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Point-Transitive Linear Spaces

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Abstract This work is a contribution to the classification of linear spaces admitting a point-transitive automorphism group. Let $\mathcal{S}$ be a regular linear space with 51 points, with lines of size 6, and G be an automorphism group of $\mathcal{S}$. We prove that G cannot be point-transitive.

Keywords linear space; design; automorphism group; point-transitive

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1. Introduction

A linear space $\mathcal{S}$ is an incidence structure $(\mathcal{P},\mathcal{L})$ consisting of a set $\mathcal{P}$ of points and a collection $\mathcal{L}$ of distinguished subsets of $\mathcal{P}$, called lines with sizes ≥ 2, such that any two points are incident with exactly one line. We assume that $\mathcal{S}$ is finite in the sense that $\mathcal{P}$ is finite. Traditionally, we define $v=|\mathcal{P}|$ and $b=|\mathcal{L}|$. Let α be a point of $\mathcal{P}$, and k be a positive integer. Then r_α^k denotes the number of lines having size k through α, b^k the number of lines of size k, and r_α the number of all lines through α, called the degree of α. If all lines have a constant size k, then we say that $\mathcal{S}$ is regular, so it is a 2-$(v,k,1)$ design. Moreover, a regular linear space is said to be non-trivial if it has at least two lines and every line contains at least three points.

An automorphism of $\mathcal{S}$ is a permutation acting on $\mathcal{P}$ which leaves $\mathcal{L}$ invariant. The full automorphism group of $\mathcal{S}$ is denoted by $\mathrm{Aut}(\mathcal{S})$ and any subgroup of $\mathrm{Aut}(\mathcal{S})$ is called an automorphism group of $\mathcal{S}$. If $G\leq \mathrm{Aut}(\mathcal{S})$ is transitive on $\mathcal{P}$ (resp., $\mathcal{L}$), then we say that G is point-transitive (resp., line-transitive). Similarly, G is said to be point-primitive (resp., point-imprimitive) if it acts primitively (resp., imprimitively) on points.

Several papers have already been devoted to the existence of the 2-$(v,k,1)$ designs. In particular, existence results for $k<6$ are known, and the existence for certain 2-$(v,6,1)$ designs are proven. A summary of these results was given in [1]. According to [2, 3], there are only a finite number of 2-$(v,6,1)$ designs which need to be considered before all existence of 2-$(v,6,1)$ designs can be proven. In fact, the existence of the 2-$(v,6,1)$ designs is unknown if and only if $v\in \{51,61,81,166,226,231,256,261,286,316,321,346,351,376,406,411,436, 441,471,501,561,591, 616,646,651,676,771,796,801\}$. Provided that $\mathcal{S}$ is a 2-$(51,6,1)$ design admitting a line-tansitive automorphism group G. Since the alternating group A_{51} is the only primitive group of degree

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51 (see [4, Table B.4]), G cannot be point-primitive by [5, Main Theorem]. Moreover, we also know that G cannot be point-imprimitive according to [6]. So no line-transitive 2-$(51,6,1)$ design exists. In this paper, we consider the 2-$(51,6,1)$ designs admitting a point-transitive automorphism group, and the following is the main result.

Theorem 1.1 *Let $\mathcal{S}$ be a 2-$(51,6,1)$ design. If G is an automorphism group of $\mathcal{S}$, then G cannot be point-transitive. That is to say: there is no point-transitive 2-$(51,6,1)$ design.*

The paper divides naturally into four parts. Section 2 presents some preliminary results and notation. Section 3 does a detailed analysis of bound of the size of $|\,\mathrm{Aut}(\mathcal{S})|$. Finally, Section 4 gives the proof of Theorem 1.1.

2. Preliminary results and notation

Let $\mathcal{S}$ be a finite linear space with v points, K be a set of positive integers such that $v \geq k$ for every $k \in K$ and the set of line-sizes of $\mathcal{S}$ is contained in K. Let α be a point of $\mathcal{P}$. Then

$$\sum_{k\in K}(k-1)r_\alpha^k = v-1 \tag{2.1}$$

and for each $k \in K$, we have

$$\sum_{\alpha\in\mathcal{P}} r_\alpha^k = k\cdot b^k. \tag{2.2}$$

In particular, if $\mathcal{S}$ is a non-trivial finite regular linear space, then the following result is well-known.

Lemma 2.1 ([5, Lemma 2.1]) *Let $\mathcal{S}$ be a non-trivial finite regular linear space. Then*

$$r=\frac{v-1}{k-1},\quad b=\frac{v(v-1)}{k(k-1)},$$

and

$$k(k-1)+1\leq v,$$

where k is the line-size of $\mathcal{S}$, and r is the number of lines through a point.

Let $\mathcal{S}=(\mathcal{P},\mathcal{L})$ be a linear space and $G \leq \mathrm{Aut}(\mathcal{S})$, Δ be a subset of $\mathcal{P}$ with $|\Delta|\geq 2$, and set $\mathcal{L}_\Delta=\{\lambda\cap\Delta : |\lambda\cap\Delta|\geq 2 \text{ for } \lambda\in\mathcal{L}\}$. Then $(\Delta,\mathcal{L}_\Delta)$ forms an incidence structure, and the induced structure is a linear space. We are interested in the case when Δ is $\mathrm{Fix}(g)$ (or $\mathrm{Fix}(H)$), the set of fixed points of $g \in G$ (or $H \leq G$) on $\mathcal{P}$. The following result gives a bound of $|\mathrm{Fix}(H)|$ for a subgroup $H \leq G$.

Lemma 2.2 ([7, Lemma 1]) *Let $\mathcal{S}$ be a finite regular linear space, G be an automorphism group of $\mathcal{S}$, and $H \neq 1$ be a subgroup of G. Then $|\mathrm{Fix}(H)|\leq r$ unless every point lies on a fixed line and then $|\mathrm{Fix}(H)|\leq r+k-3$.*

The next Lemma comes from [8], and will be of great help for our proof of Theorem 1.1.

Lemma 2.3 *If $\mathcal{S}$ is a linear space having lines of size* 3 *and* 6 (*with at least one line of size* 3 *and one line of size* 6). *Then* $v = 16$ *or* 18, *provided that* $v < 21$.

Throughout this paper, we assume that $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a 2-$(51, 6, 1)$ design, and G is a point-transitive subgroup of $\mathrm{Aut}(\mathcal{S})$. Let $|G|_p$ be the p-part of $|G|$, that is, the highest power of the prime p dividing $|G|$.

3. The order of $|\mathrm{Aut}(\mathcal{S})|$

In this section we bound the size of $|\mathrm{Aut}(\mathcal{S})|$ and show that $|\mathrm{Aut}(\mathcal{S})|$ divides $2^7 \cdot 3^4 \cdot 5^3 \cdot 17$.

Lemma 3.1 $|\mathrm{Aut}(\mathcal{S})|$ *divides* $2^m \cdot 3^n \cdot 5^3 \cdot 17$ *for two positive integers* m *and* n.

Proof Let $p \geq 5$ be a prime divisor of $|\mathrm{Aut}(\mathcal{S})|$, and g be an element of $\mathrm{Aut}(\mathcal{S})$ of order p. Then $|\mathrm{Fix}(g) \cap \lambda| = 0, 1$ or 6 for $\lambda \in \mathcal{L}$.

Suppose that $\mathrm{Fix}(g) \not\subseteq \lambda$ for each $\lambda \in \mathcal{L}$, then $\mathrm{Fix}(g)$ induces a regular linear space, that is a 2-$(|\mathrm{Fix}(g)|, 6, 1)$ design. Thus $|\mathrm{Fix}(g)| \geq 6(6-1)+1 = 31$ by Lemma 2.1. But $|\mathrm{Fix}(g)| \leq 6 + 10 - 3 = 13$ according to Lemma 2.2, a contradiction. Hence there exists a line $\lambda \in \mathcal{L}$ such that $\mathrm{Fix}(g) \subseteq \lambda$ and $|\mathrm{Fix}(g)| = 0, 1$ or 6. Therefore, the possible values of p are 5 and 17, since $51 - |\mathrm{Fix}(g)| \equiv 0 \pmod p$. Let P be a Sylow p-subgroup of $\mathrm{Aut}(\mathcal{S})$.

If $p = 5$ and $P \neq 1$, then $|\mathrm{Fix}(P)| = 1$ or 6. First we suppose that $|\mathrm{Fix}(P)| = 6$, then P acts on $\mathcal{P}\backslash\mathrm{Fix}(P)$ semiregularly, hence $|P| \mid (51-6)$, thus $|P|$ divides 5. Now suppose that $|\mathrm{Fix}(P)| = 1$. If P acts semiregularly on $\mathcal{P}\backslash\mathrm{Fix}(P)$, then $|P| \mid 5^2$. If P is not semiregular on $\mathcal{P}\backslash\mathrm{Fix}(P)$, then there exists a point $\alpha \in \mathcal{P}\backslash\mathrm{Fix}(P)$ such that $P_\alpha \neq 1$, thus $|\mathrm{Fix}(P_\alpha)| = 6$ and P_α is semiregular on $\mathcal{P}\backslash\mathrm{Fix}(P_\alpha)$, so $|P_\alpha|$ divides 5 and $|P| = |P : P_\alpha||P_\alpha|$ divides 5^3.

If $P \neq 1$ is a Sylow 17-subgroup of $\mathrm{Aut}(\mathcal{S})$, then $|\mathrm{Fix}(P)| = 0$ and P acts semiregularly on $\mathcal{P}$, thus $|P|$ divides 17. $\square$

Lemma 3.2 $|\mathrm{Aut}(\mathcal{S})|_3$ *divides* 3^4.

Proof Let T be a Sylow 3-subgroup of $\mathrm{Aut}(\mathcal{S})$. If $T \neq 1$, then T fixes a line $\lambda \in \mathcal{L}$. Thus $T/T_{(\lambda)} \leq S_6$ and then $|T : T_{(\lambda)}|$ divides 3^2. Now we suppose that $T_{(\lambda)} \neq 1$.

If $|\mathrm{Fix}(T_{(\lambda)})| \neq 6$, then $T_{(\lambda)}$ is a point-set of a linear space. If the induced linear space is regular, then $|\mathrm{Fix}(T_{(\lambda)})| \geq 31$ by Lemma 2.1, a contradiction to Lemma 2.2. Thus the induced linear space is not regular and at least has one line of size 6 and one of size 3, but it is impossible by Lemma 2.3.

Therefore, $|\mathrm{Fix}(T_{(\lambda)})| = 6$ and $T_{(\lambda)}$ acts semiregularly on $\mathcal{P} \setminus \mathrm{Fix}(T_{(\lambda)})$. Otherwise, there is another point $\beta \notin \lambda$ such that $T_{(\lambda \cup \{\beta\})} \neq 1$, then $|\mathrm{Fix}(T_{(\lambda \cup \{\beta\})})| > 6$ and $\mathrm{Fix}(T_{(\lambda \cup \{\beta\})})$ induces a linear space. If the induced linear space is regular, then $|\mathrm{Fix}(T_{(\lambda \cup \{\beta\})})| \geq 31$ by Lemma 2.1, a contradiction to Lemma 2.2. Thus the induced linear space is not regular and at least has one line of size 6 and one of size 3, but it is impossible by Lemma 2.3. So $|T_{(\lambda)}| \mid (51-6)$ and $|T|$ divides $3^2 \cdot 3^2$. $\square$

In the rest of this section, the paper deals with the maximal size of the 2-part of $|\mathrm{Aut}(\mathcal{S})|$.

Some information about the linear spaces in [8] is given. Assume that $2 \mid |\mathrm{Aut}(\mathcal{S})|$ and T is 2-subgroup of $\mathrm{Aut}(\mathcal{S})$. Let $\mathcal{D} = (\mathrm{Fix}(T), \mathcal{L}_{\mathrm{Fix}(T)})$ be the linear space induced by $\mathrm{Fix}(T)$ and then $K = \{2, 4, 6\}$ containing the set of its line-sizes. In view of (2.1), we get

$$r_\alpha^2 + 3r_\alpha^4 + 5r_\alpha^6 = |\mathrm{Fix}(T)| - 1, \tag{3.1}$$

for each $\alpha \in \mathrm{Fix}(T)$. Since a non-fixed point of T cannot be on two distinct fixed lines of it, all the non-fixed points of T which lie on its fixed lines are distinct. Thus

$$4b^2 + 2b^4 \le 51 - |\mathrm{Fix}(T)|. \tag{3.2}$$

Combining (2.2) with (3.2), we obtain

$$2\sum_{\alpha\in\mathrm{Fix}(T)} r_\alpha^2 + \frac{1}{2}\sum_{\alpha\in\mathrm{Fix}(T)} r_\alpha^4 \le 51 - |\mathrm{Fix}(T)|. \tag{3.3}$$

Now for each point $\alpha \in \mathrm{Fix}(T)$, define the weight ([8]) $\omega(\alpha)$ of α

$$\omega(\alpha) = 2r_\alpha^2 + \frac{1}{2}r_\alpha^4.$$

So that (3.3) can be written as

$$\sum_{\alpha\in\mathrm{Fix}(T)} \omega(\alpha) \le 51 - |\mathrm{Fix}(T)|. \tag{3.4}$$

If $r_\alpha^2 = x, r_\alpha^4 = y$ and $r_\alpha^6 = z$, then we say that α is of type (x, y, z).

Lemma 3.3 $|\mathrm{Aut}(\mathcal{S})|_2$ *divides* 2^7.

Proof Let $T \in \mathrm{Syl}_2(\mathrm{Aut}(\mathcal{S}))$. If $T \ne 1$, then T fixes a line $\lambda \in \mathcal{L}$. If $T_{(\lambda)} \ne 1$, then $|\mathrm{Fix}(T_{(\lambda)})| \ge 7$ since $|\mathrm{Fix}(T_{(\lambda)})| \equiv 1 \pmod 2$. Let $S = T_{(\lambda)}$ and $\beta \notin \lambda$ be a fixed point of S. Then S fixes λ_1, where λ_1 is a line through β such that $\lambda \cap \lambda_1 \ne \emptyset$. Let $\lambda \cap \lambda_1 = \{\alpha\}$. If $S_{(\lambda_1)} \ne 1$, then $|\mathrm{Fix}(S_{(\lambda_1)})| = 11$ or 13 by Lemma 2.2 and $\mathrm{Fix}(S_{(\lambda_1)})$ induces a linear space with line-sizes form $K = \{2, 4, 6\}$.

Suppose first that $|\mathrm{Fix}(S_{(\lambda_1)})| = 11$, then $\mathrm{Fix}(S_{(\lambda_1)}) = \lambda \cup \lambda_1$. The type of α is $(0, 0, 2)$, and the types of other points of $\mathrm{Fix}(S_{(\lambda_1)})$ are $(5, 0, 1)$. Thus

$$\sum_{\delta\in\mathrm{Fix}(S_{(\lambda_1)})} \omega(\delta) = 0 + 10 \times 10 = 100,$$

a contradiction to inequation (3.4).

Now Suppose that $|\mathrm{Fix}(S_{(\lambda_1)})| = 13$, and β_1, β_2 are the two fixed points which lie on neither λ nor λ_1. Then $r_{\beta_i}^4 \le 1$ and $r_{\beta_i}^6 = 0$ for $i = 1, 2$. Thus $\omega(\beta_i) = 24$ or $18 + \frac{1}{2}$. Moreover we have $\omega(\alpha) = 4$. Hence

$$\sum_{\delta\in\mathrm{Fix}(S_{(\lambda_1)})} \omega(\delta) \ge \omega(\alpha) + \omega(\beta_1) + \omega(\beta_2) \ge 41,$$

which is impossible by inequation (3.4).

Therefore, $S_{(\lambda_1)} = 1$ and $T_{(\lambda)}$ acts faithfully on $\lambda_1 \backslash \{\alpha, \beta\}$. So $T_{(\lambda)} \le S_4$ and then $|T|$ divides 2^7 since $T/T_{(\lambda)} \le S_6$. $\Box$

4. Proof of Theorem 1.1

According to Lemmas 3.1–3.3, we get $|\operatorname{Aut}(\mathcal{S})|$ divides $2^7 \cdot 3^4 \cdot 5^3 \cdot 17$. In this section, we will prove that there is no point-transitive 2-$(51, 6, 1)$ design.

Lemma 4.1 *G cannot be isomorphic to Z_{51}.*

Proof Suppose otherwise that $G = \langle g \rangle \cong Z_{51}$. Then G is regular on $\mathcal{P}$. Thus we can identify the point set $\mathcal{P}$ with G and the elements of G act by multiplication. Since $\mathcal{S}$ has 85 lines and $|\lambda^G| = 17$ or 51 for $\lambda \in \mathcal{L}$, then there at least exists one orbit λ^G such that $|\lambda^G| = 17$. Then $\lambda^{g^{17}} = \lambda$ and then λ is a union of two orbits of $\langle g^{17} \rangle$ on $\mathcal{P}$. Let $\{g^i, g^{i+17}, g^{i+34}\}$ and $\{g^j, g^{j+17}, g^{j+34}\}$ be two orbits such that

$$\lambda = \{g^i, g^{i+17}, g^{i+34}\} \cup \{g^j, g^{j+17}, g^{j+34}\},$$

where $1 \le i < j \le 51$. Then

$$\lambda^{g^{j-i}} = \{g^{2j-i}, g^{2j-i+17}, g^{2j-i+34}\} \cup \{g^j, g^{j+17}, g^{j+34}\},$$

hence $g^{j-i} \in G_\lambda$, thus $17 \mid (j-i)$. It implies that

$$\{g^i, g^{i+17}, g^{i+34}\} = \{g^j, g^{j+17}, g^{j+34}\},$$

which is impossible. Therefore, G cannot be isomorphic to Z_{51}. $\Box$

Lemma 4.2 *Assume that N is a minimal normal subgroup of G. Then $N \cong \mathrm{PSL}(2, 16)$.*

Proof $N \trianglelefteq G$ and G is point-transitive, N is $\frac{1}{2}$-transitive on $\mathcal{P}$, and the common length of orbits is $3, 17$ or 51. Let $N \cong T^\ell$ be a direct product of $\ell \ge 1$ copies of simple groups T.

If N is elementary, then $N \cong Z_3^\ell$ $(1 \le \ell \le 4)$ or $N \cong Z_{17}$. For the former case, we have $G/C_G(N) \le GL(\ell, 3)$. But for $\ell = 1, 2, 3$ and 4, $17 \nmid |GL(\ell, 3)|$, thus $|C_G(N)|$ must be divisible by 17 and $C_G(N)$ is transitive on $\mathcal{P}$. Choose $g \in C_G(N)$ and $t \in N$ such that the order of g is 17 and the order of t is 3, then $\langle g, t \rangle \cong Z_{51}$ is transitive on $\mathcal{P}$, which is impossible by Lemma 4.1. For the later case, we have $G/C_G(N) \le Z_{16}$ and similar discussion implies that there also exists a point-transitive subgroup of G which is isomorphic to Z_{51}, a contradiction.

Now suppose that T is a non-abelian simple group. If the length of orbits is 3, then N does not have an element g of order 5 or 17, otherwise $\mathrm{Fix}(g) = \mathcal{P}$. Thus $|N|$ divides $2^7 \cdot 3^4$, this implies that N is solvable, which is impossible. Hence, the common length of orbits of N on $\mathcal{P}$ is 17 or 51, and then $|T|$ is divisible by 17. Therefore, $N = T$. Using the list of non-abelian simple groups, it is easy to check that $N \cong \mathrm{PSL}(2, 16)$ or $\mathrm{PSL}(2, 17)$, for 17 divides $|N|$ and $|N|$ divides $2^7 \cdot 3^7 \cdot 5^3 \cdot 17$. According to [9, Theorem 8.27, Chapter II], $\mathrm{PSL}(2, 17)$ has no subgroup of index 17 and 51. Therefore, $N \cong \mathrm{PSL}(2, 16)$. $\Box$

Proof of Theorem 1.1 According to Lemma 4.2, if N is a minimal normal subgroup of G, then $N \cong \mathrm{PSL}(2, 16)$ and the common length of orbits of N on $\mathcal{P}$ is 17 or 51. Now suppose that the common length is 17, then N acting on each orbit is permutation isomorphic to $\mathrm{PSL}(2, 16)$

acting on projective lines, and N is 3-transitive on Δ_i for $i = 1$, 2 and 3, where Δ_i is an orbit of N on points. Let $g \in N$ be of order 2, α_i be the only one fixed point of g on Δ_i $(i = 1, 2)$, and λ be the unique line through α_1 and α_2. Then g fixes the line λ, hence λ is a union of orbits of g. Let $\Sigma = \{\beta_1, \beta_2\} \subset \lambda$ be an orbit of g. If $\Sigma \subset \Delta_1$, then for any other point $\gamma \in \Delta_1$, there exists an element $\tau \in N$ such that $\{\alpha_1, \beta_1, \beta_2\}^\tau = \{\gamma, \beta_1, \beta_2\}$, a contradiction. Similar discussion for $\Sigma \subset \Delta_2$. So $\lambda \setminus \{\alpha_1, \alpha_2\} \subset \Delta_3$, which is also impossible for N is 3-transitive on Δ_3. Therefore, $N \cong \mathrm{PSL}(2, 16)$ is point-transitive.

According to [9, Theorem 8.27, Chapter II], $\mathrm{PSL}(2, 16)$ has only one conjugacy class of groups of order 80. By MAGMA [10], the permutation representation of $\mathrm{PSL}(2, 16)$ on $\mathcal{P}$ can be obtained, and we also get the subdegrees of $\mathrm{PSL}(2, 16)$ on $\mathcal{P}$, that is $1, 1, 1$, $16, 16$ and 16.

Let $N = \mathrm{PSL}(2, 16)$, $\alpha \in \mathcal{P}$, and $\beta \neq \alpha$ be one point fixed by N_α. Let λ be the unique line through α and β. Then $\lambda^{N_\alpha} = \lambda$. Thus λ is a union of orbits of N_α, which is impossible. Therefore, G cannot be point-transitive. $\Box$

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