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On Scattering of Poles for Schrödinger Operator from the Point of View of Dirichlet Series

Xiangdong YANG*, **Bijun ZENG**
Department of Mathematics, Kunming University of Science and Technology,
Yunnan 650093*, P. R. China*

Abstract In this article, we are concerned with the scattering problem of Schrödinger operators with compactly supported potentials on the real line. We aim at combining the theory of Dirichlet series with scattering theory. New estimate on the number of poles is obtained under the situation that the growth of power series which is related to the potential is not too fast by using a classical result of Littlewood. We propose a new approach of Dirichlet series such that significant upper bounds and lower bounds on the number of poles are obtained. The results obtained in this paper improve and extend some related conclusions on this topic.

Keywords Resonances; Schrödinger operators; Dirichlet series

MR(2010) Subject Classification 34L25; 30B50

1. Introduction

We study the distribution of poles for the Schrödinger operator $H_q = -\frac{d^2}{dx^2} + q$ with a compactly supported potential q. Our work in this direction aims at combining the theory of Dirichlet series with scattering theory, extending the classical work by Zworski [1] where it is shown that the density of scattering poles is proportional to the length of the convex hull of the support of the potential.

Since the classical work by Zworski [1], a considerable amount of rigorous results on Schrödinger operators have been achieved by many researchers with the complex-analytic method. We refer to [1–9] for the general background in the field. Motivated by the complex-analytic method in [10, 11], we believe that combining theory on zeros of Dirichlet series with the theory of Schrödinger equations may yield more interesting results on the scattering of poles. The theory of Dirichlet series is one of the most fundamental pure mathematical branches, which has been comprehensively developed and has got a great amount of fruitful results. As far as we know, there are few applications of Dirichlet series in the field of scattering and differential equations. In this paper, we aim at combing Dirichlet series with scattering theory of differential equations. We hope that it may be a worthwhile endeavor which applies Dirichlet series to the filed of Schrödinger operators from real world.

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* Corresponding author
E-mail address: yangsddp@126.com (Xiangdong YANG); 1428907817@qq.com (Bijun ZENG)

In [1], it is shown that in one dimensional case

$$N(r) = 2 \times (\text{length of support of the potential}) \times \frac{r}{\pi} + o(r),$$

where $N(r)$ denotes the number of poles in a ball of radius r. In this paper, we are setting out to prove analogous statements. In particular, we want to treat compact potentials with the approach of Dirichlet series. The starting point in this endeavour is the representation of poles as the zeros of an entire function [1], then we relate the poles to zeros of Dirichlet series in a strip by conformal transformation. With some restrictions on the growth of Dirichlet series relating to poles, using a classical result of Littlewood, we prove our main results in the present paper (Theorem 3.1) which provides a new upper bound for the scattering of poles: denote by $N(u, \pi)$ the number of poles of (2.1) inside the rectangle with vertices, $-u$, u, $-u + i\pi$, $u + i\pi$, then

$$\int_{-u}^{u} N(t, \pi)\mathrm{d}t \leq A|u|,$$

where A is a positive constant depending on u. Similar lower and upper bounds for scattering of poles are obtained by application of existing results on the zeros of Dirichlet series. It is interesting to note that in last section, we prove the poles may concentrate on Julia lines.

This paper is divided into 4 sections. Section 2 is devoted to the background knowledge on scattering and Dirichlet series. We recall the preliminary knowledge on scattering. A theorem of Littlewood which is frequently used in the study of Riemann-Zeta functions is recalled for later use. We also recall some existing results on the zeros of Dirichlet series and a theorem on the value distribution of general Dirichlet series. In Section 3, we will prove the main result of this paper. We will also prove some results on lower and upper bound for distribution of poles. Finally, we prove that the poles may concentrate on Julia lines.

2. Preliminaries

At the beginning of this section, we gather some results about scattering from [1]. We shall only consider potentials q which are compactly supported.

Throughout this paper, we use the letter A to denote positive constants and it may be different at each occurrence. And we denote the complex plane by $\mathbb{C} = \{z = x + iy\}$, the lower half-plane by $\mathbb{C}_- = \{z = x + iy : y < 0\}$, the upper half-plane $\mathbb{C}_+ = \{z = x + iy : y > 0\}$, respectively. The symbol $D(a, t)$ is used to denote the disk $|z - a| < t$.

Following [1] and [6] consider the Schrödinger operator

$$H_q h = (-\frac{d^2}{dx^2} + q)h, \tag{2.1}$$

where $q(x)$ is a real-valued measurable potential which is compactly supported. Let the convex hull of $\mathrm{supp}(q)$ be the interval $[a, b]$. There exists a unique operator B_+ with kernel $B_+(x, y)$ such that

$$H_q B_+ = B_+ H_0, \tag{2.2}$$

$$B_+(x, y) = \delta(x - y) \text{ for } x > b.$$

Eq. (2.2) is equivalent to the wave equation

$$B_{xx}(x,y) - B_{yy}(x,y) - q(x)B(x,y) = 0.$$

The kernel B_+ can be decomposed as

$$\delta(x-y) + R_q(x,y),$$

where R_q satisfies

$$R_q(x,y) = 0 \quad \text{for } x > y,$$

$$R_q(x,y) = R_0(x,y) + (L_q R_q)(x,y),$$

$$R_0(x,y) = \Big(\frac{1}{2}\int_{(x+y)/2}^{b} q(s)\mathrm{d}s\Big)\theta_+(y-x),$$

$$L_q T(x,y) = \iint E(x-s, y-t)q(s)T(s,t)\mathrm{d}s\mathrm{d}t,$$

$$E(x,y) = \begin{cases} \frac{1}{2}, & x < 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\theta_+(x) = 1 \text{ if } x \geq 0 \text{ and } 0 \text{ otherwise.}$$

The solution R can be written as

$$R_q(x,y) = \Big(\sum_{k=0}^{\infty} L_q^k\Big) R_0(x,y).$$

Denote by the Fourier transform of B_+ with respect to the second variable by φ_+,

$$\varphi_+(x,\zeta) = \int e^{-iy\zeta} B_+(x,y)\mathrm{d}y$$

then

$$-\varphi_+''(x) + q\varphi_+ = \zeta^2\varphi_+,$$

$$\varphi_+(x,\zeta) = e^{-ix\zeta} \text{ for } x > b.$$

Similarly, B_- and φ_- are defined so that

$$B_-(x,y) = \delta(x-y),$$

$$\varphi_-(x,\zeta) = e^{-ix\zeta} \text{ for } x < a.$$

From [1] and [6], we know the existence of a distribution X and a function X such that

$$-i\zeta\varphi_+(x,\zeta) = \hat{X}(-\zeta)\varphi_-(x,\zeta) + \hat{Y}(\zeta)\varphi_-(x,-\zeta)$$

$$i\zeta\varphi_-(x,\zeta) = \hat{X}(\zeta)\varphi_+(x,\zeta) + \hat{Y}(\zeta)\varphi_+(x,-\zeta),$$

where

$$X(y) = \delta'(y) - \Big(\frac{1}{2}\int q(s)\mathrm{d}s\Big)\delta(y) - \frac{1}{2}\int q(s)R_q(s,s-y)\mathrm{d}s$$

$$X(y) = \frac{1}{4}q(\frac{y}{2}) + \frac{1}{2}\int q(s)R_q(s, s-y)\mathrm{d}s.$$

Thus, $\hat{X}(\zeta)$ and $\hat{Y}(\zeta)$ are entire functions. Since

$$(\hat{X}(\zeta))^- = \hat{X}(-\zeta) \text{ and } (\hat{Y}(\zeta))^- = \hat{Y}(-\zeta) \text{ for } \zeta \in \mathbf{R},$$

the unitary relation

$$|\hat{X}(\zeta)|^2 = \zeta^2 + |\hat{Y}(\zeta)|^2 \text{ for } \zeta \in \mathbf{R}$$

extends to the whole plane as

$$\hat{X}(\zeta)\hat{X}(-\zeta) = \zeta^2 + \hat{Y}(\zeta)\hat{Y}(-\zeta) \text{ for } \zeta \in \mathbf{C}.$$

From [1], we know that

$$\mathrm{supp}(X) \subset [-2(b-a), 0]$$

and

$$\mathrm{supp}(Y) \subset [2a, 2b].$$

Denote by S_q the scattering matrix of q which is as follows

$$S_q = \begin{pmatrix} \frac{i\zeta}{\hat{X}(\zeta)} & \frac{\hat{Y}(\zeta)}{\hat{X}(\zeta)} \\ \frac{\hat{Y}(-\zeta)}{\hat{X}(\zeta)} & \frac{i\zeta}{\hat{X}(\zeta)} \end{pmatrix}.$$

The scattering poles of S_q are the points $\zeta \in \mathbf{C}\backslash\{0\}$ which satisfies $\hat{X}(\zeta) = 0$. It follows from [1] that the number of poles in $\mathbf{C}\backslash\mathbf{C}_-$ is finite. Thus, we will be concentrated on the poles ζ for which $\Im\zeta < 0$. For the scattering poles, we have the following:

Lemma 2.1 *Suppose that $q \in L^1$, then*

$$\hat{X}(it) = -t - \frac{1}{2}\int q(s)\mathrm{d}s - \frac{1}{2}\int g(s)e^{ts}\mathrm{d}s,$$

where

$$g(s) = \int q(t)R_q(t, t-s)\mathrm{d}t = \int_0^{(2b+t)/2} q(t)R_q(t, t-s)\mathrm{d}t.$$

Proof See [1, p.284]. □

Let us introduce the classical Dirichlet series, that is, the series of the form

$$\sum_{n=1}^{\infty} a_n n^{-z}, \quad a_n, z \in \mathbb{C}, \tag{2.3}$$

which has been attracting attentions for a long history. This interest stems mainly from the fact that such series is closely related to the Riemann zeta function (i.e., $\sum_{n=1}^{\infty} a_n n^{-z}$, $a_n, z \in \mathbb{C}$) which is the core of the grand Riemann Hypothesis.

Recently, the mathematicians are concerned with the general Dirichlet series with real frequencies of the form

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \quad a_n, s = u + iv \in \mathbb{C}, \ 0 < \lambda_n \uparrow \infty. \tag{2.4}$$

This is a "direct" generalization of the classical ones, as in the case when $\lambda_n = \log n$, we get (2.3).

Denote by σ_c, σ_a, and σ_u the abscissa of convergence, absolute convergence, and uniform convergence of series (2.4), respectively. Then $\sigma_c \le \sigma_u \le \sigma_a$. It is well-known that between these abscissae, coefficients (a_n) and frequencies (λ_n), there are some interesting relations. In particular, if denote

$$L = \limsup_{n\to\infty} \frac{\log n}{\lambda_n}, \tag{2.5}$$

then the following inequalities always hold:

$$0 \le \sigma_a - \sigma_c \le L; \quad \limsup_{n\to\infty} \frac{\log |a_n|}{\lambda_n} \le \sigma_c.$$

We now turn to the zeros on Dirichlet series. The following results on zeros of general Dirichlet series are due to Little [12].

Lemma 2.2 *Let $f(s)$ be a general Dirichlet series defined in (2.4) which can be continued analytically in $(u \ge \frac{1}{2} - \delta_0, T - \log T \le v \le 2T + \log T)$, where δ_0 is a positive constant, and $\log(\max |f(s)| + 100) \ll \log T$. Let $f(s) \to 1$ as $\Re s \to \infty$. For $\sigma \ge 1/2$, let $N(\sigma, T)$ denote the number of zeros of $f(s)$ in $(u \ge \sigma, T \le v \le 2T)$. Then (for $\sigma \ge 1/2$), we have*

$$2\pi \int_\sigma^\infty N(t,T)\mathrm{d}t = \int_T^{2T} \log|f(\sigma + iv)|\mathrm{d}v + O(\log T).$$

Hence

$$N(1/2 + 2\delta, T) \le (A\delta)^{-1} T \log\Big(\frac{1}{T}\int_T^{2T} |f(1/2 + \delta + iv)|^A \mathrm{d}t\Big) + O(\log T)$$

holds uniformly for all positive constants A and δ.

Lemma 2.3 *Let $\sum_{n\le\sigma} |a_n| \le B(\sigma)$, $|\sum_{\sigma\le m}\sum_{n\le m} a_n|^2 \le \sigma B(\sigma)$, and $\sum_{n\le\sigma} |a_n|^2 \le \sigma B(\sigma)$, where $B(\sigma)$ depends on σ. If $B(\sigma) \ll_\varepsilon \sigma^\varepsilon$ for every $\varepsilon > 0$, then the general Dirichlet series defined in (2.4) converges uniformly over compact subsets of $\sigma > 0$, $\delta > 0$ and hence is analytic there. We have*

$$N(1/2 + \delta, T) \ll_\delta T.$$

If further $\log B(\sigma) \ll \log\log\sigma$, then we have

$$N(1/2 + \delta, T) \ll \delta^{-1} T \log \delta^{-1}$$

uniformly for $0 < \delta \le 1/2$.

We need some background knowledge on the lower bound of zeros for Dirichlet series from [13].

Definition 2.4 *Let $f(s)$ be a general Dirichlet series defined in (2.4) and $\sigma + i\tau$ be real numbers with $|f(\sigma + i\tau)| \gg \tau^\lambda$ (where λ is a positive constant independent of τ) for a set of points $\sigma + i\tau$. Then we call $\sigma + i\tau$ a Titchmarsh point and the rectangle $(u \ge \sigma - \delta, |v - \tau| \le \varepsilon)$ (where δ and ε are arbitrary small positive constants) the associated rectangle.*

Lemma 2.5 *Let $f(s)(s = u + iv)$ be a general Dirichlet series defined in (2.4). Let $\tau \ge \tau_0(\varepsilon, \delta)$*

and $\sigma_0 + i\tau(\sigma_0 = \sigma_0(\tau))$ be a Titchmarsh point for $f(s)$. Then the associated rectangle contains $\gg \log T$ zeros provided $f(s)$ is continuable analytically in the associated rectangle and there $\max|f(s)|$ is less than v^A for some constant $A > 0$. Next if $\sigma_0(\tau)$ is bounded below by a, which is a constant independent of τ and $f(s)$ can be continued analytically in $(\sigma \geq a-\delta, T \leq v \leq 2T)$, where δ_0 is a positive constant and there $\max|f(\sigma+i\tau)| \gg T^A$ and further if there are $\gg T$ well-spaced Titchmarsh points in $(\sigma \geq a-\delta, T \leq v \leq 2T)$, then $f(s)$ has $\gg T\log T$ zeros in $(\sigma \geq a-\delta, T \leq v \leq 2T)$.

We need the result on value distribution of general Dirichlet series from [14].

Lemma 2.6 *If $\{\lambda_n\}$ satisfies (2.5) and the following conditions*

$$\delta = \liminf_{n\to\infty}(\lambda_{n+1}-\lambda_n) > 0, \tag{2.6}$$

$$\overline{\lim_{n\to\infty}}\frac{\ln|a_n|}{\lambda_n} = -\infty, \tag{2.7}$$

where $\{a_n\}$ is a sequence of complex numbers, then $f(s)$ defined in (2.4) is an entire function, and for any strip of width $2\pi\delta$

$$S = \{s : |\Im s - v_0| \leq \pi\delta\},$$

$f(s)$ has one of the following properties:

(1) For $s \in S$, $\lim_{s\to-\infty}\frac{\log|f(s)|}{-u} = +\infty$.

(2) The Dirichlet series $f(s)$ has got a Julia line $\mathrm{Im} = v_1(|v_1 - v_0| \leq \pi\delta)$, i.e., $\forall a \in \mathbb{C}$, for at most one exception, $\forall\varepsilon > 0$

$$\lim_{u\to-\infty} N(u, f = a, |\Im s - v_1| < \varepsilon) = +\infty,$$

where $N(u, f = a, |\Im s - v_1| < \varepsilon)$ denotes the number of points $\{s : \mathrm{Re}s \geq u_0, |\Im s - v_1| < \varepsilon\}$ for which the function f has value a.

Let us recall a theorem of Littlewood which is frequently used to estimate the number of zeros of Dirichlet series [15].

Lemma 2.7 *Let $f(s)$ be regular in and upon the boundary of the rectangle M with vertices, $-a$, a, $-a-iT$, $a-iT$, and not zero on $\sigma = a$. Denote by $\nu(\sigma, T)$ the number of zeros $\varrho = \beta + i\gamma$ of $f(s)$ inside the rectangle with $\beta > \sigma$ including those with $\gamma = T$ but not $\gamma = 0$. Then*

$$\int_M \log f(s)\mathrm{d}s = -2\pi i\int_{-a}^{a}\nu(\sigma, T)\mathrm{d}\sigma.$$

3. Results on scattering of poles

We will deal with the scattering of poles with an approach of Dirichlet series [15, Theorem 1]. The main technique is application of Littlewood's theorem which is Lemma 2.7. The main result of this paper is as follows.

Theorem 3.1 *Suppose that $q \in L^1$, denote by*

$$c_0 = \frac{1}{2}\int q(s)\mathrm{d}s - \frac{1}{2}\int g(s)\mathrm{d}s, \tag{3.1}$$

$$c_1 = -i\Big(1 + \frac{1}{2}\int g(s)s\mathrm{d}s\Big), \tag{3.2}$$

and

$$c_n = -\frac{(-i)^n}{2}\int g(s)s^n\mathrm{d}s, \quad n \geq 2, \tag{3.3}$$

where g is defined in Lemma 2.1. If $\{c_n\} \in L^1$ and $\sum c_n \neq 0$, denote by $N(u,\pi)$ the number of poles of (2.1) inside the rectangle with vertices, $-u$, u, $-u+i\pi$, $u+i\pi$, then

$$\int_{-u}^{u} N(t,\pi)\mathrm{d}t \leq A|u|,$$

where A is some positive constant depending on u.

Proof The scattering of poles is reduced to the zeros of the entire function $\hat{X}(z)$. By Lemma 2.1, we have the expansion

$$\hat{X}(z) = \sum_{n=0}^{\infty}\frac{c_n}{n!}z^n. \tag{3.4}$$

Thus, the poles of infinite number are zeros of $\hat{X}(z)$ in $\mathbf{C}_-$. If we take the conformal transformation $-s = -u - iv = \log z$, then (3.4) can be rewritten as

$$\hat{X}(e^{-s}) = \sum_{n=0}^{\infty}\frac{c_n}{n!}e^{-sn}. \tag{3.5}$$

The scattering of poles is reduced to the zeros of (3.5) in the strip $G = \{s = u + iv,\ 0 < v < \pi\}$. Denote by M the rectangle with vertices, $-u$, u, $-u+i\pi$, $u+i\pi$, furthermore, denote by $N(u,\pi)$ the number of zeros $\varrho = \beta + i\gamma$ of $\hat{X}(e^{-s})$ inside the rectangle. Then Littlewood's theorem, i.e., Lemma 2.7 yields

$$\int_M \log \hat{X}(e^{-s})\mathrm{d}s = -2\pi i\int_{-u}^{u} N(t,\pi)\mathrm{d}t.$$

Thus, we can conclude

$$\begin{aligned}
-2\pi i\int_{-u}^{u} N(t,\pi)\mathrm{d}t = &\int_0^{\pi}\log|\hat{X}(e^{-u+iv})|\mathrm{d}v + \int_{-u}^{u}\log|\hat{X}(e^{t+i\pi})|\mathrm{d}t + \\
&\int_{\pi}^{0}\log|\hat{X}(e^{u+iv})|\mathrm{d}v + \int_{u}^{-u}\log|\hat{X}(e^{t})|\mathrm{d}t + \\
&\int_0^{\pi}\arg\hat{X}(e^{-u+iv})\mathrm{d}v + \int_{\pi}^{0}\arg\hat{X}(e^{u+iv})\mathrm{d}v \\
= &\sum_{j=1}^{6} I_j.
\end{aligned}$$

To define $\log\hat{X}(e^{-s})$, we choose the principal branch of the logarithm on the real axis as $\sigma \to \infty$; for other points s the value of the logarithm is obtained by analytic continuation.

Recall that

$$\hat{X}(e^{-u+iv}) = \sum_{n=0}^{\infty} \frac{c_n}{n!} e^{n(-u+iv)}.$$

For fixed $u \in \mathbf{R}$, there exists some constant $k(u)$ which depends on u such that

$$\frac{e^{nu}}{n!} \leq 1, \quad n \geq k(u). \tag{3.6}$$

Denote by $\log^+ a = \max\{\log a, 0\}$. By (3.6),

$$\begin{aligned}
\log|\hat{X}(e^{-u+iv})| &\leq \log^+\Big(\sum_{n=0}^{k(u)} |c_n|\frac{e^{nu}}{n!}\Big) + \log^+\Big(\sum_{n=k(u)+1}^{\infty} |c_n|\frac{e^{nu}}{n!}\Big) + \log 2 \\
&\leq \sum_{n=0}^{k(u)} \log^+(|c_n|) + \log^+(\frac{e^{nu}}{n!}) + \log^+(\sum_{n=k(u)+1}^{\infty} |c_n|) + \log k(u) + \log 2.
\end{aligned}$$

Since $\{c_n\} \in L^1$, we have

$$\log|\hat{X}(e^{-u+iv})| \leq A|u|, \tag{3.7}$$

where A is some constant depending on u. Thus, we can conclude

$$I_1 \leq A|u| + A. \tag{3.8}$$

By the same reasoning, (3.8) holds for I_2, I_3, I_4.

We will estimate the integrals I_5, I_6. If $\Re\hat{X}(e^{-u+iv})$ has m zeros for $0 \leq u \leq \pi$, we may divide the interval $[0, \pi]$ into at most $m+1$ subintervals in each of which $\Re\hat{X}(e^{-u+iv})$ has constant sign. Then

$$|\arg \hat{X}(e^{-u+iv})| \leq (m+1)\pi. \tag{3.9}$$

Choose $R = 2|u|$ and without loss of generality, we may suppose that $|u|$ is bigger than π. Denote by $n(r)$ the number of zeros of $\hat{X}(e^{-u+iv})$ in $|s| \leq R$. Applying Jesen's formula to $\hat{X}(e^{-u+iv})$ yields

$$\int_0^{2R} \frac{n(r)}{r} \mathrm{d}r = \frac{1}{2\pi}\int_0^{2\pi} \log|\hat{X}(e^{2Re^{i\theta}})|\mathrm{d}\theta - \log|\hat{X}(1)|.$$

Since $\sum c_n \neq 0$ which is $\hat{X}(1) \neq 0$, we deduce

$$n(R) \leq \frac{1}{2\pi \log 2}\int_0^{2\pi} \log|\hat{X}(e^{2Re^{i\theta}})|\mathrm{d}\theta - \frac{\log|\hat{X}(1)|}{\log 2}.$$

By (3.7), we can conclude $n(R) \leq AR$. Since the interval $(-u, u)$ is contained in the disk $|s| \leq R$, we have $m \leq n(R)$. Thus, we have

$$I_5 \leq A|u| + A. \tag{3.10}$$

The same reasoning can apply to I_6 and the estimate is the same as (3.10). $\square$

Let us apply the zeros of Dirichlet series to get more interesting results on the upper or lower bounds on the distribution of poles.

Theorem 3.2 *Suppose that $q \in L^1$, and $\{c_n\}$ are defined in (3.1), (3.2) and (3.3). If $\{\frac{c_n}{n!}\} \in L^1$*

and $\hat{X}(e^{-s}) \to a_0(\neq 0)$ as $\Re s \to \infty$, denote by $N(\sigma, T)$ the poles of (2.1) in $(u \geq \sigma, T \leq v \leq 2T)$ $(0 < T \leq \frac{\pi}{2})$. Then, for $\sigma \geq 1/2$, we have

$$N(1/2 + 2\delta, T) \ll A_1 \delta^{-1} T + O(\log T)$$

holds uniformly for some positive constants A_1 and δ.

Proof Recall the definition of $\hat{X}(e^{-s})$,

$$\hat{X}(e^{-s}) = \sum_{n=0}^{\infty} \frac{c_n}{n!} e^{-sn}.$$

It is just a general Dirichlet series defined in (2.4) as long as $\lambda_n = n$. Thus, scattering of poles has now been reduced to the zeros of a general Dirichlet series.

Since $\{\frac{c_n}{n!}\} \in L^1$, for $(u \geq 1/2 + 2\delta, T \leq v \leq 2T)$ $(0 < T \leq \frac{\pi}{2})$, we have

$$|\hat{X}(e^{-s})| \leq \sum_{n=0}^{\infty} \frac{|c_n|}{n!}.$$

We also have

$$\limsup \frac{\log \frac{c_n}{n!}}{n} \leq 0 \leq \sigma_c,$$

which verifies that Dirichlet series $\hat{X}(e^{-s})$ can be continued analytically in $(u \geq \frac{1}{2} - \delta_0, T - \log T \leq v \leq 2T + \log T)$. Thus, without loss of generality,

$$\log(\max |\hat{X}(e^{-s})| + 100) \ll T,$$

otherwise, we may consider $A\hat{X}(e^{-s})$ where A is some constant. Since $\hat{X}(e^{-s}) \to a_0(\neq 0)$ as $\Re s \to \infty$, without loss of generality, we may suppose that $\hat{X}(e^{-s}) \to 1$ as $\Re s \to \infty$. The conclusion follows by applying Lemma 2.2. $\square$

Theorem 3.3 *Suppose that $q \in L^1$, and $\{c_n\}$ are defined in (3.1), (3.2) and (3.3). If*

$$\sum_{n \leq \sigma} \frac{|c_n|}{n!} \leq B(\sigma), \quad \sum_{n \leq \sigma} (\frac{|c_n|}{n!})^2 \leq \sigma B(\sigma), \quad \sum_{\sigma \leq m} |\sum_{n \leq m} \frac{c_n}{n!}|^2 \leq \sigma B(\sigma),$$

where $B(\sigma)$ depends on σ. If $B(\sigma) \ll_\varepsilon \sigma^\varepsilon$ for every $\varepsilon > 0$, denote by $N(\sigma, T)$ the poles of (2.1) in $(u \geq \sigma, T \leq v \leq 2T)$ $(0 < T \leq \frac{\pi}{2})$. Then, for $\sigma \geq 1/2$, we have

$$N(1/2 + \delta, T) \ll_\delta T.$$

If further $\log B(\sigma) \ll \log \log \sigma$, then we have

$$N(1/2 + \delta, T) \ll \delta^{-1} T \log \delta^{-1}$$

uniformly for $0 < \delta \leq 1/2$.

Proof The conclusion follows by applying Lemma 2.3 and the proof of Theorem 3.2. $\square$

Theorem 3.4 *Suppose that $q \in L^1$, and $\{c_n\}$ are defined in (3.1), (3.2) and (3.3). If $\tau \geq \tau_0(\varepsilon, \delta)$ $(\sigma_0 + i\tau(\sigma_0 = \sigma_0(\tau))$ is a Titchmarsh point for $\hat{X}(e^{-s})$, denote by $N(\sigma, T)$ the number of poles* *in the associated rectangle, if $\hat{X}(e^{-s})$ is continuable analytically in the associated rectangle then*

$$N(\sigma, T) \gg \log T.$$

Further, if $\sigma_0(\tau)$ are bounded below, $\max|\hat{X}(e^{-s})| \gg T^A$, moreover, if there are $\gg T$ well-spaced Titchmarsh points in $(\sigma \geq a-\delta, T \leq v \leq 2T)$, then we have

$$N(\sigma, T) \gg T \log T$$

in $(\sigma \geq a-\delta, T \leq v \leq 2T)$.

Proof The conclusion follows by applying Lemma 2.5 and the proof of Theorem 3.2. □

The next result will show that there may exist Julia lines for the scattering of poles.

Theorem 3.5 *Suppose that $q \in L^1$, and $\{c_n\}$ are defined in (3.1), (3.2) and (3.3). Denote by $N(u, |\Im s - v_1| < \varepsilon)$ the poles of (2.1) in the strip $\{s : |\Im s - v_1| < \varepsilon\}$. If*

$$\overline{\lim_{n\to\infty}} \frac{\ln|c_n|}{n!n} = -\infty, \tag{3.11}$$

and for $s \in S_1$. If

$$\lim_{s\to-\infty} \frac{\log|\hat{X}(e^{-s})}{-u} < +\infty, \tag{3.12}$$

where

$$S_1 = \{s : |\Im s - v_0| \leq \pi\},$$

then there exist v_1 and u_0 such that $\hat{X}(e^{-s}) \neq 0$ on $H = \{s : \mathrm{Re}s \geq u_0, |\Im s - v_1| < \varepsilon)\}$, otherwise, $\forall\varepsilon > 0$,

$$\lim_{u\to-\infty} N(u, |\Im s - v_1| < \varepsilon) = +\infty,$$

for $s \in H$.

Proof Recall the definition of $\hat{X}(e^{-s})$,

$$\hat{X}(e^{-s}) = \sum_{n=0}^{\infty} \frac{c_n}{n!} e^{-sn},$$

it is a Dirichlet series defined in (2.4) with frequencies $\lambda_n = n$, which verifies the conditions (2.5), (2.6). It is obvious that (3.11) is just (2.7).

Since (3.12) holds, the conclusion follows from Lemma 2.6. □

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