations, we get

$$R(X, \varphi X)X = \nabla_X \nabla_{\varphi X} X - \nabla_{\varphi X} \nabla_X X - \nabla_{[X,\varphi X]} X = (1-\lambda^2)\varphi X - 2\nabla_\xi X. \tag{3.16}$$

Comparing (3.15) with (3.16), we obtain

$$\nabla_\xi X = (\frac{\lambda^2 - 1}{2} - \frac{r}{4})\varphi X. \tag{3.17}$$

Therefore, we have

$$[\xi, X] = (\frac{(\lambda - 1)^2}{2} - \frac{r}{4})\varphi X. \tag{3.18}$$

Now we compute $R(\xi, X)\xi$ in two ways. By (3.9), we have

$$R(\xi, X)\xi = -\kappa X. \tag{3.19}$$

On the other hand, by direct calculations, we obtain

$$\begin{aligned} R(\xi, X)\xi =& \nabla_\xi \nabla_X \xi - \nabla_X \nabla_\xi X - \nabla_{[\xi,X]}\xi \\ =& (\lambda - 1)(\frac{\lambda^2 - 1}{2} - \frac{r}{4})X + (1+\lambda)(\frac{(\lambda - 1)^2}{2} - \frac{r}{4})X. \end{aligned} \tag{3.20}$$

Comparing (3.19) with (3.20), we find

$$r = 2(\lambda^2 - 1) = 2\kappa. \tag{3.21}$$

From (3.15) and (3.19) it is easy to get that

$$K(X, \xi) = \kappa \ \text{ and } \ K(X, \varphi X) = -\kappa. \tag{3.22}$$

Case 2 Suppose that h is of η_3 type. We choose the local orthonormal φ-basis $\{X, \varphi X, \xi\}$, where $g(X, X) = -1$, $hX = \lambda\varphi X$, $h\varphi X = -\lambda X$, thus $\mathrm{Tr}h^2 = -2\lambda^2$ and $\lambda = \sqrt{-(1+\kappa)} \neq 0$. By similar methods we have

$$\begin{aligned} &\nabla_X X = \lambda\xi;\ \nabla_{\varphi X}\varphi X = \lambda\xi;\ \nabla_{\varphi X} X = -\xi;\ \nabla_X \varphi X = \xi;\ [X, \varphi X] = 2\xi \\ &[\xi, X] = -\lambda X + (1 - (\frac{\lambda^2 + 1}{2} + \frac{r}{4}))\varphi X;\quad r = 2\kappa = \mathrm{const}. \end{aligned} \tag{3.23}$$

Therefore, there still holds

$$K(X, \xi) = \kappa \ \text{ and } \ K(X, \varphi X) = -\kappa. \tag{3.24}$$

Thus, we complete the proof of Theorem 3.6. □

Remark 3.7 Note that for $\kappa \neq 0, -1$, since $r = 2\kappa = \mathrm{tr}l = \mathrm{const.}$, by (3.4), it follows that $a = 0$. Thus $QX = b\eta(X)\xi = 2\kappa\eta(X)\xi$.

Definition 3.8 *A paracontact metric structure (φ, ξ, η, g) is said to be locally φ-symmetric if*

$\varphi^2(\nabla_W R)(X,Y,Z)=0$, for any vector fields X,Y,Z,W orthogonal to ξ.

Theorem 3.9 *Let (M^3,φ,ξ,η,g) be a paracontact metric manifold with $Q\varphi=\varphi Q$. Then M^3 is locally φ-symmetric if and only if the scalar curvature r of M^3 is constant.*

The proof of Theorem 3.9 is similar to the proof of [1, Theorem 3.4], we omit here.

Corollary 3.10 *Let (M^3,φ,ξ,η,g) be a paracontact metric manifold with $Q\varphi=\varphi Q$. If $\mathrm{tr}l\neq -2$, then M^3 is locally φ-symmetric.*

Proof By the proof of Theorem 3.6, if $\mathrm{tr}l\neq 0,-2$, then $r=2\kappa$ is constant; And by Remark 3.2, we know if $\mathrm{tr}l=0$, M^3 is flat. Considering of the proof of Theorem 3.9, the Corollary 3.10 follows. □

4. On paracontact metric manifolds with $l\varphi=\varphi l$

In this section we shall mainly consider paracontact metric manifolds with $l\varphi=\varphi l$, and give some conditions under which $l\varphi=\varphi l$ is equivalent to $Q\varphi=\varphi Q$.

Proposition 4.1 *Let (M^3,φ,ξ,η,g) be a paracontact metric manifold. Then the following conditions are equivalent: $l\varphi=\varphi l\Leftrightarrow \nabla_\xi h=0\Leftrightarrow \nabla_\xi\tau=0$.*

Remark 4.2 It is easy to get $\nabla_\xi l=0$ from $\nabla_\xi h=0$, but we can only get $(\nabla_\xi h)^2=0$ from $\nabla_\xi l=0$.

On a 3-dimensional paracontact metric manifold, by (2.7) and (2.12), if $l\varphi=\varphi l$, it is easy to get that $lX=\frac{\mathrm{tr}l}{2}\varphi^2X$. On the other hand, replacing $Y=Z=\xi$ in (2.9), we have

$$lX=QX-\eta(X)Q\xi+(\mathrm{tr}l)X-\eta(QX)\xi-\frac{r}{2}(X-\eta(X)\xi). \tag{4.1}$$

Thus we easily get

$$QX=aX+b\eta(X)\xi+\eta(X)Q\xi+\eta(QX)\xi, \tag{4.2}$$

where $a=\frac{1}{2}(r-\mathrm{tr}l)$, $b=-\frac{1}{2}(r+\mathrm{tr}l)$.

From (4.1), it is easy to get the following useful lemma

Lemma 4.3 *Let (M^3,φ,ξ,η,g) be a paracontact metric manifold. If for any $X\in\mathcal{D}$, it always holds $QX\in\mathcal{D}$, then $Q\varphi=\varphi Q$ is equivalent to $l\varphi=\varphi l$.*

Lemma 4.4 *Let (M^3,φ,ξ,η,g) be a paracontact metric manifold with $l\varphi=\varphi l$. If h is of η_1 type and M^3 is not para-Sasakian, suppose $\{e,\varphi e,\xi\}$ is the φ-basis such that $he=\lambda e$ $(\lambda\neq 0)$, $g(e,e)=-1$. Then*

(1) $\nabla_e\xi=(\lambda-1)\varphi e$;

(2) $\nabla_{\varphi e}\xi=-(\lambda+1)e$;

(3) $\nabla_\xi e=0$;

(4) $\nabla_\xi\varphi e=0$;

(5) $\nabla_e e=\frac{1}{2\lambda}[\eta(Qe)-\varphi e(\lambda)]\varphi e$;

(6) $\nabla_{\varphi e}\varphi e=-\frac{1}{2\lambda}[\eta(Q\varphi e)+e(\lambda)]e$;

(7) $\nabla_e \varphi e = \frac{1}{2\lambda}[\eta(Qe) - \varphi e(\lambda)]e + (1-\lambda)\xi$;

(8) $\nabla_{\varphi e} e = -\frac{1}{2\lambda}[\eta(Q\varphi e) + e(\lambda)]\varphi e - (1+\lambda)\xi$.

Proof Since h is of η_1 type and M^3 is not para-Sasakian. We choose the φ-basis $\{e, \varphi e, \xi\}$ such that $he = \lambda e (\lambda \neq 0), -g(e,e) = g(\varphi e, \varphi e) = 1$. Using (2.3) gives

(1) $\nabla_e \xi = (\lambda - 1)\varphi e$;

(2) $\nabla_{\varphi e} \xi = -(\lambda + 1)e$;

(3) $\nabla_\xi e = A\varphi e$;

(4) $\nabla_\xi \varphi e = Ae$;

(5) $\nabla_e e = B\varphi e$;

(6) $\nabla_{\varphi e} \varphi e = Ce$;

(7) $\nabla_e \varphi e = Be + (1-\lambda)\xi$;

(8) $\nabla_{\varphi e} e = C\varphi e - (1+\lambda)\xi$. (4.3)

By (2.9), it follows that

$$R(e, \varphi e)\xi = \eta(Q\varphi e)e - \eta(Qe)\varphi e. \tag{4.4}$$

On the other hand, using Proposition 3.5 and (4.3), we obtain

$$R(e, \varphi e)\xi = -(e(\lambda) + 2\lambda C)e - (\varphi e(\lambda) + 2\lambda B)\varphi e. \tag{4.5}$$

Comparing (4.4) and (4.5), we obtain $B = \frac{1}{2\lambda}[\eta(Qe) - \varphi e(\lambda)]$, $C = -\frac{1}{2\lambda}[\eta(Q\varphi e) + e(\lambda)]$.

Since $\nabla_\xi l = 0, \nabla_\xi h = 0$, from Remark 4.2, differentiating $he = \lambda e (\lambda \neq 0)$ along ξ, we get $\xi(\lambda)e + 2\lambda A\varphi e = 0$. Because e and φe are linearly independent, we certainly have $\xi(\lambda) = 0$ and $A = 0$. Thus, we complete the proof of Lemma 4.4. □

Lemma 4.5 *Let $(M^3, \varphi, \xi, \eta, g)$ be a paracontact metric manifold with $l\varphi = \varphi l$. If h is of η_2 type and $\{e_1, e_2, \xi\}$ is the pseudo orthonormal basis such that $he_1 = e_2, he_2 = 0, g(e_1, e_2) = g(\xi, \xi) = 1, g(e_1, e_1) = g(e_1, e_3) = g(e_2, e_3) = 0$. Without loss of generality, we can assume that $\varphi e_1 = e_1, \varphi e_2 = -e_2$. Then*

(1) $\nabla_{e_1} \xi = -(e_1 + e_2)$;

(2) $\nabla_{e_2} \xi = e_2$;

(3) $\nabla_\xi e_1 = 0$;

(4) $\nabla_\xi e_2 = 0$;

(5) $\nabla_{e_1} e_1 = Be_1 + \xi$;

(6) $\nabla_{e_2} e_2 = -\frac{1}{2}\eta(Qe_1)e_2$;

(7) $\nabla_{e_1} e_2 = -Be_2 + \xi$;

(8) $\nabla_{e_2} e_1 = \frac{1}{2}\eta(Qe_1)e_1 - \xi$,

where $B = g(\nabla_{e_1} e_1, e_2)$.

Lemma 4.6 *Let $(M^3, \varphi, \xi, \eta, g)$ be a paracontact metric manifold with $l\varphi = \varphi l$. If h is of η_3 type and $\{e, \varphi e, \xi\}$ is the φ-basis such that $he = \lambda\varphi e, h\varphi e = -\lambda e, g(e, e) = -1$. Then*

(1) $\nabla_e \xi = -\lambda e - \varphi e$;

(2) $\nabla_{\varphi e} \xi = -e + \lambda\varphi e$;

(3) $\nabla_\xi e=0$;

(4) $\nabla_\xi \varphi e=0$;

(5) $\nabla_e e=-\frac{1}{2\lambda}[\eta(Q\varphi e)+\varphi e(\lambda)]\varphi e-\lambda\xi$;

(6) $\nabla_{\varphi e}\varphi e=\frac{1}{2\lambda}[\eta(Qe)-e(\lambda)]e-\lambda\xi$;

(7) $\nabla_e \varphi e=\frac{1}{2\lambda}[\eta(Q\varphi e)+\varphi e(\lambda)]\varphi e+\xi$;

(8) $\nabla_{\varphi e} e=\frac{1}{2\lambda}[\eta(Qe)-e(\lambda)]\varphi e-\xi$.

The proofs of Lemmas 4.5 and 4.6 are similar to that of Lemma 4.4, we omit them, but it is worth noticing that in the case when h is of η_2 type, $\eta(Qe_2)=0$ always holds.

Remark 4.7 By (2.12), we get if h is of η_1 type, then $h^2e=\lambda^2 e$ ($\lambda\ge 0$), then $\mathrm{tr}l=2(\lambda^2-1)\ge -2$; If h is of η_2 type, then $h^2e_i=0$, then $\mathrm{tr}l=-2$; If h is of η_3 type, then $h^2e=-\lambda^2e$, then $\mathrm{tr}l=-2(\lambda^2+1)<-2$. It follows that $\mathrm{tr}l=-2$ if and only if M^3 is para-Sasakian or h is of η_2 type.

Corollary 4.8 *Let (M^3,φ,ξ,η,g) be a paracontact metric manifold with $l\varphi=\varphi l$. We have* $\xi(\mathrm{tr}l)=0$.

Proof By Remark 4.2 we know $\nabla_\xi l=0$ holds, and by the proof of Lemma 3.1, we get $\xi(\mathrm{tr}l)=0$. □

Proposition 4.9 *Let (M^3,φ,ξ,η,g) be a paracontact metric manifold with $l\varphi=\varphi l$. If it also satisfies $\nabla_\xi(QX)$ is parallel to X for any vector field $X\in\mathcal{D}$. Then $Q\varphi=\varphi Q$ if and only if* $\mathrm{tr}l=\mathrm{const}$ $(\neq 0)$.

Proof Firstly, by Lemma 3.1 we know that if $Q\varphi=\varphi Q$, then $\mathrm{tr}l=\mathit{const}.$ holds everywhere on M^3. Now we only need to explain $\mathrm{tr}l=\mathrm{const.}\neq 0$. Otherwise, if $\mathrm{tr}l=\mathrm{const.}=0$. Then $lX=0$, by (4.1) and (4.2), we get $\forall X\in\mathcal{D}$, $QX=\frac{r}{2}X$, r is constant. Using (3.17) we directly get $\nabla_\xi QX=\frac{r}{2}\nabla_\xi X=\frac{r}{4}(\lambda^2-1-\frac{r}{2})\varphi X$, which is not parallel to X, and this is contradiction with conditions.

Now we prove the converse part of the Theorem, we discuss the question according to h of different types.

Case 1 If h is of η_1 type and M^3 is non-para-Sasakian. Let $\{e,\varphi e,\xi\}$ be the φ-basis such that $he=\lambda e$ ($\lambda\neq 0$), $-g(e,e)=g(\varphi e,\varphi e)=1$, then, by the Jacobi's identity for $e,\varphi e,\xi$ and using Lemma 4.4 we get

$$-\eta(\nabla_\xi Qe)+\xi(\varphi e(\lambda))-(\lambda-1)(\eta(Q\varphi e)+e(\lambda))+2\lambda e(\lambda)=0;$$

$$-\eta(\nabla_\xi Q\varphi e)+\xi(e(\lambda))+(\lambda-1)(\eta(Qe)-\varphi e(\lambda))+2\lambda\varphi e(\lambda)=0.$$

Since

$$\xi(\varphi e(\lambda))=[\xi,\varphi e]+\varphi e(\xi(\lambda))=(1+\lambda)e(\lambda);\quad \xi(e(\lambda))=(1-\lambda)\varphi e(\lambda).$$

From above we get

$$-\eta(\nabla_\xi Qe)-(\lambda+1)\eta(Q\varphi e)+2\lambda e(\lambda)=0;$$

$$-\eta(\nabla_\xi Q\varphi e) + (\lambda - 1)\eta(Qe) + 2\varphi e(\lambda) = 0.$$

From the above two equalities we get

$$e(\lambda) = \frac{1}{2\lambda}(\eta(\nabla_\xi Qe) + (\lambda + 1)\eta(Q\varphi e)),$$

$$\varphi e(\lambda) = \frac{1}{2}(\eta(\nabla_\xi Q\varphi e) - (\lambda - 1)\eta(Qe)).$$

If $\mathrm{tr}l = \text{const.}$, by $\mathrm{tr}l = 2(\lambda^2 - 1)$, it follows $e(\mathrm{tr}l) = 4\lambda e(\lambda) = 0$, thus $e(\lambda) = 0,\ \varphi e(\lambda) = 0$. Thus the condition $\nabla_\xi Qe$ is parallel to e for any vector field $e \in \mathcal{D}$, it follows that $\eta(Qe) = \eta(Q\varphi e) = 0$. By Lemma 4.3, it immediately follows that $Q\varphi = \varphi Q$.

What is more, if M^3 is para-Sasakian, then, $h = 0$. By (2.3) and (2.11), we obtain $R(X,Y)\xi = -(\eta(Y)X - \eta(X)Y)$, thus $\xi \in \mathcal{N}(\kappa = -1)$. By Proposition 3.3, we know that $Q\varphi = \varphi Q$.

Case 2 If h is of η_2 type, by the Jacobi's identity for e_1, e_2, ξ and using Lemma 4.5, we get $\eta(Qe_1) + \xi(\eta(Qe_1)) = 0$, that is to say, $\eta(Qe_1) + \eta(\nabla_\xi Qe_1) = 0$. Since $\nabla_\xi Qe_1$ is parallel to e_1, $\eta(Qe_1) = 0$. Recall that $\eta(Qe_2) = 0$ in the case when h is η_2 type, by Lemma 4.3, it immediately follows that $Q\varphi = \varphi Q$.

Case 3 The proof of h being of η_3 type is similar to the case of h being of η_1 type, we omit here.

Thus, we complete the proof. □

5. Classifications under $l\varphi = \varphi l$ and η-parallel Ricci tensor

In analogy with the contact metric case [3], we now introduce the following definition.

Definition 5.1 *A paracontact metric manifold has η-parallel Ricci tensor if and only if*

$$g((\nabla_Z Q)\varphi X, \varphi Y) = 0 \tag{5.1}$$

for any vector fields X, Y and for Z orthogonal to ξ.

Theorem 5.2 *Let $(M^3, \varphi, \xi, \eta, g)$ be a paracontact metric manifold with $l\varphi = \varphi l$. If M^3 has η-parallel Ricci tensor, then M^3 is flat or a para-Sasakian space form.*

Proof Assuming that M^3 is not para-Sasakian. By (2.3) and (4.2), we get

$$\begin{aligned}(\nabla_Y Q)\varphi Z =&(\nabla_Y Q)\varphi Z\\ =&\nabla_Y(a\varphi Z + \eta(Q\varphi Z)\xi) - (a\nabla_Y\varphi Z + b\eta(\nabla_Y\varphi Z)\xi + \eta(\nabla_Y\varphi Z)Q\xi + \eta(Q\nabla_Y\varphi Z)\xi)\\ =&Y(a)\varphi Z + g(-\varphi Y + \varphi hY, a\varphi Z)\xi + \eta((\nabla_Y Q)\varphi Z)\xi +\\ &\eta(Q\varphi Z)(-\varphi Y + \varphi hY) - b\eta(\nabla_Y\varphi Z)\xi - \eta(\nabla_Y\varphi Z)Q\xi.\end{aligned} \tag{5.2}$$

By (5.1), for any vector field W it holds $g((\nabla_Y Q)\varphi Z, \varphi W) = 0$, substituting (5.2) into which, it follows

$$Y(a) = \eta(Q\varphi Z)(\varphi Y - \varphi hY) + \eta(\nabla_Y\varphi Z)Q\xi. \tag{5.3}$$

Now we give the following discussion based on the different type of h.

If h is of η_1 type. Substituting $Y=Z=e$ in (5.3) and by Lemma 4.4, we obtain

$$\begin{aligned} e(a)\varphi e =&(1-\lambda)\eta(Q\varphi e)\varphi e+(1-\lambda)Q\xi- \\ &(1-\lambda)\eta(Qe)e+2(1-\lambda)\eta(Q\varphi e)\varphi e+(1-\lambda)(\mathrm{tr}l)\xi. \end{aligned} \tag{5.4}$$

Thus we get

$$e(a)=2(1-\lambda)\eta(Q\varphi e),\ (1-\lambda)\eta(Qe)=0,\ (1-\lambda)\mathrm{tr}l=0. \tag{5.5}$$

Substituting $e,\varphi e$ instead of Y,Z in (5.3) and using Lemma 4.4, we have

$$e(a)e=(1-\lambda)\eta(Qe)\varphi e. \tag{5.6}$$

It follows that

$$e(a)=0,\quad (1-\lambda)\eta(Qe)=0. \tag{5.7}$$

Replacing $Y=\varphi e$, $Z=e$ or $Y=Z=\varphi e$ in (5.3) and using Lemma 4.4, we get, respectively,

$$\varphi e(a)=0,\quad (1+\lambda)\eta(Q\varphi e)=0, \tag{5.8}$$

or

$$\varphi e(a)=2(1+\lambda)\eta(Qe),\quad (1+\lambda)\eta(Q\varphi e)=0,\ (1+\lambda)\mathrm{tr}l=0. \tag{5.9}$$

From the equations (5.5) and (5.7)–(5.9), we get $\eta(Qe)=\eta(Q\varphi e)=0$, $\mathrm{tr}l=0$. By Lemma 4.3, we obtain $Q\varphi=\varphi Q$ and $\mathrm{tr}l=0$, therefore, M^3 is flat.

If h is of η_2 type. Replacing $Y=e_2$, $Z=e_1$ in (5.3) and using Lemma 4.5, we obtain

$$e_2(a)=0,\ \eta(Qe_1)=0,\ \mathrm{tr}l=0. \tag{5.10}$$

Remember that $\eta(Qe_2)=0$ in case when h is of η_2 type, thus we get $\eta(Qe_1)=\eta(Qe_2)=0$ and $\mathrm{tr}l=0$, and M^3 is flat.

If h is of η_3 type. The proof is similar to the case when h is of η_1 type, and we omit it here.

Now we consider the case of M^3 being a para-Sasakian manifold satisfying (5.1).

Since $h=0$, it follows $\nabla_X\xi=-\varphi X$. Combining with (2.11), we obtain $R(X,Y)\xi=-(\eta(Y)X-\eta(X)Y)$, that is to say, $\xi\in\mathcal{N}(-1)$. By Proposition 3.3, it follows $Q\varphi=\varphi Q$ and M^3 is η-Einstein, then $QX=aX+b\eta(X)\xi$. Choosing the φ-basis $\{e,\varphi e,\xi\}$ and using (5.1) for (1) $X=\varphi e$, $Y=Z=e$, and (2) $X=Y=e$, $Z=\varphi e$, it follows

$$g((\nabla_e Q)e,\varphi e)=0\ \text{ and }\ g((\nabla_{\varphi e}Q)\varphi e,\varphi e)=0. \tag{5.11}$$

Also, we have

$$g((\nabla_\xi Q)\xi,\varphi e)=0. \tag{5.12}$$

By the well known formula

$$-(\nabla_e Q)e+(\nabla_{\varphi e}Q)\varphi e+(\nabla_\xi Q)\xi=\frac{1}{2}\mathrm{grad}\, r. \tag{5.13}$$

By (5.11)–(5.13), we get $\varphi e(r)=0$. Similarly, we have $e(r)=0$. What is more, since $Q\varphi=\varphi Q$, by the proof of Lemma 3.1, we know $\xi(r)=0$, therefore, $r=\mathrm{const.}$.

On the other hand, since $K(e,\varphi e) = \text{tr}l - \frac{r}{2}$, and on para-Sasakian manifold, it holds $\text{tr}l = -2$, and $r = 2(\lambda^2 - 1) = -2$, thus we get $K(e,\varphi e) = K(e,\xi) = -1$ on M^3. Therefore, M^3 is a para-Sasakian space form. Thus, we complete the proof. □

6. Classifications under $l\varphi = \varphi l$ and cyclic η-parallel curvature

First, we give the definition of cyclic η-parallel Ricci tensor in analogy with the contact metric case [3].

Definition 6.1 *A paracontact metric manifold has cyclic η-parallel Ricci tensor if and only if*

$$g((\nabla_Z Q)X, Y) + g((\nabla_Y Q)Z, X) + g((\nabla_X Q)Y, Z) = 0 \tag{6.1}$$

for any vector fields X, Y, Z orthogonal to ξ.

Proposition 6.2 *Let $(M^3, \varphi, \xi, \eta, g)$ be a paracontact metric manifold with $l\varphi = \varphi l$. If M^3 has cyclic η-parallel Ricci tensor, then $Q\varphi = \varphi Q$.*

Proof If M^3 is para-Sasakian, by the proof of Theorem 5.2, we know $Q\varphi = \varphi Q$. Now let M^3 be non-para-Sasakian. We discuss the question in several conditions according to h of different type.

If h is of η_1 type, choosing the φ-basis $\{e, \varphi e, \xi\}$ and by (4.2) and Lemma 4.4, we get

$$Qe = ae + \eta(Qe)\xi. \tag{6.2}$$

For the definition of cyclic η-parallel Ricci tensor, if we let $X = Y = Z = e$, it follows $g((\nabla_e Q)e, e) = 0$. Using (6.2) and after direct computation, we obtain $e(a) = 0$. If we let $X = Y = Z = \varphi e$, it follows $g((\nabla_{\varphi e} Q)\varphi e, \varphi e) = 0$. By similar method as before we have $\varphi e(a) = 0$. Next, substituting $X = Y = e$, $Z = \varphi e$ and $X = e$, $Y = Z = \varphi e$, we get $\varphi e(a) = 4\lambda\eta(Qe)$ and $e(a) = -4\lambda\eta(Q\varphi e)$, respectively. Thus $\varphi e(a) = 4\lambda\eta(Qe) = 0$ and $e(a) = -4\lambda\eta(Q\varphi e) = 0$, and since M^3 is para-Sasakian, $\lambda \neq 0$, it follows $\eta(Qe) = \eta(Qe) = 0$. By Lemma 4.3, we obtain $Q\varphi = \varphi Q$.

If h is of η_2 type, choosing the pseudo orthonormal basis $\{e_1, e_2, \xi\}$ and by (4.2) and Lemma 4.5, we get

$$Qe_1 = ae + \eta(Qe_1)\xi. \tag{6.3}$$

By the definition of cyclic η-parallel Ricci tensor and after some calculations, we get

$$g((\nabla_{e_1} Q)e_1, e_1) = -2\eta(Qe_1) = 0.$$

Thus we get $\eta(Qe_1) = 0$ and $\eta(Qe_2) = 0$ always holds in the case when h is of η_2 type. By Lemma 4.3, we obtain $Q\varphi = \varphi Q$.

If h is of η_3 type, using the same method as η_1 type, we can obtain: $e(a) = 2\lambda\eta(Qe)$ if $X = Y = Z = e$, and $\varphi e(a) = -2\lambda\eta(Q\varphi e)$ if $X = Y = Z = \varphi e$; $e(a) = -2\lambda\eta(Qe)$ if $X = e, Y = Z = \varphi e$, and $\varphi e(a) = 2\lambda\eta(Q\varphi e)$ if $X = Y = e, Z = \varphi e$; Thus we get $\eta(Qe) = \eta(Qe) = 0$ and therefore $Q\varphi = \varphi Q$. Thus, we complete the proof. □

Using Proposition 6.2 and Theorem 3.6, we can get the following classification theorem:

Theorem 6.3 *Let (M^3,φ,ξ,η,g) be a paracontact metric manifold with $l\varphi=\varphi l$. If M^3 has cyclic η-parallel Ricci tensor, then M^3 is either flat, para-Sasakian, h is of η_2 type or of constant ξ-sectional curvature $\kappa<1$ and constant φ-sectional curvature $-\kappa$.*

Acknowledgements We thank the referees and the editor for their careful reading and helpful suggestions.

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