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Majorization Problems for Some Subclasses of Starlike Functions

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Abstract In this article, we study several majorizaton properties for the subclasses SL^*, $SR^*(a,b)$, $BS^*(\alpha)$, S_c^*, S_q^* and S_R^* of starlike functions. Meanwhile, some special cases of our main results are given.

Keywords analytic function; subordination; majorization; starlike function

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1. Introduction and definitions

Let $\mathbb{C}$ be complex plane and assume that $\mathcal{A}$ denote the class of analytic and univalent functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \tag{1.1}$$

in the open unit disk

$$\mathbb{U}=\{z:z\in\mathbb{C}\ \text{ and }\ |z|<1\}.$$

In 1967, Macgregor [1] introduced the following notion of majorization.

Definition 1.1 *Let f and g be analytic in $\mathbb{U}$. We say that f is majorized by g in $\mathbb{U}$ and write*

$$f(z)\ll g(z),\quad z\in\mathbb{U},$$

if there exists a function $\varphi(z)$, analytic in $\mathbb{U}$, satisfying

$$|\varphi(z)|\le 1\ \text{ and }\ f(z)=\varphi(z)g(z),\quad z\in\mathbb{U}. \tag{1.2}$$

Later, Roberston [2] gave the concept of quasi-subordination, which is shown as below.

Definition 1.2 *For two analytic functions f and g in $\mathbb{U}$, we say f is quasi-subordinate to g in $\mathbb{U}$ and write*

$$f(z)\prec_q g(z),\quad z\in\mathbb{U},$$

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if there exist two analytic functions $\varphi(z)$ *and* $\omega(z)$ *in* $\mathbb{U}$, *such that* $\frac{f(z)}{\varphi(z)}$ *is analytic in* $\mathbb{U}$ *and*

$$|\varphi(z)| \le 1, \quad \omega(0) = 0 \ \ \textit{and} \ \ |\omega(z)| \le |z| < 1, \quad z \in \mathbb{U},$$

satisfying

$$f(z) = \varphi(z) g(\omega(z)), \quad z \in \mathbb{U}. \tag{1.3}$$

Remark 1.3 (i) For $\varphi(z) \equiv 1$ in (1.3), we have

$$f(z) = g(\omega(z)), \quad z \in \mathbb{U}$$

and say f is subordinate to g in $\mathbb{U}$, denoted by [3]

$$f(z) \prec g(z), \quad z \in \mathbb{U};$$

(ii) For $\omega(z) = z$ in (1.3), the quasi-subordination (1.3) becomes the majorization (1.2).

In 1991, Ma and Minda [4] introduced the following function class $S^*(\phi)$, namely, Ma-Minda type starlike functions, which is defined by using the above subordination principle:

$$S^*(\phi) := \{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z), \quad z \in \mathbb{U}\},$$

where $\phi(z)$ is analytic and univalent in $\mathbb{U}$ and for which $\phi(\mathbb{U})$ is convex with $\phi(0) = 1$ and $\Re(\phi(z)) > 0$ for $z \in \mathbb{U}$.

We notice that, for choosing the suitable function $\phi(z)$, the class $S^*(\phi)$ reduces to some different subclasses of starlike functions, for instance,

(i) If we take

$$\phi(z) = \frac{1 + Az}{1 + Bz}, \quad -1 \le B < A \le 1; \ z \in \mathbb{U},$$

then we obtain the class

$$S^*(A, B) := \{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, \quad -1 \le B < A \le 1; \ z \in \mathbb{U}\},$$

which was introduced by Janowski [5]. As a special case, for $A = 1 - 2\alpha$ and $B = -1$, we have the class $S^*(1 - 2\alpha, -1) = S^*(\alpha)$ of starlike function of order α $(0 \le \alpha < 1)$. Further, for $A = 1$ and $B = -1$, we have the familiar class $S^*(1, -1) = S^*$ of starlike function in $\mathbb{U}$.

(ii) If we set

$$\phi(z) = \sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1 - z}{1 + 2(\sqrt{2} - 1)z}}, \quad z \in \mathbb{U},$$

then we have the class

$$SL^* := \{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1 - z}{1 + 2(\sqrt{2} - 1)z}}, \quad z \in \mathbb{U}\}, \tag{1.4}$$

which was introduced and investigated by Mendiratta et al. [6] and implies that $\frac{zf'(z)}{f(z)}$ lies in the region bounded by the left-half of lemniscate of Bernoulli given by $\gamma_1 := \{w \in \mathbb{C} : \Re(w) > 0, \ |(w - \sqrt{2})^2 - 1| < 1\}$.

(iii) If we let

$$\phi(z) = \sqrt[a]{b(1 + z)}, \quad a \ge 1; \ b \ge \frac{1}{2}; \ z \in \mathbb{U},$$

then we show the class

$$SR^*(a,b) := \{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \sqrt[a]{b(1+z)}, \quad a \geq 1;\ b \geq \frac{1}{2};\ z \in \mathbb{U}\}, \tag{1.5}$$

which was introduced and investigated by Sokol [7] and implies that $\frac{zf'(z)}{f(z)}$ lies in the region bounded by the right-half of lemniscate of Bernoulli given by $\gamma_2 := \{w \in \mathbb{C} : \Re(w) > 0,\ |w^a - b| < b\}$. Specially, for $a = 2$ and $b = 1$, we have

$$SR^* := SR^*(2,1) = \{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \sqrt{1+z}, \quad z \in \mathbb{U}\},$$

which was introduced and studied by Sokol and Stankiewicz [8].

(iv) If we choose

$$\phi(z) = 1 + \frac{z}{1-\alpha z^2}, \quad 0 \leq \alpha < 1;\ z \in \mathbb{U},$$

then we get the class, related to the Booth lemniscate

$$BS^*(\alpha) := \{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{z}{1-\alpha z^2}, \quad 0 \leq \alpha < 1;\ z \in \mathbb{U}\}, \tag{1.6}$$

which was introduced and studied by Piejko and Sokol [9] (see also [10]).

(v) If we choose

$$\phi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2, \quad z \in \mathbb{U},$$

then we obtain the class, associated with a cardioid

$$S_c^* := \{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2, \quad z \in \mathbb{U}\}, \tag{1.7}$$

which was introduced and studied by Sharma et al. [11].

(vi) If we take

$$\phi(z) = z + \sqrt{1+z^2}, \quad z \in \mathbb{U},$$

then we get the class

$$S_q^* := \{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec z + \sqrt{1+z^2}, \quad z \in \mathbb{U}\}, \tag{1.8}$$

which was introduced and discussed by Raina and Sokol [12].

(vii) If we put

$$\phi(z) = 1 + \frac{z}{k}(\frac{k+z}{k-z}),\ k = \sqrt{2}+1, \quad z \in \mathbb{U},$$

then we have the class, connected with a rational function

$$S_R^* := \{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{z}{k}(\frac{k+z}{k-z}),\ k = \sqrt{2}+1, \quad z \in \mathbb{U}\}, \tag{1.9}$$

which was introduced and discussed by Kumar and Ravichandran [13].

A majorization problem for the normalized class S^* of starlike functions has been investigated by MacGregor [1] and Altintas et al. [14] (see also [15]). Recently, many researchers have studied several majorization problems for univalent and multivalent functions or meromorphic and multivalent meromorphic functions, which are all subordinate to the same function

$\phi(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$, involving various different operators, the interested reader can, for example, see [16–24]. However, in this paper, we investigate the problems of majorization for the subclasses SL^*, $SR^*(a,b)$, $BS^*(\alpha)$, S_c^*, S_q^* and S_R^* of starlike functions, which are, respectively, subordinate to a different function $\phi(z)$ and also not restricted to any operators.

2. Main results

At first, we give and prove majorization property for the class SL^*.

Theorem 2.1 *Let the function $f \in \mathcal{A}$ and suppose that $g \in SL^*$. If $f(z)$ is majorized by $g(z)$ in $\mathbb{U}$, that is, that $f(z) \ll g(z)$ $(z \in \mathbb{U})$, then, for $|z| \le r_1$, we have $|f'(z)| \le |g'(z)|$, where r_1 is the smallest positive root of the equation:*

$$(2\sqrt{2}-1)r^5 + (17-10\sqrt{2})r^4 - 6r^3 + 2(6\sqrt{2}-11)r^2 + (6\sqrt{2}-1)r + 1 - 2\sqrt{2} = 0. \tag{2.1}$$

Proof Since $g \in SL^*$, then, according to (1.4) and subordination relationship, we get

$$\frac{zg'(z)}{g(z)} = \sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-\omega(z)}{1+2(\sqrt{2}-1)\omega(z)}}, \tag{2.2}$$

where $\omega(z) = c_1 z + c_2 z^2 + \cdots$ is bounded and analytic in $\mathbb{U}$, satisfying [25]

$$\omega(0) = 0 \ \text{ and } \ |\omega(z)| \le |z|, \quad z \in \mathbb{U}. \tag{2.3}$$

From (2.2) and (2.3), we easily obtain

$$|\frac{g(z)}{g'(z)}| \le \frac{|z|}{\sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1+|z|}{1-2(\sqrt{2}-1)|z|}}}. \tag{2.4}$$

Also, because $f(z)$ is majorized by $g(z)$ in $\mathbb{U}$, we find, from Definition 1.1, that

$$f(z) = \varphi(z)g(z).$$

Differentiating the above equality on both sides with respect to z, we obtain

$$f'(z) = \varphi'(z)g(z) + \varphi(z)g'(z) = g'(z)(\varphi'(z)\frac{g(z)}{g'(z)} + \varphi(z)). \tag{2.5}$$

On the other hand, observing that the Schwarz function φ satisfies the inequality [26]

$$|\varphi'(z)| \le \frac{1-|\varphi(z)|^2}{1-|z|^2}, \quad z \in \mathbb{U}. \tag{2.6}$$

Using (2.4) and (2.6) in (2.5), we get

$$|f'(z)| \le \big[|\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|^2} \cdot \frac{|z|}{\sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1+|z|}{1-2(\sqrt{2}-1)|z|}}}\big]|g'(z)|,$$

which, by taking

$$|z| = r, \ |\varphi(z)| = \rho, \quad 0 \le \rho \le 1, \tag{2.7}$$

reduces to the inequality

$$|f'(z)| \le \Phi_1(r,\rho)|g'(z)|,$$

where

$$\Phi_1(r,\rho) = \frac{r(1-\rho^2)}{(1-r^2)[\sqrt{2}-(\sqrt{2}-1)\sqrt{\frac{1+r}{1-2(\sqrt{2}-1)r}}]} + \rho.$$

In order to determine r_1, we must choose

$$r_1 = \max\{r \in [0,1) : \Phi_1(r,\rho) \leq 1, \quad \forall \rho \in [0,1]\}$$

or

$$r_1 = \max\{r \in [0,1) : \Psi_1(r,\rho) \geq 0, \quad \forall \rho \in [0,1]\},$$

where

$$\Psi_1(r,\rho) = (1-r^2)[\sqrt{2}-(\sqrt{2}-1)\sqrt{\frac{1+r}{1-2(\sqrt{2}-1)r}}] - r(1+\rho).$$

Obviously, for $\rho = 1$, the function $\Psi_1(r,\rho)$ takes its minimum value, namely,

$$\min\{\Psi_1(r,\rho) : \rho \in [0,1]\} = \Psi_1(r,1) := \psi_1(r),$$

where

$$\psi_1(r) = (1-r^2)[\sqrt{2}-(\sqrt{2}-1)\sqrt{\frac{1+r}{1-2(\sqrt{2}-1)r}}] - 2r.$$

Further, because $\psi_1(0) = 1 > 0$ and $\psi_1(1) = -2 < 0$, there exists r_1, such that $\psi_1(r) \geq 0$ for all $r \in [0, r_1]$, where r_1 is the smallest positive root of the equation (2.1). This completes the proof of Theorem 2.1. □

Next, we discuss majorization property for the class $SR^*(a,b)$.

Theorem 2.2 *Let the function $f \in \mathcal{A}$ and suppose that $g \in SR^*(a,b)$. If $f(z) \ll g(z)$ $(z \in \mathbb{U})$, then, for $|z| \leq r_2$, we have $|f'(z)| \leq |g'(z)|$, where $r_2 := r_2(a,b)$ is the smallest positive root of the equation:*

$$b(1-r)(1-r^2)^a - 2^a r^a = 0, \quad a \geq 1;\ b \geq \frac{1}{2}. \tag{2.8}$$

Proof Because $g \in SR^*(a,b)$, from (1.5) and subordination relationship, we show that

$$\frac{zg'(z)}{g(z)} = \sqrt[a]{b(1+\omega(z))}, \quad a \geq 1;\ b \geq \frac{1}{2}, \tag{2.9}$$

where $\omega(z)$ is defined as (2.3).

By (2.3) and (2.9), it is easy to get

$$|\frac{g(z)}{g'(z)}| \leq \frac{|z|}{\sqrt[a]{b(1-|z|)}}. \tag{2.10}$$

In view of (2.6) as well as (2.10) in (2.5), we have

$$|f'(z)| \leq [|\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|^2} \cdot \frac{|z|}{\sqrt[a]{b(1-|z|)}}]|g'(z)|.$$

Now, according to (2.7) and just as the proof of Theorem 2.1, we can get the required result (2.8) and so complete the proof of Theorem 2.2. □

As a special case of Theorem 2.2, when $a = 2$ and $b = 1$, we get the following corollary.

Corollary 2.3 *Let the function $f \in \mathcal{A}$ and assume that $g \in SR^*$. If $f(z) \ll g(z)$ $(z \in \mathbb{U})$, then, for $|z| \le r_3$, we have $|f'(z)| \le |g'(z)|$, where $r_3 := r_2(2,1)$ is the smallest positive root of the equation:*

$$r^5 - r^4 - 2r^3 + 6r^2 + r - 1 = 0.$$

Applying the same as the proof of Theorems 2.1 or 2.2, and combining (1.6)–(1.9), we can obtain the following majorization results for the classes $BS^*(\alpha)$, S_c^*, S_q^* and S_R^*, respectively. Here, we choose to omit the details of the proofs.

Theorem 2.4 *Let the function $f \in \mathcal{A}$ and suppose that $g \in BS^*(\alpha)$. If $f(z) \ll g(z)$ $(z \in \mathbb{U})$, then, for $|z| \le r_4$, we have $|f'(z)| \le |g'(z)|$, where $r_4 := r_4(\alpha)$ is the smallest positive root of the equation:*

$$\alpha r^4 - (2\alpha + 1)r^3 - (\alpha + 1)r^2 - r + 1 = 0, \quad 0 \le \alpha < 1.$$

Taking $\alpha = \frac{1}{2}$ in Theorem 2.4, we obtain the following corollary.

Corollary 2.5 *Let the function $f \in \mathcal{A}$ and assume that $g \in BS^*(\frac{1}{2})$. If $f(z) \ll g(z)$ $(z \in \mathbb{U})$, then, for $|z| \le r_5$, we have $|f'(z)| \le |g'(z)|$, where $r_5 := r_4(\frac{1}{2})$ is the smallest positive root of the equation:*

$$r^4 - 4r^3 - 3r^2 - 2r + 2 = 0.$$

Theorem 2.6 *Let the function $f \in \mathcal{A}$ and suppose that $g \in S_c^*$. If $f(z) \ll g(z)$ $(z \in \mathbb{U})$, then, for $|z| \le r_6$, we have $|f'(z)| \le |g'(z)|$, where r_6 is the smallest positive root of the equation:*

$$2r^4 - 4r^3 - 5r^2 - 2r + 3 = 0.$$

Theorem 2.7 *Let the function $f \in \mathcal{A}$ and suppose that $g \in S_q^*$. If $f(z) \ll g(z)$ $(z \in \mathbb{U})$, then, for $|z| \le r_7$, we have $|f'(z)| \le |g'(z)|$, where r_7 is the smallest positive root of the equation:*

$$2r^6 - r^4 + 4r^2 - 1 = 0.$$

Theorem 2.8 *Let the function $f \in \mathcal{A}$ and suppose that $g \in S_R^*$. If $f(z) \ll g(z)$ $(z \in \mathbb{U})$, then, for $|z| \le r_8$, we have $|f'(z)| \le |g'(z)|$, where r_8 is the smallest positive root of the equation:*

$$r^4 + 2(\sqrt{2} + 1)r^3 - 3r^2 - 2(3\sqrt{2} + 2)r + 3 + 2\sqrt{2} = 0.$$

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