Journal of Mathematical Research with Applications
Nov., 2018, Vol. 38, No. 6, pp. 557–566
DOI:10.3770/j.issn:2095-2651.2018.06.002
Http://jmre.dlut.edu.cn

Divisibilities and Congruences Identities on Traces of Singular Moduli

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Abstract Zagier found that traces of singular moduli are the Fourier coefficients of certain modular forms of weight 3/2. As a result, formulas and congruences of these traces are obtained in various situations. Recently, Ahlgen proved a uniform relationship for traces of singular moduli by using the relationship of modular forms with the action of Hecke operators. On the basis of these results, we get some interesting divisibilities and congruences identities on traces of singular moduli and Hurwitz-Kronecker class number.

Keywords traces of singular moduli; divisibilities; congruences; Hecke operators

MR(2010) Subject Classification 11F37; 11F03

1. Introduction

The modular function $j(z)$ is defined by

$$j(z) = \frac{E_4(z)^3}{\Delta(z)} = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots, \tag{1.1}$$

where $q := e^{2\pi i\tau}$, $\tau \in \mathbb{H}$ (=the complex upper half-plane). Throughout this paper, let $d \equiv 0, 3 \pmod 4$ be a positive integer, so that $-d$ is a negative discriminant. Denote by $\mathcal{Q}_d$ the set of positive definite integer binary quadratic forms $Q(x,y) = ax^2 + bxy + cy^2 = [a,b,c]$ of discriminant $-d = b^2 - 4ac$ with the usual action of the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$.

For each $Q \in \mathcal{Q}_d$, its associated unique element α_Q is the root of $Q(x,1) = 0$, where $\alpha_Q \in \mathbb{H}$. Then the value of $j(\alpha_Q)$ is well-known as the singular moduli, which is an algebraic integer that only depends on the Γ-equivalence class of Q.

Singular modulis have been studied intensively since the time of Kronecker and Weber [1]. After that, Gross and Zagier obtained formulas for their norms, and for the norms of their differences. Zagier [2] obtained the result for their traces, and a number of generalizations, which are closely related to the theorem of Bocherds [3] of which he gave a new proof and a generalization. In a word, the formula for traces of the singular values of j can be stated in two equivalent ways. Firstly, these traces are the Fourier coefficients of a certain modular form of weight 3/2. On the other hand, they are the solutions of a certain pair of recursion aligns.

Received November 9, 2017; Accepted July 6, 2018
Supported by the National Natural Science Foundation of China (Grant No. 61402335), the Natural Science Foundation of Shaanxi Province (Grant No. 2016JM1004), the Scientific Research Program Funded by Shaanxi Provincial Education Department (Grant No. 17JK0266) and the Natural Science Foundation of Weinan Normal University (Grant Nos. 17ZRRC01; 17YKS09).
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Singular modulis are algebraic integers which play prominent roles in number theory and other subjects. For example, Hilbert class fields of imaginary quadratic fields are generated by singular moduli, and isomorphism classes of elliptic curves with complex multiplication are distinguished by singular moduli.

More precisely, if $h(-d)$ denotes the class number of $-d$, i.e., the number of Γ-equivalence classes of primitive quadratic forms in $\mathcal{Q}_d$, then each of the corresponding $h(-d)$ values of $j(\alpha_Q)$ is an algebraic integer of exact degree $h(-d)$ and they form a full set of conjugates, so that the sum of these numbers is the trace. In other words, $Q(j(\alpha_Q))$ is a ring class field extension whose degree over $Q(\alpha_Q)$ is given by the class number $h(-d)$ (if $-d$ is a fundamental discriminant, then $Q(j(\alpha_Q))$ is the Hilbert class field of $Q(\alpha_Q)$).

For example, $h(-3) = h(-4) = h(-7) = h(-8) = 1$ and $h(-15) = 2$ and the corresponding j-values are $j(\frac{1+i\sqrt{3}}{2}) = 0$, $j(i) = 1728$, $j(\frac{1+i\sqrt{7}}{2}) = -3375$, $j(i\sqrt{2}) = 8000$, $\frac{1+i\sqrt{15}}{2} = \frac{-191025-85995\sqrt{5}}{2}$, $j(\frac{1+i\sqrt{15}}{4}) = \frac{-191025+85995\sqrt{5}}{2}$.

In order to give the formula for traces of singular moduli, firstly, define the isotropy numbers $\omega_Q \in \{1, 2, 3\}$ as

$$\omega_Q := \begin{cases} 2, & \text{if } Q \sim_\Gamma [a, 0, a], \\ 3, & \text{if } Q \sim_\Gamma [a, a, a], \\ 1, & \text{otherwise}, \end{cases} \tag{1.2}$$

where ω_Q is the order of the stabilizer of Q in Γ. We sum over all forms in $\mathcal{Q}_d$ and weight the number $J(\alpha_Q)$ by the factor $1/\omega_Q$, where $J(\tau) := j(\tau) - 744$.

The Hurwitz-Kronecker class number $H(d)$ is the number of equivalence classes of forms of the discriminant $-d$ under the action of Γ, which is weighted by ω_Q and defined as

$$H(d) := \sum_{Q \in \mathcal{Q}_d/\Gamma} \frac{1}{\omega_Q}, \tag{1.3}$$

for example, $H(3) = \frac{1}{3}$, $H(4) = \frac{1}{2}$, $H(15) = 2$.

Following Zagier, we define traces of singular moduli of discriminant $-d$ as

$$\mathrm{Tr}(d) := \sum_{Q \in \mathcal{Q}_d/\Gamma} \frac{j(\alpha_Q) - 744}{\omega_Q}, \tag{1.4}$$

for instance, $\mathrm{Tr}(3) = \frac{0-744}{3} = -248$, $\mathrm{Tr}(4) = \frac{1728-744}{2} = 492$, $\mathrm{Tr}(15) = -191025 - 2 \cdot 744 = -192513$.

On the other hand, by using the notation above, we define the modified class polynomials as

$$\mathcal{H}_d(X) := \prod_{Q \in \mathcal{Q}_d/\Gamma} (X - j(\alpha_Q))^{\frac{1}{\omega_Q}}. \tag{1.5}$$

We can interpret $H(d)$ and $\mathrm{Tr}(d)$ as the first two Fourier coefficients of the logarithmic derivative of $\mathcal{H}_d(j(z))$ whose exponents are the coefficients in a weakly holomorphic modular form $f_d = q^{-d} + O(q)$ of weight 1/2 and level 4. Zagier proved these results by considering a family of weight 3/2 modular forms g_D that are the "dual" to the modular forms f_d and by directly relating the traces with the Hecke traces of singular moduli to the coefficents of the modular forms g_D. Zagier's famous results have inspired the works of singular moduli by many authors.

Ahlgen [4] followed Zagier and required a sequence of modular function $J_m(\tau)$ on $\mathrm{PSL}_2(\mathbb{Z})$. Here we denote $J_0(\tau) := 1$ for $m \geq 1$, let $J_m(\tau)$ be the unique modular function on $\mathrm{PSL}_2(\mathbb{Z})$ which is holomorphic on $\mathbb{H}$, and has an expansion of the form $J_m(\tau) = q^{-m} + O(q)$. Each J_m can be expressed as a polynomial in j, for example, we have

$$\begin{aligned}
&J_0(\tau) = 1,\\
&J_1(\tau) = j(\tau) - 744 = q^{-1} + 196884q + \cdots,\\
&J_2(\tau) = j(\tau)^2 - 1488j(\tau) + 159768 = q^{-2} + 42987520q + \cdots.
\end{aligned}$$

Otherwise, define $J_m := J|T_0(m)$, where $T_0(m)$ is the Hecke operator with index m of weight 0. Then for $m \geq 0$, define the m-th Hecke traces of the singular moduli of discriminant $-d$ as

$$\mathrm{Tr}_m(d) := \sum_{Q \in \mathcal{Q}_d/\Gamma} \frac{J_m(\alpha_Q)}{\omega_Q}. \tag{1.6}$$

If $m \geq 1$, then $\mathrm{Tr}_m(d)$ is an integer, while $\mathrm{Tr}_0(d) = H(d)$ which is the usual Hurwitz class number, and $\mathrm{Tr}_1(d) = \mathrm{Tr}(d)$.

Jenkins [5] generalized traces of singular moduli even further by adding a twist. Let D be a fundamental discriminant. Denote the genus character χ_D to be the character assigning a quadratic form $Q = [a, b, c]$ of discriminant divisible by D, then the value of it is defined as

$$\chi_D(Q) := \begin{cases} 0, & \text{if } (a, b, c, D) > 1; \\ (\frac{D}{n}), & \text{if } (a, b, c, D) = 1, \end{cases} \tag{1.7}$$

where n is any integer represented by Q and coprime to D. This is independent of the choice for the integer n. For a form Q of discriminant $-dD$, then $\chi_D = \chi_-d$ if D and $-d$ are both fundamental discriminants.

Let us define the twisted m-th Hecke traces of singular moduli as

$$\mathrm{Tr}_m(D, d) := \sum_{Q \in \mathcal{Q}_d/\Gamma} \frac{\chi_D(Q) J_m(\alpha_Q)}{\omega_Q} \tag{1.8}$$

and $\mathrm{Tr}_m(1, d) = \mathrm{Tr}_m(d)$.

Recently, there are many results on these traces of singular moduli. The goal of this paper is to give some properties of the Hurwitz-Kronecker class number $H(d)$ and traces of singular moduli. Exactly speaking, by considering the condition of odd case of $p^{2n+1}d$ in the coefficients $B_1(D, d)$ of q^d in g_D, we get an exact formula of them. Then we obtain an exact formula of the twisted traces as well. At last, some divisibilities and congruences identities on traces of singular moduli and Hurwitz-Kronecker class number are established.

2. Zagier's traces of singular moduli

Zagier related traces of singular moduli with the coefficients of a certain weakly holomorphic modular form of weight 3/2.

Let $M'_{\lambda+\frac{1}{2}}$ be the space of weight $\lambda + \frac{1}{2}$ weakly holomorphic modular forms on $\Gamma_0(4)$ with

Fourier expansion satisfying the Kohnen's plus space condition

$$f(z)=\sum_{(-1)^{\lambda}n\equiv 0,1 \,(\mathrm{mod}\,4)} a(n)q^n. \tag{2.1}$$

Then for any $0<D\equiv 0,1 \pmod 4$, let $g_D(z)$ be the unique element of $M'_{\frac{3}{2}}$ with Fourier expansion

$$g_D(z)=q^{-D}+B_1(D,0)+\sum_{0<d\equiv 0,3\,(\mathrm{mod}\ 4)} B_1(D,d)q^d, \tag{2.2}$$

and for any $0\le d\equiv 0,3 \pmod 4$, let $f_d(z)$ be the unique element of $M'_{\frac{1}{2}}$ with Fourier expansion

$$f_d(z)=q^{-d}+\sum_{0<D\equiv 0,1\,(\mathrm{mod}\,4)} A_1(D,d)q^D, \tag{2.3}$$

where all of the coefficients $A_1(D,d)$ and $B_1(D,d)$ of f_d and g_D are integers.

Applying Hecke operators to the weakly holomorphic modular forms f_d and g_D, we define

$$A_m(D,d) = \text{the coefficients of } q^D \text{ in } f_d(z)|T_{\frac{1}{2}}(m^2),$$

$$B_m(D,d) = \text{the coefficients of } q^d \text{ in } g_D(z)|T_{\frac{3}{2}}(m^2).$$

Borcherds [6] obtained a striking theorem for describing the full Fourier expansion of $\mathcal{H}_d(j(z))$ in terms of the coefficients of weakly holomorphic modular forms of weight 1/2. Specifically, he proved the following result.

Let $0<d\equiv 0,3 \pmod 4$. Then

$$\mathcal{H}_d(j(z))=q^{-H(d)}\prod_{n=1}^{\infty}(1-q^n)^{A(n^2,d)}, \tag{2.4}$$

where $A(D,d)$ is the coefficient of q^D in a certain weakly holomorphic modular form f_d of weight 1/2 on the group $\Gamma_0(4)$.

Then Zagier [2] proved the following relationships between the coefficients of the weakly holomorphic modular forms f_d and g_D.

With the notation defined above, then

(1) There holds

$$A_m(D,d)=-B_m(D,d); \tag{2.5}$$

(2) If $m\ge 1$, then

$$A_m(1,d)=\sum_{n|m} nA_m(n^2,d). \tag{2.6}$$

Zagier also proved the following result by relating traces of singular moduli to the coefficients of these modular forms.

$$\mathrm{Tr}(d)=A_1(1,d)=-B_1(1,d), \tag{2.7}$$

where $-d<0$ is a discriminant.

Suppose that p is an odd prime and s is a positive integer. When p is an integer or ramified in particular quadratic number fields, Ahlgren and Ono [7] presented many congruences for traces of singular moduli modulo p^s. In addition, they gave a result about the trace of p^2d when p splits in $Q(\sqrt{-d})$ as follows.

$$\mathrm{Tr}(p^2d) \equiv 0 \ (\mathrm{mod}\, p). \tag{2.8}$$

Edixhoven [8] extended their observation and proved if $(\frac{-d}{p}) = 1$, then

$$\mathrm{Tr}(p^{2n}d) \equiv 0 \ (\mathrm{mod}\, p^n), \quad n \geq 1. \tag{2.9}$$

Bolyan [9] exactly computed the formula for $\mathrm{Tr}(p^{2n}d)$ in a new way, and obtained a result of Edixhoven's when $p = 2$.

Jenkins [10] gave the following theorem by using the properties of Hecke operators.

If p is an odd prime, d, D, n are positive integers such that $-d, D \equiv 0, 1 \ (\mathrm{mod}\, 4)$, then

$$\begin{aligned} B_1(D, p^{2n}d) =& p^n B_1(p^{2n}D, d) + \sum_{k=0}^{n-1} (\frac{D}{p})^{n-k-1} (B_1(\frac{D}{p^2}, p^{2k}d) - p^{k+1} B_1(p^{2k}D, \frac{d}{p^2})) + \\ & \sum_{k=0}^{n-1} (\frac{D}{p})^{n-k-1} ((\frac{D}{p}) - (\frac{-d}{p})) p^k B_1(p^{2k}D, d), \end{aligned} \tag{2.10}$$

where $B_1(M, N) = 0$ if M or N is not integer.

He also obtained two corollaries of them in [9]. Using the notation above yields

(1) If $(\frac{D}{p}) = (\frac{-d}{p}) \neq 0$, then $B_1(D, p^{2n}d) = p^n B_1(p^{2n}D, d)$;

(2) If $(\frac{-d}{p}) = 1$, then $\mathrm{Tr}(p^{2n}d) = -p^n B_1(p^{2n}, d)$.

Remark 2.1 Jenkins' exact formulas for coefficients of $B_1(D, p^{2n}d)$ of $q^{p^{2n}d}$ in g_D and $\mathrm{Tr}(p^{2n}d)$ are the generalization results for congruences of traces of singular moduli which are given by Ahlgren-Ono and Edixhoven.

Guerzhoy [11] proved that if $p \in \{3, 5, 7, 13\}$ and $-d < -4$ is a fundamental discriminant, then,

$$(1 - (\frac{-d}{p}))H(-d) = \frac{p-1}{24} \lim_{n\to\infty} \mathrm{Tr}(p^{2n}d). \tag{2.11}$$

After that, Jenkins and Ono [12] established the following congruences relationship between the Hurwitz-Kronecker class number $H(d)$ and the traces of singular moduli $\mathrm{Tr}(p^{2n}d)$.

Suppose that $-d < -4$ is a fundamental discriminant, $n \geq 1$ is an integer, if $p \in \{2, 3\}$ and $(\frac{-d}{p}) = -1$ or $p \in \{5, 7, 13\}$ and $(\frac{-d}{p}) \neq 1$, then

$$\frac{24}{p-1}(1 - (\frac{-d}{p}))H(-d) \equiv \mathrm{Tr}(p^{2n}d) \quad (\mathrm{mod}\ p^n). \tag{2.12}$$

Here, we obtain an exact formula for the coefficient $B_1(D, p^{2n+1}d)$ of $q^{p^{2n+1}}d$ in g_D by using the Hecke operators. Some congruences and divisibilities identities are obtained as well.

Theorem 2.2 *Let p be an odd prime, n a positive integer, $m \geq 0$ an integer, $0 \leq d, p^m d \equiv$*

$0,3 \pmod 4$ *and* $0 < D \equiv 0,1 \pmod 4$. *Then we have*

$$B_1(D, p^{2n+m}d) = p^n B_1(p^{2n}D, p^m d) + \sum_{k=0}^{n-1} (\frac{D}{p})^{n-k-1}(B_1(\frac{D}{p^2}, p^{2k+m}d) - p^{k+1}B_1(p^{2k}D, p^{m-2}d)) + \sum_{k=0}^{n-1} (\frac{D}{p})^{n-k-1}((\frac{D}{p}) - (\frac{-p^m d}{p}))p^k B_1(p^{2k}D, d), \tag{2.13}$$

where $B_1(M, N) = 0$ *if* M *or* N *is not integer.*

Proof Firstly, we need to check $m = 1$ is true.

Using formulas of the Hecke operators, thanks to Zagier [2], for p is a prime and $pd \equiv 0,3 \pmod 4$, then we have

$$A_p(D, pd) = pA_1(p^2D, pd) + (\frac{D}{p})A_1(D, pd) + A_1(\frac{D}{p^2}, pd), \tag{2.14}$$

and

$$B_p(D, pd) = B_1(D, p^3d) + pB_1(D, \frac{d}{p}), \tag{2.15}$$

combining with the formula (2.5), we get

$$B_p(D, pd) = pB_1(p^2D, pd) + (\frac{D}{p})B_1(D, pd) + B_1(\frac{D}{p^2}, pd), \tag{2.16}$$

applying (2.15) to (2.16), we have

$$B_1(D, p^3d) = pB_1(p^2D, pd) + (\frac{D}{p})B_1(D, pd) + B_1(\frac{D}{p^2}, pd) - pB_1(D, \frac{d}{p}). \tag{2.17}$$

Making the induction to n, we can see that the case $n = 1$ is just as (2.17), then we can assume the theorem holds up to $n - 1$, then we get the conclusion by using the induction hypothesis $n - 2$ times. After that the theorem follows for $m = 1$. At last, by combining with the result (2.10), the first result is reached for all $m \geq 0$. □

We relate the relationship between $H(d)$ and $\mathrm{Tr}(d)$ to get congruences for $H(d)$ modulo p^n as follows.

Theorem 2.3 *Let* p *be an odd prime,* n *a positive integer, and* $d, pd \equiv 0,3 \pmod 4$ *with* $(\frac{-d}{p}) = 1$. *Then we have*

$$24H(p^{2n}) \equiv M_c(d, p^{2n}d, p; n) \pmod{p^n}, \tag{2.18}$$

in particular, under these hypotheses p^n *divides* $24H(p^{2n}d)$ *if and only if* p^n *divides* $M_c(d, p^{2n}d, p; n)$, *where the function* $M_c(d, p^{2n}d, p; n)$ *is defined by*

$$M_c(d, p^{2n}d, p; n) = \Big[(1+i)\sum_{c\equiv 0(4)} K_{\frac{3}{2}}(-1, p^{2n}d; c) + 2\sum_{\substack{c\equiv 0(4)\\ (\frac{c}{4})\text{odd}}} e(\frac{p^{2n}d}{2})\varepsilon_{\frac{c}{4}} S(-\overline{4}, \overline{4}p^{2n}d; \frac{c}{4})\Big] c^{-\frac{1}{2}}\sinh(\frac{4\pi p^n\sqrt{d}}{c}), \tag{2.19}$$

where $K_{\frac{3}{2}}(m,n;c)$ is the generalized Kloosterman sum which is defined as, for $\lambda \in \mathbb{Z}$

$$K_{\lambda+\frac{1}{2}}(m,n;c) = \sum_{\substack{a(c)\\(a,c)=1}} (\frac{c}{a})^{2\lambda+1}\varepsilon_a^{2\lambda+1}e(\frac{ma+n\overline{a}}{c}),$$

and

$$S(m,n;s) = \sum_{a(s)} (\frac{a}{s})e(\frac{ma+n\overline{a}}{s}),$$

is the Salié sum, $e(z)=e^{2\pi iz}$, and the ε_v is given by

$$\varepsilon_v = \begin{cases} 1, & v \equiv 1 \pmod 4, \\ i, & v \equiv 3 \pmod 4. \end{cases}$$

Proof For $D \equiv 0,1 \pmod 4$, Bruinier, Jenkins, and Ono [13] proved that the Fourier coefficient $B_1(D,d)$ for positive integer index d with $d \equiv 0,3 \pmod 4$, which is given by

$$B_1(D,d) = 24\delta_{\square,D}H(d) - (1+i)\sum_{c\equiv 0(4)} (1+\delta_{\text{odd}}(\frac{c}{4})) \times \frac{K_{\frac{3}{2}}(-D,d;c)}{\sqrt{Dc}}\sinh(\frac{4\pi}{c}\sqrt{Dd}), \quad (2.20)$$

where $\delta_{\square,D}=1$ if n is a square, and $\delta_{\square,D}=0$ otherwise. The $\delta_{\text{odd}}(\nu)$ on the integer is defined by

$$\delta_{\text{odd}}(\nu) = \begin{cases} 1, & \text{if } \nu \text{ is odd}, \\ 0, & \text{otherwise}. \end{cases}$$

Let $D=1$. Replacing d by $p^{2n}d$ in $B_1(n,d)$, we have

$$B_1(1,p^{2n}d) = 24H(p^{2n}d) - (1+i)\sum_{c\equiv 0(4)} (1+\delta_{\text{odd}}(\frac{c}{4})) \times \frac{K_{\frac{3}{2}}(-1,p^{2n}d;c)}{\sqrt{c}}\sinh(\frac{4\pi}{c}p^n\sqrt{d}). \quad (2.21)$$

Namely, we get

$$\begin{aligned} B_1(1,p^{2n}d) =& 24H(p^{2n}d) - (1+i)\sum_{c\equiv 0(4)} \frac{K_{\frac{3}{2}}(-1,p^{2n}d;c)}{\sqrt{c}}\sinh(\frac{4\pi}{c}p^n\sqrt{d}) - \\ & (1+i)\sum_{\substack{c\equiv 0(4)\\(\frac{c}{4})\text{odd}}} \frac{K_{\frac{3}{2}}(-1,p^{2n}d;c)}{\sqrt{c}}\sinh(\frac{4\pi}{c}p^n\sqrt{d}). \end{aligned} \quad (2.22)$$

Since $(\frac{c}{4})$ is odd, we can rewrite the third term in (2.22) by using the formula of Kloosterman sum $K_{\frac{3}{2}}(m,n;c)$ (see [14]) as

$$K_{\frac{3}{2}}(-1,p^{2n}d;c) = (\cos\frac{\pi(p^{2n}d-1)}{2} - \sin\frac{\pi(p^{2n}d-1)}{2})(1-i)\varepsilon_{\frac{c}{4}}S(-\overline{4},\overline{4}p^{2n}d;\frac{c}{4}). \quad (2.23)$$

Since $pd \equiv 0 \pmod 4$, the cosine-sine term is 1 unless $pd \equiv 3 \pmod 4$, in which case it is -1. Thus, we have

$$K_{\frac{3}{2}}(-1,p^{2n}d;c) = (1-i)e(\frac{p^{2n}d}{2})\varepsilon_{\frac{c}{4}}S(-\overline{4},\overline{4}p^{2n}d;\frac{c}{4}). \quad (2.24)$$

Inserting (2.24) into (2.22), we get

$$B_1(1,p^{2n}d) = 24H(p^{2n}d) - (1+i)\sum_{c\equiv 0(4)} K_{\frac{3}{2}}(-1,p^{2n}d;c)c^{-\frac{1}{2}}\sinh(\frac{4\pi}{c}p^n\sqrt{d}) - 2\sum_{\substack{c\equiv 0(4)\\ (\frac{c}{4})\text{odd}}} e(\frac{p^{2n}d}{2})\varepsilon_{\frac{c}{4}} S(-\overline{4},\overline{4}p^{2n}d;\frac{c}{4})c^{-\frac{1}{2}}\sinh(\frac{4\pi}{c}p^n\sqrt{d}). \tag{2.25}$$

Considering the result in (2.12) while $(\frac{-d}{p})=1$, we get

$$\mathrm{Tr}(p^{2n}d) = -B_1(1,p^{2n}d) \equiv 0 \pmod{p^n}. \tag{2.26}$$

Inserting (2.26) into (2.25), then Theorem 2.3 follows immediately. □

Corollary 2.4 *With the notations in Theorem 2.3, we have*

$$\mathrm{Tr}(p^{2n+1}d) = M_c(d,p^{2n+1}d,p;n) - 24H(p^{2n+1}d). \tag{2.27}$$

Theorem 2.5 *Suppose $-d<-4$ and $d\equiv 0,3 \pmod 4$, and n is a positive integer. If $p\in\{2,3\}$ and $(\frac{-d}{p})=-1$ or $p\in\{5,7,13\}$ and $(\frac{-d}{p})\neq 1$, then we have*

$$\frac{24}{p-1}(1-(\frac{-d}{p}))H(-d) + 24H(p^{2n}d) \equiv M_c(d,p^{2n}d,p;n) \pmod{p^n}. \tag{2.28}$$

In particular, under these hypotheses p^n divides $\frac{24}{p-1}(1-(\frac{-d}{p}))H(-d)+24H(p^{2n}d)$ if and only if p^n divides $M_c(d,p^{2n}d,p;n)$, where $M_c(d,p^{2n}d,p;n)$ is defined in (2.19).

Proof Since $d\equiv 0,3 \pmod 4$, n is a positive integer, and p is a prime, following Zagier [2] and combining Theorem 2.3, we can get the desired result. □

By using the [15, Theorem 2.3 and Corollary 3], we can get the following congruences for the $\mathrm{Tr}(p^{2n+1})$.

Corollary 2.6 *Let $-d<-4$ and $pd\equiv 0,3 \pmod 4$, and n be a positive integer. If $p=3$ and $(\frac{-d}{p})=-1$ or $p\in\{5,7,13\}$ and $(\frac{-d}{p})\neq 1$, then we have*

$$Tr(p^{2n+1}) \equiv \frac{24}{p-1}H(-pd) \pmod{p^n}. \tag{2.29}$$

In particular, under these hypotheses p^n divides $Tr(p^{2n+1})$ if and only if $\frac{24}{p-1}H(-pd)$.

3. Twisted traces of singular moduli

Following the notation above, the coefficients of f_d of a certain weakly holomorphic modular forms of weight 1/2 index positive D on group $\Gamma_0(4)$ can be interpreted in terms of twisted traces in the following manner by Zagier [2].

If m,d,D are positive integers, and $-d,D\equiv 0,1 \pmod 4$, then

$$\mathrm{Tr}_m(D,d) = A_m(D,d)\sqrt{D}. \tag{3.1}$$

Combining with the identity (3.1), we have

$$\mathrm{Tr}_m(D,d) = -B_m(D,d)\sqrt{D}. \tag{3.2}$$

By the way, we point out that $\mathrm{Tr}_m(d)$ is a special case of $\mathrm{Tr}_m(D,d)$ while $D=1$. In this view, we mainly talk about the cases when $D>1$ in this section.

For m,n,d,D are positive integers and $D>1$, with $-d,D\equiv 0,1 \pmod 4$, p is a prime, we have

(1) If $(\frac{-d}{p})=(\frac{D}{p})\neq 0$ or $p\|d$ and $p\|D$, then

$$\mathrm{Tr}_m(D,p^2nd)\equiv 0 \pmod{p^n}. \tag{3.3}$$

(2) If $pd\equiv 0,3 \pmod 4$, $p^2\nmid D$, then

$$\mathrm{Tr}_m(D,p^{2n+1}d)\equiv (\frac{D}{p})\mathrm{Tr}_m(D,p^{2n-1}d) \pmod{p^n}. \tag{3.4}$$

(3) If $p^2\nmid D$ and $p\nmid d$, then

$$\mathrm{Tr}_m(D,p^{2n}d)\equiv [(\frac{D}{p})-(\frac{-d}{p})]\sum_{k=1}^{n}(\frac{-d}{p})^{k+1}\mathrm{Tr}_m(D,p^{2n-2k}d) \pmod{p^n}. \tag{3.5}$$

Jenkins [5] proved an exact expression for twisted traces in an infinite series as follows.

Suppose m,n,d,D are positive integers and $D>1$, where $-d,D\equiv 0,1 \pmod 4$ are fundamental discriminants, then

$$\mathrm{Tr}_m(D,d)=\sum_{0<c\equiv 0(4)} S_{D,d}(m,c)\sinh(\frac{4\pi m\sqrt{dD}}{c}), \tag{3.6}$$

where $S_{D,d}(m,c)$ is

$$S_{D,d}(m,c)=\sum_{\substack{x(c)\\ x^2\equiv -Dd(c)}} \chi_D(\frac{c}{4},x,\frac{x^2+Dd}{c})e(\frac{2mx}{c}), \tag{3.7}$$

and $e(z)=e^{2\pi iz}$.

Here we give an exact formula for the twisted traces of singular moduli, which also is a new relationship between $\mathrm{Tr}_m(D,d)$, $\mathrm{Tr}_m(d)$ and $H(d)$, while $D>1$. Our result is as follows.

Theorem 3.1 *Let m,d,D be positive integers and $D>1$, where $-d,D\equiv 0,1 \pmod 4$ are fundamental discriminants, and $dD\equiv 0,3 \pmod 4$ is a discriminant. Then we have*

$$Tr_m(D,d)=(\frac{D}{n})[Tr_m(Dd)+24H(Dd)\sigma(m)], \tag{3.8}$$

where n is any integer represented by quadratic forms $Q=[a,b,c]$ of discriminant D and coprime to D.

Proof If $dD\equiv 0,3 \pmod 4$, Duke [16] obtained

$$\mathrm{Tr}_m(Dd)=-24H(Dd)\sigma(m)+\sum_{0<c\equiv 0(4)} S_{Dd}(m,c)\sinh(\frac{4\pi m\sqrt{Dd}}{c}), \tag{3.9}$$

where $S_{Dd}(m,c)$ is defined as

$$S_{Dd}(m,c)=\sum_{x^2\equiv -Dd(c)} e(\frac{2mx}{c}). \tag{3.10}$$

Recall that $\chi_D(\frac{c}{4},x,\frac{x^2+Dd}{c})$ of $S_{D,d}(m,c)$ is defined by

$$\chi_D(\frac{c}{4},x,\frac{x^2+Dd}{c}):=\begin{cases}0, & \text{if } (\frac{c}{4},x,\frac{x^2+Dd}{c},D)>1,\\ (\frac{D}{n}), & \text{if } (\frac{c}{4},x,\frac{x^2+Dd}{c},D)=1.\end{cases} \tag{3.11}$$

If $(\frac{c}{4},x,\frac{x^2+Dd}{c},D)=1$, then

$$S_{D,d}(m,c)=(\frac{D}{n})\sum_{\substack{x(c)\\ x^2\equiv -Dd(c)}} e(\frac{2mx}{c}). \tag{3.12}$$

Considering the condition, we have

$$S_{D,d}(m,c)=(\frac{D}{n})S_{Dd}(m,c), \tag{3.13}$$

then we get

$$\mathrm{Tr}_m(D,d)=\sum_{0<c\equiv 0(4)}(\frac{D}{n})S_{Dd}(m,c)\sinh(\frac{4\pi m\sqrt{dD}}{c}). \tag{3.14}$$

Using the formula (3.11), we have

$$\mathrm{Tr}_m(Dd)+24H(Dd)\sigma(m)=\sum_{0<c\equiv 0(4)}S_{Dd}(m,c)\sinh(\frac{4\pi m\sqrt{dD}}{c}). \tag{3.15}$$

Inserting (3.15) into (3.14) leads to the conclusion of the theorem when $\chi_D(\frac{c}{4},x,\frac{x^2+Dd}{c})=1$ in $S_{D,d}(m,c)$. □

Acknowledgements We thank the referees for their time and comments.

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