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Existence of Ground States for Fractional Kirchhoff Equations

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Abstract In this paper, we study a fractional Kirchhoff problem and establish the existence of nontrivial nonnegative ground states under some suitable conditions.

Keywords ground states; fractional Kirchhoff equation; variational methods

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1. Introduction

Consider the fractional Kirchhoff problem

$$\Big(a+b\int_{\mathbb{R}^N}|(-\triangle)^{s/2}u|^2\mathrm{d}x\Big)(-\triangle)^s u+\lambda V(x)u=f(x,u) \text{ in } \mathbb{R}^N, \tag{1.1}$$

where $s\in(0,1)$, $N>2s$, $\lambda>0$ is a real parameter, a,b are positive constants, and $(-\triangle)^s$ is a fractional Laplacian operator defined by

$$(-\triangle)^s u(x)=c_{N,s}\ \text{P.V.}\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y,\quad x\in\mathbb{R}^N.$$

Here P.V. is the principal value and $c_{N,s}$ is a normalization constant.

Notice that the stationary Kirchhoff variational model in bounded regular domains of $\mathbb{R}^N$, which takes into account the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string, was first proposed by Fiscella and Valdinoci [1].

When $\lambda=0$ and $s=1$, problem (1.1) becomes the Kirchhoff type problem

$$-\Big(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x\Big)\triangle u=f(x,u) \text{ in } \mathbb{R}^N.$$

This problem is analogous to the stationary case of equations that arise in the study of string or membrane vibrations, namely,

$$u_{tt}-\Big(a+b\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\Big)\triangle u=f(x,u),$$

where u denotes the displacement, $f(x,u)$ denotes the external force, b denotes the initial tension and a is a number related to the intrinsic properties of the string. The above problem was first proposed by Kirchhoff in 1883 to describe the transversal oscillations of stretched strings.

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When $\lambda = 0$ and $a = 1$, $b = 0$, problem (1.1) becomes the fractional Laplacian problem

$$(-\triangle)^s u = f(x, u) \text{ in } \mathbb{R}^N.$$

In recent years, a great attention has been focused on the study of the fractional Laplacian equation [2–5]. In the context of fractional quantum mechanics, nonlinear fractional Schrödinger equation has been proposed by Laskin [6, 7], as an extension of the Feynman path integral. Literatures on fractional and nonlocal operators and on their applications are quite large, we refer the readers to [3,8] and the references therein. For the basic properties of fractional Sobolev spaces we refer to [9].

Nowadays, fractional Sobolev spaces and corresponding nonlocal equations were widely studied in various contexts, such as optimization, soft thin films, anomalous diffusion, ultra-relativistic limits of quantum mechanics, flame propagation, materials science and water waves, multiple scattering, molecular dynamics, turbulence models, minimal surfaces, anomalous diffusion, conservation laws, quasi-geostrophic flows, crystal dislocation, semipermeable membranes, finance, stratified materials and the thin obstacle problem [10–14].

In most papers on fractional Kirchhoff equations [15–20], to ensure the boundedness of Palais-Smale or Cerami sequences and the mountain pass geometry of the associated Euler-Lagrange functional, the Ambrosetti-Rabinowitz condition or other similar conditions are often assumed:

$$\text{There exists } \mu > 4 \text{ such that } 0 < \mu F(x, k) \le f(x, k)k,\ (x, k) \in \mathbb{R}^N \times \mathbb{R}^+. \tag{A.R.}$$

Recently, without considering the (A.R.) condition, Ref. [21] investigated the existence of radial solutions for a fractional Kirchhoff-type problem by variational methods combined with a cut-off function technique.

Inspired by [21], in this paper, by using Nehari manifold, we obtain the existence of ground states for the fractional Kirchhoff equation (1.1) without the (A.R.) condition. Throughout this work, we set $F(x, k) = \int_0^k f(x, t)\,\mathrm{d}t$ and assume that $V(x)$ and $f(x, k)$ satisfy the following conditions:

(V) $V \in C\left(\mathbb{R}^N, \mathbb{R}\right)$, $V(x) \ge 0$ and there exists $v_0 > 0$ such that the Lebesgue measure of the set $\mathcal{V}_0 = \left\{x \in \mathbb{R}^N : V(x) < v_0\right\}$ is finite.

(f$_1$) $f \in C^1\left(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}\right)$, $f(x, k) \equiv 0$ for all $k \le 0$, and $f(x, k) = o(k)$ uniformly for x as $k \to 0$.

(f$_2$) $f(x, k)/k^3$ is strictly increasing for $k > 0$.

(f$_3$) There exist $a_1 > 0$, $\tau > \max\{1, N/2s\}$ such that $|f(x, k)|^\tau \le a_1 \mathcal{F}(x, k)k^\tau$ for all $(x, k) \in \mathbb{R}^N \times \mathbb{R}^+$ with k large enough, where $\mathcal{F}(x, k) = \frac{1}{4} f(x, k)k - F(x, k)$.

(f$_4$) There exist $\mu > 4$ and a constant $C > 0$ such that

$$F(x, k) \ge Ck^\mu,\ \forall (x, k) \in \mathbb{R}^N \times \mathbb{R}^+.$$

Remark 1.1 Under our assumptions the following conclusions hold:

(i) By (f_2), $\mathcal{F}(x, k) = \frac{1}{4} f(x, k)k - F(x, k)$ is strictly increasing for $k > 0$. Thus,

$$\mathcal{F}(x, k) > \mathcal{F}(x, 0) = 0 \text{ for } k > 0.$$

(ii) If f satisfies (f_1), (f_3) and (f_4), then $|f(x,k)|^{\tau-1} \leq \frac{1}{4}a_1 k^{\tau+1}$ for $k > 0$ large enough. Hence there exists $a_2 > 0$, such that the following growth restriction condition holds:

$$|f(x,k)| \leq a_2(k + k^{p-1}), \tag{1.2}$$

where $p = 2\tau/(\tau-1) \in (2, 2_s^*)$ $(2_s^* = 2N/(N-2s))$.

(iii) Under (1.2) and the (A.R.) condition, (f_3) holds for $\tau \in (N/2s, p/(p-2))$, $\tau > 1$ (See [22, Lemma 2.2]).

We state our result as follows:

Theorem 1.2 *Assume (V) and (f_1)–(f_4) hold. There exists $\Lambda > 0$ such that Eq. (1.1) has at least one nontrivial nonnegative ground state u_λ for $\lambda \geq \Lambda$.*

The rest of this paper is organized as follows. In Section 2, some notations and preliminaries are presented. In Section 3, using Nehari manifold and Ekeland variational principle, we obtain the existence of ground states for problem (1.1).

2. Preliminaries

Consider the Sobolev space $H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : [u]_s < \infty\}$, where $[u]_s$ denotes the so-called Gagliardo semi-norm

$$[u]_s := \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\Big)^{1/2}.$$

In light of [9, Proposition 3.6], the following characterization holds

$$[u]_s^2 = 2C(N,s)^{-1}\int_{\mathbb{R}^N} |(-\triangle)^{s/2}u(x)|^2\mathrm{d}x,$$

where $2C(N,s)^{-1}$ is a normalization constant. For the sake of simplicity, throughout the paper we omit the normalization constant.

Denote the best Sobolev embedding constant as

$$\bar{S} := \inf_{|u|_{L^{2_s^*}}=1} [u]_s^2.$$

Let us denote the inner product and norm of $H^s(\mathbb{R}^N)$ as follows:

$$(u,v)_s = \int_{\mathbb{R}^N} \big((-\triangle)^{s/2}u \cdot (-\triangle)^{s/2}v + uv\big)\mathrm{d}x.$$

$$\|u\|_s = (u,u)_s^{1/2}.$$

Set

$$E = \Big\{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2\mathrm{d}x < \infty\Big\}$$

with inner product

$$(u,v) = \int_{\mathbb{R}^N} \big(a(-\triangle)^{s/2}u \cdot (-\triangle)^{s/2}v + V(x)uv\big)\mathrm{d}x$$

and the associated norm

$$\|u\| = (u,u)^{1/2}.$$

For $\lambda > 0$, we denote the associated inner product and norm as follows:

$$(u,v)_\lambda = \int_{\mathbb{R}^N} \big(a(-\triangle)^{s/2}u \cdot (-\triangle)^{s/2}v + \lambda V(x)uv\big)\mathrm{d}x,$$

$$\|u\|_\lambda = (u,u)_\lambda^{1/2}.$$

Set $E_\lambda = (E, \|\cdot\|_\lambda)$. By (V), the definition of $\bar{S}$ and the Hölder inequality, we obtain that

$$\begin{aligned}
\|u\|_s^2 =& [u]_s^2 + \int_{\{x\in\mathbb{R}^N: V(x)<v_0\}} u^2\mathrm{d}x + \int_{\{x\in\mathbb{R}^N: V(x)\ge v_0\}} u^2\mathrm{d}x \\
\le& [u]_s^2 + \Big(\int_{\{x\in\mathbb{R}^N: V(x)<v_0\}} u^{2_s^*}\mathrm{d}x\Big)^{2/2_s^*} \big|\{x\in\mathbb{R}^N : V(x)<v_0\}\big|^{(2_s^*-2)/2_s^*} + \\
& \frac{1}{\lambda v_0}\int_{\mathbb{R}^N} \lambda V(x)u^2\mathrm{d}x \\
\le& [u]_s^2 + \bar{S}^{-1}\big|\{x\in\mathbb{R}^N : V(x)<v_0\}\big|^{(2_s^*-2)/2_s^*}[u]_s^2 + \frac{1}{\lambda v_0}\int_{\mathbb{R}^N}\lambda V(x)u^2\mathrm{d}x \\
\le& \max\Big\{\frac{1}{a}(1+\bar{S}^{-1}\big|\{x\in\mathbb{R}^N : V(x)<v_0\}\big|^{(2_s^*-2)/2_s^*}), \frac{1}{\lambda v_0}\Big\}\times \\
& \int_{\mathbb{R}^N}\big(a|(-\Delta)^{s/2}u(x)|^2 + \lambda V(x)u^2\big)\mathrm{d}x,
\end{aligned}$$

which implies that the embedding $E_\lambda \hookrightarrow H^s(\mathbb{R}^N)$ is continuous. By [9, Theorem 6.7], $H^s(\mathbb{R}^N)$ continuously embeds into $L^q(\mathbb{R}^N)$ for $q\in[2,2_s^*]$ and compactly embeds into $L^q_{loc}(\mathbb{R}^N)$ for $q\in[2,2_s^*)$. Thus, there exists $c_q>0$ (independent of $\lambda\ge 1$) such that

$$|u|_q \le c_q\|u\|_\lambda, \tag{2.1}$$

where $|\cdot|_q$ with $q\in[2,2_s^*]$ denotes the usual norm in $L^q(\mathbb{R}^N)$.

The energy functional associated with Eq. (1.1) is defined by

$$I_\lambda(u) = \frac{1}{2}\|u\|_\lambda^2 + \frac{b}{4}[u]_s^4 - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x.$$

It is easy to show that $I_\lambda \in C^1$.

Definition 2.1 *(i) We say any sequence $\{u_n\}\subset E_\lambda$ is a $(PS)_c$ sequence for I_λ if $I_\lambda(u_n)\to c$ and $I'_\lambda(u_n)\to 0$ in E_λ^{-1}.*

(ii) We say that a C^1 functional I_λ satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence for I_λ has a convergent subsequence.

We assume that the conditions (V) and (f_1)–(f_4) are satisfied from now on.

Lemma 2.2 *Every $(PS)_c$ sequence of I_λ is bounded in E_λ.*

Proof Let $\{u_n\}\subset E_\lambda$ be a $(PS)_c$ sequence of I_λ, that is,

$$I_\lambda(u_n)\to c, \quad I'_\lambda(u_n)\to 0 \text{ in } E_\lambda^{-1}.$$

By Remark 1.1 (i), we obtain that

$$\begin{aligned} c+1+\|u_n\|_\lambda &\geq I_\lambda(u_n) - \frac{1}{4}\langle I'_\lambda(u_n), u_n\rangle \\ &= \frac{1}{4}\|u_n\|_\lambda^2 + \int_{\mathbb{R}^N}(\frac{1}{4}f(x,u_n)u_n - F(x,u_n))\mathrm{d}x \\ &> \frac{1}{4}\|u_n\|_\lambda^2 \end{aligned}$$

for n large enough. Thus, $\{u_n\}$ is bounded in E_λ. □

Lemma 2.3 *Let $\{u_n\} \subset E_\lambda$ be a $(PS)_c$ sequence of I_λ. There exists a $u \in E_\lambda$ such that $I'_\lambda(u) = 0$; if $u \neq 0$, then*

$$[u_n]_s^2 \to [u]_s^2. \tag{2.2}$$

Proof By Lemma 2.2, $\{u_n\}$ is bounded in E_λ. Thus, up to a subsequence $\{u_n\}$, we may assume that there exists a $u \in E_\lambda$ such that $u_n \rightharpoonup u$ and there is a constant $A \in \mathbb{R}^+$ such that $[u_n]_s^2 \to A^2$. If $u \equiv 0$, the conclusion holds. If $u \neq 0$, we claim that $[u]_s^2 = A^2$. In fact, by the weakly lower semi-continuity of a norm, we get

$$[u]_s^2 \leq A^2.$$

Suppose $[u]_s^2 < A^2$. Since $I'_\lambda(u_n) \to 0$, we have for any $\varphi \in E_\lambda$,

$$\begin{aligned} &\int_{\mathbb{R}^N}\big(a(-\triangle)^{s/2}u\cdot(-\triangle)^{s/2}\varphi + \lambda V(x)u\varphi\big)\mathrm{d}x + bA^2\int_{\mathbb{R}^N}(-\triangle)^{s/2}u\cdot(-\triangle)^{s/2}\varphi\mathrm{d}x - \\ &\quad\int_{\mathbb{R}^N} f(x,u)\varphi\mathrm{d}x = 0. \end{aligned}$$

If we choose $\varphi = u$, then we get $\langle I'_\lambda(u), u\rangle < 0$. By (f_1) and (1.2), $\langle I'_\lambda(tu), tu\rangle > 0$ for small $t > 0$. Therefore, there exists $t_0 \in (0,1)$, such that $\langle I'_\lambda(t_0u), t_0u\rangle = 0$ and $I_\lambda(t_0u) = \max_{t\in[0,1]} I_\lambda(tu)$. Since $\mathcal{F}(x,u)$ is strictly increasing for $u > 0$, we get that

$$\begin{aligned} c &\leq I_\lambda(t_0u) - \frac{1}{4}\langle I'_\lambda(t_0u), t_0u\rangle = \frac{t_0^2}{4}\|u\|_\lambda^2 + \int_{\mathbb{R}^N}\mathcal{F}(x,t_0u)\mathrm{d}x \\ &< \frac{1}{4}\|u\|_\lambda^2 + \int_{\mathbb{R}^N}\mathcal{F}(x,u)\mathrm{d}x \leq \liminf_{n\to\infty}\big(I_\lambda(u_n) - \frac{1}{4}\langle I'_\lambda(u_n), u_n\rangle\big) \\ &= c, \end{aligned}$$

this is a contradiction. Then, $[u]_s^2 = A^2$ and $I'_\lambda(u) = 0$. □

Lemma 2.4 *Let $\beta \in [2, 2_s^*)$. There exists a subsequence $\{u_{n_j}\}$ such that for each $\varepsilon > 0$, there exists $r_\varepsilon > 0$ satisfying*

$$\limsup_{j\to\infty}\int_{B_j\setminus B_r}|u_{n_j}|^\beta\mathrm{d}x \leq \varepsilon$$

for all $r \geq r_\varepsilon$, where $B_j = \{x \in \mathbb{R}^N : |x| \leq j\}$.

Proof Refer to [23, 24]. □

Let $\eta : [0,\infty) \to [0,1]$ be a smooth function satisfying $\eta(t) = 1$ if $t \leq 1$; $\eta(t) = 0$ if $t \geq 2$. Define

$$\hat{u}_j(x) = \eta(2|x|/j)\,u(x).$$

It is clear that

$$\|u-\hat{u}_j\|_\lambda \to 0 \ \text{ as } \ j\to\infty. \tag{2.3}$$

Lemma 2.5 *Let $\{u_{n_j}\}$ and $\{\hat{u}_j\}$ be defined as above. Then*

$$\lim_{j\to\infty}\int_{\mathbb{R}^N}\big(f(x,u_{n_j})-f(x,u_{n_j}-\hat{u}_j)-f(x,\hat{u}_j)\big)\,\psi \mathrm{d}x=0$$

uniformly for $\psi\in E$ with $\|\psi\|\le 1$.

Proof ([23, 24]) Together with the results of Lemma 2.4 for both $\beta=2$ and $\beta=p$, Remark 1.1 (ii), (2.1) and local compactness of Sobolev embedding imply that for any $r>0$,

$$\lim_{j\to\infty}\int_{B_r}\left(f\left(x,u_{n_j}\right)-f\left(x,u_{n_j}-\hat{u}_j\right)-f\left(x,\hat{u}_j\right)\right)w\mathrm{d}x=0 \tag{2.4}$$

uniformly for $w\in E$ with $\|w\|\le 1$. For any $\varepsilon>0$, there is $r_\varepsilon>0$ such that

$$\int_{\mathbb{R}^N\setminus B_r}|u|^\beta \mathrm{d}x<\varepsilon \tag{2.5}$$

for any $r\ge r_\varepsilon$. Then by (2.5), we have

$$\limsup_{j\to\infty}\int_{B_j\setminus B_r}|\hat{u}_j|^\beta\mathrm{d}x\le\int_{\mathbb{R}^N\setminus B_r}|u|^\beta\mathrm{d}x<\varepsilon$$

for any $r\ge r_\varepsilon$. Using Lemma 2.4, for $\beta=2$ and $\beta=p$, Remark 1.1 (ii) and (2.4), we get

$$\begin{aligned}
&\limsup_{j\to\infty}\Big|\int_{\mathbb{R}^N}\left(f\left(x,u_{n_j}\right)-f\left(x,u_{n_j}-\hat{u}_j\right)-f\left(x,\hat{u}_j\right)\right)w\mathrm{d}x\Big|\\
&=\limsup_{j\to\infty}\Big|\int_{B_j\setminus B_r}\left(f\left(x,u_{n_j}\right)-f\left(x,u_{n_j}-\hat{u}_j\right)-f\left(x,\hat{u}_j\right)\right)w\mathrm{d}x\Big|\\
&\le C_1\limsup_{j\to\infty}\int_{B_j\setminus B_r}\left(\left|u_{n_j}\right|+|\hat{u}_j|\right)|w|\mathrm{d}x+\\
&\quad C_2\limsup_{j\to\infty}\int_{B_j\setminus B_r}(\left|u_{n_j}\right|^{p-1}+|\hat{u}_j|^{p-1})|w|\mathrm{d}x\\
&\le C_1\limsup_{j\to\infty}(\left|u_{n_j}\right|_{L^2(B_j\setminus B_r)}+|\hat{u}_j|_{L^2(B_j\setminus B_r)})|w|_{L^2(\mathbb{R}^N)}+\\
&\quad C_2\limsup_{j\to\infty}(\left|u_{n_j}\right|^{p-1}_{L^p(B_j\setminus B_r)}+|\hat{u}_j|^{p-1}_{L^p(B_j\setminus B_r)})|w|_{L^p(\mathbb{R}^N)}\\
&\le C_3\varepsilon^{1/2}+C_4\varepsilon^{(p-1)/p},
\end{aligned} \tag{2.6}$$

thus we obtain the conclusion. □

Lemma 2.6 *Let $\{u_{n_j}\}$ and $\{\hat{u}_j\}$ be defined as above. Then we have the following conclusions:*

(i) $I_\lambda(u_{n_j}-\hat{u}_j)\to c-I_\lambda(u)$;

(ii) $I'_\lambda(u_{n_j}-\hat{u}_j)\to 0$.

Proof By (2.2) and (2.3), we get that

$$[u_{n_j}]_s^2-[\hat{u}_j]_s^2\to 0. \tag{2.7}$$

Thus, by (2.7), we obtain that

$$
\begin{aligned}
I_\lambda(u_{n_j}-\hat{u}_j) =&\frac{1}{2}\left\|u_{n_j}-\hat{u}_j\right\|_\lambda^2+\frac{b}{4}[u_{n_j}-\hat{u}_j]_s^4-\int_{\mathbb{R}^N} F\left(x, u_{n_j}-\hat{u}_j\right) \mathrm{d} x \\
=&\frac{1}{2}\left\|u_{n_j}-\hat{u}_j\right\|_\lambda^2+\frac{b}{4}(\left[u_{n_j}\right]_s^2-[\hat{u}_j]_s^2)^2-\int_{\mathbb{R}^N} F\left(x, u_{n_j}-\hat{u}_j\right) \mathrm{d} x+o(1) \\
=&I_\lambda(u_{n_j})-I_\lambda(\hat{u}_j)+\frac{b}{2}[\hat{u}_j]_s^2([\hat{u}_j]_s^2-\left[u_{n_j}\right]_s^2)+ \\
&\int_{\mathbb{R}^N}\left(F\left(x, u_{n_j}\right)-F\left(x, u_{n_j}-\hat{u}_j\right)-F\left(x, \hat{u}_j\right)\right) \mathrm{d} x+o(1) \\
=&I_\lambda(u_{n_j})-I_\lambda(\hat{u}_j)+\int_{\mathbb{R}^N}\left(F\left(x, u_{n_j}\right)-F\left(x, u_{n_j}-\hat{u}_j\right)-F\left(x, \hat{u}_j\right)\right) \mathrm{d} x+o(1).
\end{aligned}
\tag{2.8}
$$

By (2.3) and the Brézis-Lieb lemma [25], we have that

$$
\int_{\mathbb{R}^N}\left(F\left(x, u_{n_j}\right)-F\left(x, u_{n_j}-\hat{u}_j\right)-F\left(x, \hat{u}_j\right)\right) \mathrm{d} x \rightarrow 0. \tag{2.9}
$$

By $I_\lambda(u_{n_j}) \rightarrow c$, $I_\lambda(\hat{u}_j) \rightarrow I_\lambda(u)$ as $j \rightarrow \infty$, (2.8) and (2.9), we obtain that

$$
I_\lambda(u_{n_j}-\hat{u}_j) \rightarrow c-I_\lambda(u).
$$

Now we prove $I_\lambda^{\prime}(u_{n_j}-\hat{u}_j) \rightarrow 0$. Indeed, by $\hat{u}_j \rightarrow u$ and $u_{n_j} \rightharpoonup u$ in E_λ, we obtain that

$$
\begin{aligned}
\left\langle I_\lambda^{\prime}(u_{n_j}-\hat{u}_j), w\right\rangle =&\left\langle I_\lambda^{\prime}(u_{n_j}), w\right\rangle-\langle I_\lambda^{\prime}(\hat{u}_j), w\rangle+ \\
&\int_{\mathbb{R}^N}\left(f\left(x, u_{n_j}\right)-f\left(x, u_{n_j}-\hat{u}_j\right)-f\left(x, \hat{u}_j\right)\right) w d x+o(1)\|w\|
\end{aligned}
\tag{2.10}
$$

for any $w \in E$ with $\|w\| \leq 1$. By (2.6), we get

$$
\int_{\mathbb{R}^N}\left(f\left(x, u_{n_j}\right)-f\left(x, u_{n_j}-\hat{u}_j\right)-f\left(x, \hat{u}_j\right)\right) w \mathrm{d} x \rightarrow 0 \tag{2.11}
$$

uniformly for $w \in E$ with $\|w\| \leq 1$. By $I_\lambda^{\prime}(u_{n_j}) \rightarrow 0$, $I_\lambda^{\prime}(\hat{u}_j) \rightarrow I_\lambda^{\prime}(u)=0$, (2.10) and (2.11), we conclude that $I_\lambda^{\prime}(u_{n_j}-\hat{u}_j) \rightarrow 0$. □

Lemma 2.7 *There exists $\Lambda>0$ such that for $\lambda \geq \Lambda$, I_λ satisfies the $(PS)_c$ condition.*

Proof Let $\{u_{n_j}\}$ be defined as above. By Lemma 2.2, $\{u_{n_j}\}$ is bounded in E_λ. Thus, up to a subsequence $\{u_{n_j}\}$, such that $u_{n_j} \rightharpoonup u$ in E_λ. By (2.3), $\hat{u}_j \rightarrow u$ in E_λ. Then, $w_j:=u_{n_j}-\hat{u}_j=(u_{n_j}-u)+(u-\hat{u}_j) \rightharpoonup 0$ in E_λ. By (V), $w_j \rightarrow 0$ in $L^2(\mathcal{V}_0)$. Thus,

$$
|w_j|_2^2=\int_{\{V(x) \geq v_0\}} w_j^2 \mathrm{d} x+\int_{\{V(x)<v_0\}} w_j^2 \mathrm{d} x \leq \frac{\|w_j\|_\lambda^2}{\lambda v_0}+o(1). \tag{2.12}
$$

Moreover, for $2<s_0<p<2_s^*$, by (2.12), the Hölder inequality and the Sobolev inequality, we get that

$$
\begin{aligned}
|w_j|_{s_0}^{s_0} & \leq|w_j|_2^{2(p-s_0)/(p-2)}|w_j|_p^{p(s_0-2)/(p-2)} \\
& \leq c_p^{p(s_0-2)/(p-2)}(\lambda v_0)^{-(p-s_0)/(p-2)}\|w_j\|_\lambda^{s_0}+o(1).
\end{aligned}
\tag{2.13}
$$

By (f_1), for any $\varepsilon>0$, there exists $\delta>0$ such that if $|u| \leq \delta$ for all $x \in \mathbb{R}^N$, we have $f(x, u) \leq \varepsilon|u|$.

By (2.12), we have that

$$\int_{|w_j|\le\delta} f(x,w_j)w_j\mathrm{d}x \le \varepsilon\int_{|w_j|\le\delta} w_j^2\mathrm{d}x \le \varepsilon|w_j|_2^2 \le \frac{\varepsilon}{\lambda v_0}\|w_j\|_\lambda^2 + o(1). \tag{2.14}$$

By Lemma 2.6, we have that

$$\int_{\mathbb{R}^N}\mathcal{F}(x,w_j)\mathrm{d}x + \frac{1}{4}\|w_j\|_\lambda^2 = I_\lambda(w_j) - \frac{1}{4}\langle I'_\lambda(w_j), w_j\rangle \to c - I_\lambda(u).$$

Then,

$$\int_{\mathbb{R}^N}\mathcal{F}(x,w_j)\mathrm{d}x \le c - I_\lambda(u). \tag{2.15}$$

By Remark 1.1 (i) and Lemma 2.3, we obtain that

$$I_\lambda(u) = I_\lambda(u) - \frac{1}{4}\langle I'_\lambda(u), u\rangle = \frac{1}{4}\|u\|_\lambda^2 + \int_{\mathbb{R}^N}\big(\frac{1}{4}f(x,u)u - F(x,u)\big)\mathrm{d}x > 0. \tag{2.16}$$

Thus, by (2.15) and (2.16), we conclude that

$$\int_{\mathbb{R}^N}\mathcal{F}(x,w_j)\mathrm{d}x < c. \tag{2.17}$$

By (f_3), (2.13) with $s_0 = 2\tau/(\tau-1)$, (2.17) and the Hölder inequality, we have

$$\begin{aligned}\int_{|w_j|>\delta} f(x,w_j)w_j\mathrm{d}x &\le a_1^{1/\tau}\int_{\mathbb{R}^N}(\mathcal{F}(x,w_j))^{1/\tau}w_j^2\mathrm{d}x\\ &\le a_1^{1/\tau}\Big(\int_{\mathbb{R}^N}\mathcal{F}(x,w_j)\mathrm{d}x\Big)^{1/\tau}|w_j|_{s_0}^2\\ &\le a_1^{1/\tau}c^{1/\tau}|w_j|_{s_0}^2\\ &\le a_1^{1/\tau}c^{1/\tau}c_p^{2p(s_0-2)/s_0(p-2)}(\lambda v_0)^{-2(p-s_0)/s_0(p-2)}\|w_j\|_\lambda^2 + o(1),\end{aligned} \tag{2.18}$$

where $2(p-s_0)/s_0(p-2) > 0$. By (2.14), (2.18) and $\langle I'_\lambda(w_j), w_j\rangle \to 0$, we have

$$\begin{aligned}o(1) = \langle I'_\lambda(w_j), w_j\rangle &= \|w_j\|_\lambda^2 + b[w_j]_s^4 - \int_{\mathbb{R}^N} f(x,w_j)w_j\mathrm{d}x\\ &\ge \|w_j\|_\lambda^2 - \int_{\mathbb{R}^N} f(x,w_j)w_j\mathrm{d}x\\ &\ge \big(1 - \frac{\varepsilon}{\lambda v_0} - \frac{a_1^{1/\tau}c_p^{2p(s_0-2)/s_0(p-2)}c^{1/\tau}}{(\lambda v_0)^{2(p-s_0)/s_0(p-2)}}\big)\|w_j\|_\lambda^2 + o(1).\end{aligned}$$

Set $\Lambda = \Lambda(\varepsilon, v_0, s_0, a_1, \tau, p, c_p, c) > 0$ large enough, when $\lambda \ge \Lambda$, we have $w_j \to 0$ in E_λ, i.e., $u_{n_j} \to \hat{u}_j$ in E_λ. By (2.3), $u_{n_j} \to u$ in E_λ. □

3. Proof of Theorem 1.2

Define $N_\lambda = \{u \in E_\lambda\setminus\{0\} : \langle I'_\lambda(u), u\rangle = 0\}$, $c_\lambda = \inf_{N_\lambda} I_\lambda(u)$. The following lemma implies $N_\lambda \neq \varnothing$.

Lemma 3.1 *Assume* (V) *and* (f_1)–(f_4) *hold. For any* $u \in E_\lambda\setminus\{0\}$, *there exists a unique* $t(u) > 0$ *such that* $t(u)u \in N_\lambda$.

Proof Clearly, $I_\lambda(0) = 0$. For $u \in E_\lambda\setminus\{0\}$, by ($f_1$) and (1.2), for every $\varepsilon > 0$, there exists $c_\varepsilon > 0$

such that

$$F(x,u)\le \frac{\varepsilon}{2}u^2+\frac{c_\varepsilon}{p}u^p \quad \text{for all} \quad (x,u)\in\mathbb{R}^N\times\mathbb{R}^+.$$

By (2.1), we get that

$$I_\lambda(u)=\frac{1}{2}\|u\|_\lambda^2+\frac{b}{4}[u]_s^4-\int_{\mathbb{R}^N}F(x,u)\mathrm{d}x\ge\frac{1}{2}\|u\|_\lambda^2-\frac{\varepsilon}{2}c_2^2\|u\|_\lambda^2-\frac{c_\varepsilon}{p}c_p^p\|u\|_\lambda^p.$$

Pick $\varepsilon c_2^2=1/2$, we obtain that

$$I_\lambda(u)\ge\frac{1}{4}\|u\|_\lambda^2-C\|u\|_\lambda^p,$$

where C is a constant independent of λ. Thus, there exist $\rho>0$ and $\alpha>0$, independent of λ, such that

$$\inf_{\|u\|_\lambda=\rho}I_\lambda(u)\ge\alpha>0.$$

Define the function $g(t):=I_\lambda(tu),\ t\in[0,+\infty)$. By (f$_4$), we have

$$g(t)\le\frac{t^2}{2}\|u\|_\lambda^2+\frac{bt^4}{4}[u]_s^4-Ct^\mu\int_{\mathbb{R}^N}u^\mu\mathrm{d}x\to-\infty \text{ as } t\to+\infty.$$

Moreover,

$$\begin{aligned}g'(t)=0\Leftrightarrow tu\in N_\lambda&\Leftrightarrow t^2\|u\|_\lambda^2+bt^4[u]_s^4=\int_{\mathbb{R}^N}f(x,tu)tu\mathrm{d}x\\&\Leftrightarrow\frac{\|u\|_\lambda^2}{t^2}+b[u]_s^4=\frac{1}{t^3}\int_{\mathbb{R}^N}f(x,tu)u\mathrm{d}x.\end{aligned}$$

By (f$_2$), the right hand side is an increasing function of t, the left hand side is a decreasing function of t.

By the above discussion, we get that there exists a unique $t=t(u)>0$ such that $t(u)u\in N_\lambda$. $\square$

Lemma 3.2 *Assume (V) and (f$_1$)–(f$_4$) hold. I_λ is coercive and bounded below on N_λ.*

Proof For $u\in N_\lambda$, by Remark 1.1 (i), we have

$$\begin{aligned}I_\lambda(u)&=I_\lambda(u)-\frac{1}{4}\langle I_\lambda'(u),u\rangle=\frac{1}{4}\|u\|_\lambda^2+\int_{\mathbb{R}^N}\big(\frac{1}{4}f(x,u)u-F(x,u)\big)\mathrm{d}x\\&\ge\frac{1}{4}\|u\|_\lambda^2.\end{aligned}$$

Thus, I_λ is coercive and bounded below on N_λ. $\square$

Define $c_\lambda=\inf_{u\in N_\lambda}I_\lambda(u)$. By Lemma 3.2, there exists a constant $\delta>0$ such that $c_\lambda>\delta$.

Lemma 3.3 *Under the assumptions of Theorem 1.2. For each $u\in N_\lambda$, there exist $\epsilon>0$ and a differentiable function $\xi:B(0,\epsilon)\subset H^s(\mathbb{R}^N)\to\mathbb{R}^+$ such that $\xi(0)=1$, the function $\xi(v)(u-v)\in N_\lambda$ and*

$$\langle\xi'(0),v\rangle=\frac{2(u,v)_\lambda+4b[u]_s^2\int_{\mathbb{R}^N}(-\Delta)^{s/2}u(-\Delta)^{s/2}v\mathrm{d}x-\int_{\mathbb{R}^N}f(x,u)vdx-\int_{\mathbb{R}^N}f'(x,u)uv\mathrm{d}x}{\langle\Psi_\lambda'(u),u\rangle}$$

for all $v\in H^s(\mathbb{R}^N)$, where $\Psi_\lambda(u)=\langle I_\lambda'(u),u\rangle$.

Proof The following argument is similar to [26, Lemma 3.1]. For $u \in N_\lambda$, define a function $F : \mathbb{R} \times H^s(\mathbb{R}^N) \to \mathbb{R}$ by

$$\begin{aligned} F(\xi, w) &= \langle I'_\lambda (\xi (u-w)), \xi (u-w) \rangle \\ &= \xi^2 \|u-w\|_\lambda^2 + b\xi^4 [u-w]_s^4 - \int_{\mathbb{R}^N} f(x, \xi(u-w)) \xi (u-w) \mathrm{d}x. \end{aligned}$$

Then $F(1,0) = \langle I'_\lambda (u), (u) \rangle = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}\xi} F(1,0) = 2\|u\|_\lambda^2 + 4b[u]_s^4 - \int_{\mathbb{R}^N} f(x,u) u \mathrm{d}x - \int_{\mathbb{R}^N} f'(x,u) u^2 \mathrm{d}x \neq 0$$

$$\begin{aligned} \Big\langle \frac{\mathrm{d}}{\mathrm{d}w} F(1,0), v \Big\rangle = -2(u,v)_\lambda - 4[u]_s^2 \int_{\mathbb{R}^N} (-\triangle)^{s/2} u \cdot (-\triangle)^{s/2} v \mathrm{d}x + \\ \int_{\mathbb{R}^N} f(x,u) v \mathrm{d}x + \int_{\mathbb{R}^N} f'(x,u) uv \mathrm{d}x \end{aligned}$$

for all $v \in B(0,\epsilon)$. Therefore, according to the implicit function theorem, there exist $\epsilon > 0$ and a differentiable function $\xi : B(0,\epsilon) \subset H^s(\mathbb{R}^N) \to \mathbb{R}^+$ such that $\xi(0) = 1$,

$$\langle \xi'(0), v \rangle = \frac{2(u,v)_\lambda + 4[u]_s^2 \int_{\mathbb{R}^N} (-\triangle)^{s/2} u \cdot (-\triangle)^{s/2} v \mathrm{d}x - \int_{\mathbb{R}^N} f(x,u) v \mathrm{d}x - \int_{\mathbb{R}^N} f'(x,u) uv \mathrm{d}x}{2\|u\|_\lambda^2 + 4b[u]_s^4 - \int_{\mathbb{R}^N} f(x,u) u \mathrm{d}x - \int_{\mathbb{R}^N} f'(x,u) u^2 \mathrm{d}x},$$

and $F(\xi(v), v) = 0$ for every $v \in B(0,\epsilon)$. It is equivalent to

$$\langle I'_\lambda (\xi(v)(u-v)), \xi(v)(u-v) \rangle = 0$$

for every $v \in B(0,\epsilon)$, i.e., $\xi(v)(u-v) \in N_\lambda$. $\square$

Proposition 3.4 *Under the assumptions of Theorem 1.2. There exists a minimizing sequence* $\{u_n\} \subset N_\lambda$ *such that*

$$I_\lambda(u_n) = c_\lambda + o(1),\ I'_\lambda(u_n) = o(1) \text{ in } E_\lambda^{-1}.$$

Proof By Lemma 3.2 and Ekeland variational principle [27], there exists a minimizing sequence $\{u_n\} \subset N_\lambda$ such that

$$I_\lambda(u_n) \le c_\lambda + \frac{1}{n} \tag{3.1}$$

and

$$I_\lambda(u_n) < I_\lambda(u_0) + \frac{1}{n} \|u_n - u_0\|_\lambda \tag{3.2}$$

for each $u_0 \in N_\lambda$. By taking n large enough, we get that

$$I_\lambda(u_n) = \frac{1}{4}\|u_n\|_\lambda^2 - \int_{\mathbb{R}^N} \big(F(x,u_n) - \frac{1}{4} f(x,u_n)u_n\big) = c_\lambda + \frac{1}{n} < 2c_\lambda. \tag{3.3}$$

By (3.3) and Remark 1.1 (i), we obtain that

$$\frac{1}{4}\|u_n\|_\lambda^2 < 2c_\lambda. \tag{3.4}$$

Thus,

$$\|u_n\|_\lambda < (8c_\lambda)^{1/2}. \tag{3.5}$$

Next we will show that

$$\|I'_\lambda(u_n)\|_{E_\lambda^{-1}} \to 0 \quad \text{as} \quad n \to \infty.$$

Indeed, by Lemma 3.3, for $u_n \in N_\lambda$, there exists the function $\xi_n : B(0,\epsilon_n) \subset H^s(\mathbb{R}^N) \to \mathbb{R}^+$ for $\epsilon_n > 0$, such that $\xi_n(0) = 1$ and $\xi_n(v)(u_n - v) \in N_\lambda$ for all $v \in H^s(\mathbb{R}^N)$. Fixed $n \in \mathbb{N}$, choose $0 < \rho < \epsilon_n$. Let $u \in H^s(\mathbb{R}^N)$ with $u \neq 0$ and $u_\rho = \frac{\rho u}{\|u\|_\lambda}$. Set

$$P_\rho := \xi_n(u_\rho)(u_n - u_\rho).$$

Since $P_\rho \in N_\lambda$, we conclude from (3.2) that

$$I_\lambda(P_\rho) - I_\lambda(u_n) \geq -\frac{1}{n}\|u_n - P_\rho\|_\lambda.$$

By the mean value theorem, we get that

$$\langle I'_\lambda(u_n), P_\rho - u_n\rangle + o\left(\|u_n - P_\rho\|_\lambda\right) \geq -\frac{1}{n}\|u_n - P_\rho\|_\lambda.$$

Thus,

$$\begin{aligned}
&-\langle I'_\lambda(u_n), u_\rho\rangle + (\xi_n(u_\rho) - 1)\langle I'_\lambda(u_n), (u_n - u_\rho)\rangle \\
&\geq -\frac{1}{n}\|u_n - P_\rho\|_\lambda + o\left(\|u_n - P_\rho\|_\lambda\right).
\end{aligned} \tag{3.6}$$

It follows from $P_\rho \in N_\lambda$ and (3.6) that

$$\begin{aligned}
&-\rho\Big\langle I'_\lambda(u_n), \frac{u}{\|u\|_\lambda}\Big\rangle + (\xi_n(u_\rho) - 1)\langle I'_\lambda(u_n) - I'_\lambda(P_\rho), u_n - u_\rho\rangle \\
&\geq -\frac{1}{n}\|u_n - P_\rho\|_\lambda + o\left(\|u_n - P_\rho\|_\lambda\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\Big\langle I'_\lambda(u_n), \frac{u}{\|u\|_\lambda}\Big\rangle \leq &\frac{\|u_n - P_\rho\|_\lambda}{n\rho} + \frac{o\left(\|u_n - P_\rho\|_\lambda\right)}{\rho} + \\
&\frac{\xi_n(u_\rho) - 1}{\rho}\langle I'_\lambda(u_n) - I'_\lambda(P_\rho), u_n - u_\rho\rangle.
\end{aligned} \tag{3.7}$$

Clearly, we have

$$\|u_n - P_\rho\|_\lambda \leq \rho|\xi_n(u_\rho)| + |\xi_n(u_\rho) - 1|\,\|u_n\|_\lambda$$

and

$$\lim_{n\to\infty}\frac{|\xi_n(u_\rho) - 1|}{\rho} \leq \|\xi'_n(0)\|.$$

Let $\rho \to 0$ in (3.7). Then by (3.5), we could find a constant $C > 0$, independent of ρ, such that

$$\Big\langle I'_\lambda(u_n), \frac{u}{\|u\|_\lambda}\Big\rangle \leq \frac{C}{n}(1 + \|\xi'_n(0)\|). \tag{3.8}$$

In the following we show that $\|\xi'_n(0)\|$ is uniformly bounded in n. In fact, by Lemma 3.3, (3.5) and the Hölder inequality, we have

$$\langle \xi'_n(0), v\rangle \leq \frac{d\|v\|_\lambda}{|\langle \Psi'_\lambda(u_n), u_n\rangle|} \text{ for some } d > 0.$$

In the following we only need to prove that

$$|\langle \Psi'_\lambda (u_n), u_n\rangle| > C$$

for some $C > 0$ and n large enough. If not, assume that there exists a subsequence $\{u_n\}$, such that

$$\langle \Psi'_\lambda (u_n), u_n\rangle = o(1) \text{ as } n \to \infty. \tag{3.9}$$

Together with (3.9), the fact that $u_n \in N_\lambda$ yields

$$\int_{\mathbb{R}^N} (f'(x, u_n) u_n - 3f(x, u_n)) \,\mathrm{d}x = -2\|u_n\|_\lambda^2 + o(1) < 0.$$

However, by (f_2),

$$\int_{\mathbb{R}^N} (f'(x, u_n) u_n - 3f(x, u_n)) \,\mathrm{d}x > 0.$$

This is a contradiction. Thus, by (3.8), we get

$$\Big\langle I'_\lambda(u_n), \frac{u}{\|u\|_\lambda}\Big\rangle \le \frac{C}{n}.$$

Therefore, we complete the proof. □

Proof of Theorem 1.2 By Lemma 3.2 and Ekeland variational principle [27], there exists a minimizing sequence $\{u_n\}$ for I_λ on N_λ such that

$$I_\lambda(u_n) = c_\lambda + o(1),\ I'_\lambda(u_n) = o(1) \text{ in } E_\lambda^{-1}.$$

By Lemma 2.7, for $\lambda \ge \Lambda$, there exists a subsequence $\{u_n\}$ and $u_\lambda \in E_\lambda$ such that

$$u_n \to u_\lambda \text{ in } E_\lambda.$$

Thus, $u_\lambda \in N_\lambda$ and $I_\lambda(u_\lambda) = c_\lambda$. By ($f_1$), u_λ is a nontrivial nonnegative ground state for Eq. (1.1). □

References

[1] A. FISCELLA, E. VALDINOCI. *A critical Kirchhoff type problem involving a nonlocal operator*. Nonlinear Anal., 2014, **94**: 156–170.

[2] L. ACETO, P. NOVATI. *Rational approximation to the fractional Laplacian operator in reaction-diffusion problems*. SIAM J. Sci. Comput., 2017, **39**(1): A214–A228.

[3] G. AUTUORI, P. PUCCI. *Elliptic problems invilving the fractional Laplacian in $\mathbb{R}^N$*. J. Differential Equations, 2013, **255**: 2340–2362.

[4] V. TARASOV. *Exact discretization of fractional Laplacian*. Comput. Math. Appl., 2017, **73**(5): 855–863.

[5] Hui ZHOU, Yang LIU, P. AGARWAL. *Solvability for fractional p-Laplacian differential equations with multipoint boundary conditions at resonance on infinite interval*. J. Appl. Math. Comput., 2017, **53**(1-2): 51–76.

[6] N. LASKIN. *Fractional quantum mechanics and Lévy path integrals*. Phys. Lett. A, 2000, **268**(4-6): 298–305.

[7] N. LASKIN. *Fractional Schrödinger equation*. Phys. Rev. E (3), 2002, **66**(5): 1–7.

[8] Xiaojun CHANG, Zhiqiang WANG. *Nodal and multiple solutions of nonlinear problems involving the fractional Laplacian*. J. Differential Equations, 2014, **256**(8): 2965–2992.

[9] E. DI NEZZA, G. PALATUCCI, E. VALDINOCI. *Hitchhiker's guide to the fractional Sobolev spaces*. Bull. Sci. Math., 2012, **136**(5): 521–573.

[10] D. APPLEBAUM. *Lévy processes-from probablity to finance and quantum groups*. Notices Amer. Math. Soc., 2004, **51**: 1336–1347.

[11] L. CAFFARELLI, L. SILVESTER. *An extension problem related to the fractional Laplacian*. Comm. Partial Differential Equations, 2007, **32**(7-9): 1245–1260.

[12] G. MOLICA BISCI. *Fractional equations with bounded primitive*. Appl. Math. Lett., 2014, **27**: 53–58.

[13] P. PUCCI, Mingqi XIANG, Binlin ZHANG. *Existence and multiplicity of entire solutions for fractional p-Kirchhoff equations*. Adv. Nonlinear Anal., 2016, **5**(1): 27–55.

[14] Mingqi XIANG, Binlin ZHANG, V. T. D. RADULESCU. *Existence of solutions for perturbed fractional p-Laplacian equationa*. J. Differential Equations, 2016, **260**(2): 1392–1413.

[15] Wenjing CHEN. *Multiplicity of solutions for a fractional Kirchhoff type problem*. Commun. Pure Appl. Anal., 2015, **14**(5): 2009–2020.

[16] Hua JIN, Wenbin LIU. *Fractional Kirchhoff equation with a general critical nonlinearity*. Appl. Math. Lett., 2017, **74**: 140–146.

[17] Mingqi XIANG, G. MOLICA BISCI, Guohua TIAN, et al. *Infinitely many solutions for the stationary Kirchhoff problems involving the fractional p-Laplacian*. Nonlinearity, 2016, **29**(2): 357–374.

[18] Mingqi XIANG, Binlin ZHANG, Xiuying GUO. *Infinitely many solutions for a fractional Kirchhoff type problem via fountain theorem*. Nonlinear Anal., 2015, **120**: 299–313.

[19] Mingqi XIANG, Binlin ZHANG, V. T. D. RADULSCU. *Existence of solutions for a binonlocal fractional p-Kirchhoff type problem*. Comput. Math. Appl., 2016, **71**: 255–266.

[20] Xia ZHANG, Chao ZHANG. *Existence of solutions for critical fractional Kirchhoff problems*. Math. Methods Appl. Sci., 2017, **40**(5): 1649–1665.

[21] Mingqi XIANG, Binlin ZHANG, Miaomiao YANG. *A fractional Kirchhoff-type problem in* $\mathbb{R}^N$ *without the (AR) condition*. Complex Var. Elliptic Equ., 2016, **61**(11): 1481–1493.

[22] Yanheng DING, A. SZUKIN. *Bound states for semilinear Schrödinger equations with sign-changing potential*. Calc. Var. Partial Differential Equations, 2007, **29**(3): 397–419.

[23] Yanheng DING, Fanghua LIN. *Solutions of perturbed Schrödinger equations with critical nonlinearity*. Calc. Var. Partial Differential Equations, 2007, **30**(2): 231–249.

[24] Qilin XIE, Shiwang MA. *Existence and concentration of positive solutions for Kirchhoff-type problems with a steep well potential*. J. Math. Anal. Appl., 2015, **431**(2): 1210–1223.

[25] H. BRÉZIS, E. LIEB. *A relation between pointwise convergence of functions and convergence of functionals*. Proc. Amer. Math. Soc., 1983, **88**(3): 486–490.

[26] T. F. WU. *Multiplicity results for a semi-linear elliptic equation involving sign-changing weight function*. Rocky Mountain J. Math., 2009, **39**(3): 995–1011.

[27] I. EKELAND. *On the variational principle*. J. Math. Anal. Appl., 1974, **47**: 324–353.