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Fujita-type Phenomenon of the Nonlocal Diffusion Equations with Localized Source

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Abstract In this paper, we investigate the Cauchy problem for the nonlocal diffusion system with localized source $u_t = J * u - u + a(x)v^p$, $v_t = J * v - v + a(x)u^q$. We first prove that the Fujita curve is $(pq)_c = 1 + \max\{p+1, q+1\}$ based on whether there exist global solutions, that is, if $1 < pq \le (pq)_c$, then every nonnegative solution blows up in finite time, but for $pq > (pq)_c$, there exist both global and non-global solutions to the problem. Furthermore, we establish the secondary critical curve on the space-decay of initial value at infinity.

Keywords nonlocal diffusion system; Fujita critical curve; secondary critical curve; global existence; blow-up

MR(2010) Subject Classification 35K70; 35B05; 35B40

1. Introduction

In this paper, we consider the Cauchy problem for a nonlocal diffusion system

$$\begin{cases} u_t = J * u - u + a(x)v^p, & x \in \mathbb{R},\ t > 0, \\ v_t = J * v - v + a(x)u^q, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $J \in C_c(\mathbb{R})$ is nonnegative, radially symmetric and decreasing, with unit integral, $*$ stands for usual convolution in $\mathbb{R}$, and $a(x)$ is a nonnegative continuous function with compact support containing the origin, and $u_0,\ v_0 \in L^r(\mathbb{R}) \bigcap L^\infty(\mathbb{R})$ are radially symmetric and decreasing in $(0,\infty)$ with $r > 1$. As we all know, the nonlocal diffusion system (1.1) can be used to describe a variety of nonlocal diffusion processes for generation populations [1–4], deblurring-denoising of images [5], etc.

For nonlinear nonlocal diffusion systems, García-Melián and Quirós [6] proved that the following problem

$$\begin{cases} u_t = J * u - u + u^p, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N \end{cases} \tag{1.2}$$

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has the same critical exponent $p_c = 1 + \frac{2}{N}$ as the classical nonlinear heat equation

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.3}$$

Namely, (i) If $1 < p \le p_c$, the solution of (1.2) blows up in finite time for every nonnegative and nontrivial initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, (ii) But for $p > p_c$, there exist both global and blow-up solutions depending on the size of the initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. The study of the critical exponent p_c originated from Fujita [7]. Moreover, to describe the critical space-decay rate of initial data in the co-existence parameter region of global and blow-up solutions, Lee and Ni [8] introduced the second critical exponent for the system (1.3). That is, assuming $u_0(x) \sim |x|^{-a},\ |x| \to \infty$ in the case $p > p_c$, there exist both global and blow-up solutions of (1.3) for $a > a_0 = \frac{2}{p-1}$ and $0 < a < a_0$.

For the coupled heat system

$$\begin{cases} u_t = \Delta u + v^p, \quad v_t = \Delta v + u^q, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

Escobedo and Herrero [9] achieved the critical Fujita curve as $(pq)_c = 1 + \frac{2}{N}\max\{p+1, q+1\}$, namely, if $1 < pq \le (pq)_c$, every solution blows up in finite time; but for $pq > (pq)_c$, there exist both global and blow-up solutions. Letting $u_0(x) \sim |x|^{-a}$, $v_0(x) \sim |x|^{-b}$, $|x| \to \infty$, Mochizuki [10] proved in the coexistence region $pq > (pq)_c$ that there exist global solutions if $a > a_0 = \frac{2(p+1)}{pq-1}$ and $b > b_0 = \frac{2(q+1)}{pq-1}$, while every solution of (1.4) blows up in finite time when $0 < a < a_0$ or $0 < b < b_0$.

Concerning the coupled nonlocal diffusion system

$$\begin{cases} u_t = J * u - u + v^p, & x \in \mathbb{R}^N,\ t > 0, \\ v_t = J * v - v + u^q, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.5}$$

where $p,\ q > 1$, u_0 and v_0 are nonnegative and bounded. Yang [11] has achieved that the critical Fujita curve of (1.5) is $(pq)_c = 1 + \frac{2}{N}\max\{p+1, q+1\}$. That is, every nonnegative nontrivial solution of the system (1.5) blows up in finite time if $1 < pq \le (pq)_c$, but for $pq > (pq)_c$, there exist both global and blow-up solutions depending on the size of the initial data. Letting $u_0(x) = \lambda\psi(x)$, $v_0(x) = \mu\varphi(x)$, $\lambda, \tau > 0$, assuming $pq > (pq)_c$, if $a > a_0 = \frac{2(p+1)}{pq-1}$ and $b > b_0 = \frac{2(q+1)}{pq-1}$ with $\limsup_{|x|\to\infty} |x|^b\psi(x) < \infty$ and $\limsup_{|x|\to\infty} |x|^b\varphi(x) < \infty$, there exist both global and non-global solutions. While any solution of (1.5) blows up in finite time for $0 < a < a_0$ or $0 < b < b_0$.

For the single nonlocal diffusion equation

$$\begin{cases} u_t = J * u - u + a(x)u^p, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.6}$$

where $a(x)$ is a nonnegative continuous function with compact support containing the origin. Yang and Zhou [12] deduced that the critical Fujita exponent of (1.6) is $p_c = 2$ if $N = 1$, and the critical exponent is $p_c = 1$ if $N > 1$. That is, there is no nontrivial global solution if $1 < p \le p_c$;

but for $p > p_c$, there exist oth global and blow-up solutions depending on the size of the initial data. Letting $N = 1$, $u_0(x) = \lambda\varphi(x)$, $\lambda > 0$. Assuming $p > p_c = 2$, if $0 < b < b_0 = \frac{1}{p-1}$ with $\liminf_{|x|\to\infty} |x|^b\psi(x) > 0$, then any solution of (1.6) blows up in finite time, but for $b_0 < b < 1$ with $\limsup_{|x|\to\infty} |x|^b\psi(x) < \infty$, there exist both global and non-global solutions relying on the size of the initial data. Letting $N > 1$, $u_0 = \lambda\varphi(x)$, $\lambda > 0$ and $\limsup_{|x|\to\infty} |x|^b\psi(x) < \infty$, then the solution of (1.6) is global with λ small enough.

Motivated by these works, we will deal with the Cauchy problem (1.1) and study its Fujita curve. Now we state the main results in this paper.

Theorem 1.1 *The critical Fujita curve of the system (1.1) is*

$$(pq)_c = 1 + \max\{p+1, q+1\},$$

that is, the system (1.1) has no nontrivial global solutions if $1 < pq \le (pq)_c$; *but for* $pq > (pq)_c$, *there exist global solutions for small initial data.*

To study the secondary critical curve, we need the following notation:

$$\mathbb{I}_b = \{\phi(x) \in C_b(\mathbb{R}) \mid \phi(x) \ge 0,\ \liminf_{|x|\to\infty} |x|^b\phi(x) > 0\},$$

$$\mathbb{I}^b = \{\phi(x) \in C_b(\mathbb{R}) \mid \phi(x) \ge 0,\ \limsup_{|x|\to\infty} |x|^b\phi(x) < \infty\},$$

where $C_b(\mathbb{R})$ denotes the whole bounded continuous functions in $\mathbb{R}$. Hence we have the following results.

Theorem 1.2 *Assume* (u, v) *is a solution to system (1.1),* $pq > (pq)_c = 1 + \max\{p+1, q+1\}$, $u_0(x) = \lambda\psi(x)$, $v_0(x) = \tau\varphi(x)$ *with* $\lambda, \tau > 0$, *and* $a_0 = \frac{p+1}{pq-1}$, $b_0 = \frac{q+1}{pq-1}$.

(i) If $0 < a < a_0$ *or* $0 < b < b_0$, $\psi \in \mathbb{I}_a$, $\varphi \in \mathbb{I}_b$, *then the solution of (1.1) blows up in finite time.*

(ii) If $a > a_0$ *and* $b > b_0$, $\psi \in \mathbb{I}^a, \varphi \in \mathbb{I}^b$, *then there exists a global solution of (1.1) provided that* λ *and* τ *are small enough.*

Remark 1.3 Theorem 1.2 implied that the secondary critical curve of system (1.1) is $\min\{a - a_0, b - b_0\} = 0$.

In the following, we will prove the Theorem 1.1 in Section 2, while we prove the Theorem 1.2 in Section 3.

2. Critical Fujita curve

In this section, we will deal with the critical Fujita curve to the system (1.1).

Proof of Theorem 1.1 Let ϕ_R be the principal eigenfunction to the problem

$$\begin{cases} J * \phi_R - \phi_R = -\lambda_1(B_R)\phi_R, & x \in B_R = (-R, R), \\ \phi_R = 0, & x \in R \setminus B_R, \end{cases}$$

normalized with $\|\phi_R\|_{L^1(\mathbb{R})}=1$ and $\phi_R>0$ in B_R. Denote

$$\Phi(t)=\int_{-R}^{R}u\phi_R\mathrm{d}x,\quad \Psi(t)=\int_{-R}^{R}v\phi_R\mathrm{d}x.$$

By Fubini's theorem, we have that

$$\begin{aligned}\Phi'(t)&\ge\int_{-R}^{R}\int_{-R}^{R}J(x-y)(\phi_R(y)-\phi_R(x))\mathrm{d}yu(x)\mathrm{d}x+\int_{-R}^{R}av^p\phi_R\mathrm{d}x\\&=-\lambda_1(B_R)\Phi(t)+\int_{-R}^{R}av^p\phi_R\mathrm{d}x.\end{aligned}$$

By the comparison principle, the solution (u,v) is also radially decreasing under the assumption that both $u_0(x)$ and $v_0(x)$ are radially symmetric and decreasing. We assume that $a(x)$ achieve its maximum at $x=0$, then let $\delta\in(0,1)$, we can get that

$$\begin{aligned}\int_{-R}^{R}av^p\phi_R\mathrm{d}x&\ge\frac{1}{2}\int_{-\delta}^{\delta}av^p\phi_R\mathrm{d}x+\frac{a(\delta)}{2}\int_{-\delta}^{\delta}v^p\phi_R\mathrm{d}x\\&\ge\frac{a(\delta)}{2}v^p(\delta,t)\int_{-\delta}^{\delta}\phi_R\mathrm{d}x\Big(\int_{-R}^{-\delta}+\int_{\delta}^{R}\Big)\phi_R\mathrm{d}x+\frac{a(\delta)}{2}\int_{-\delta}^{\delta}v^p\phi_R\mathrm{d}x\\&\ge\frac{a(\delta)}{2}\int_{-\delta}^{\delta}\phi_R\mathrm{d}x\Big(\int_{-R}^{-\delta}+\int_{\delta}^{R}\Big)v^p\phi_R\mathrm{d}x+\frac{a(\delta)}{2}\int_{-\delta}^{\delta}\phi_R\mathrm{d}x\int_{-\delta}^{\delta}v^p\phi_R\mathrm{d}x\\&\ge\frac{a(\delta)}{2}\int_{-\delta}^{\delta}\phi_R\mathrm{d}x\int_{-R}^{R}v^p\phi_R\mathrm{d}x,\end{aligned}\tag{2.1}$$

and hence

$$\Phi'(t)\ge-\lambda_1(B_R)\Phi(t)+\frac{a(\delta)}{2}\int_{-\delta}^{\delta}\phi_R\mathrm{d}x\Psi^p(t).$$

Similarly, we can also derive that

$$\Psi'(t)\ge-\lambda_1(B_R)\Psi(t)+\frac{a(\delta)}{2}\int_{-\delta}^{\delta}\phi_R\mathrm{d}x\Phi^q(t).$$

Consequently, let $\mu=\frac{a(\delta)}{2}\int_{-\delta}^{\delta}\phi_R\mathrm{d}x$. We have

$$\begin{cases}\Phi'(t)\ge-\lambda_1(B_R)\Phi(t)+\mu\Psi^p(t),\\ \Psi'(t)\ge-\lambda_1(B_R)\Psi(t)+\mu\Phi^q(t).\end{cases}$$

We assume that $F(t)=\mu^{\frac{p+1}{pq-1}}\Phi(t),\ G(t)=\mu^{\frac{q+1}{pq-1}}\Psi(t)$, then we can achieve that

$$\begin{cases}F'(t)\ge-\lambda_1(B_R)F(t)+G^p(t),\\ G'(t)\ge-\lambda_1(B_R)G(t)+F^q(t).\end{cases}$$

Without loss of generality, let $p\ge q$. We can get from [13] that the solution of (1.1) blows up in finite time if

$$F(0)>A\lambda_1(B_R)^{\frac{p+1}{pq-1}}\text{ or }G(0)>B\lambda_1(B_R)^{\frac{q+1}{pq-1}},\tag{2.2}$$

where and in the following, both A and B are constants. If $R_0 \in (0, R)$, we can get that

$$\begin{aligned} &\Big(\frac{a(\delta)}{2}\int_{-\delta}^{\delta} R\phi_R \mathrm{d}x\Big)^{\frac{p+1}{pq-1}} \int_{-R}^{R} Ru_0\phi_R \mathrm{d}x \\ &\geq \Big(\frac{a(\delta)}{2}\int_{-\delta}^{\delta} R\phi_R(\frac{Rx}{R_0})\mathrm{d}x\Big)^{\frac{p+1}{pq-1}} \int_{-R}^{R} Ru_0\phi_R(\frac{Rx}{R_0})\mathrm{d}x \end{aligned} \tag{2.3}$$

with ϕ_R is radially decreasing [6]. Let Γ be the principal eigenfunction to the problem

$$\begin{cases} -\Delta\Gamma = \mu_1\Gamma, & x \in B_1 = (-1, 1), \\ \Gamma = 0, & x \in \partial B_1 \end{cases}$$

with $\|\Gamma\|_{B_1} = 1$. By [14, Theorem 1.5] and [15, Proposition 3.2], we have

$$R\phi_R(\frac{Rx}{R_0}) \to \Gamma(\frac{x}{R_0}) \ \text{ in } L^1(B_1) \text{ as } R \to \infty. \tag{2.4}$$

Via Fatou's lemma with (2.3) and (2.4), letting $R \to \infty$ first and then $R_0 \to \infty$, we get

$$\liminf_{R\to\infty} \Big(\frac{a(\delta)}{2}\int_{-\delta}^{\delta} R\phi_R \mathrm{d}x\Big)^{\frac{p+1}{pq-1}} \int_{-R}^{R} Ru_0\phi_R \mathrm{d}x \geq (\frac{\delta a(\delta)}{2})^{\frac{p+1}{pq-1}} \Gamma^{\frac{pq+p}{pq-1}}(0) \int_{-R_1}^{R_1} u_0 \mathrm{d}x \geq C \tag{2.5}$$

for every $R_1 > 0$, where and in the sequel, C represents positive constants independent of R. We also know by [14, Theorem 1.5] that

$$\lim_{R\to\infty} R^2\lambda_1(B_R) = \mu_1, \tag{2.6}$$

and hence

$$\lim_{R\to\infty} (\lambda_1(B_R))^{\frac{p+1}{pq-1}} R^{\frac{pq+p}{pq-1}} = \lim_{R\to\infty} (R^2\lambda_1(B_R))^{\frac{p+1}{pq-1}} R^{\frac{pq-p-2}{pq-1}} = 0$$

provided $pq < (pq)_c = 1 + \max\{p + 1, q + 1\}$. This with (2.5) can get that

$$\Big(\frac{a(\delta)}{2}\int_{-\delta}^{\delta} R\phi_R \mathrm{d}x\Big)^{\frac{p+1}{pq-1}} \int_{-R}^{R} Ru_0\phi_R \mathrm{d}x > (\lambda_1(B_R))^{\frac{p+1}{pq-1}} R^{\frac{pq+p}{pq-1}} \tag{2.7}$$

with R large enough. Namely,

$$\Big(\frac{a(\delta)}{2}\int_{-\delta}^{\delta} \phi_R \mathrm{d}x\Big)^{\frac{p+1}{pq-1}} \Phi(0) > A\lambda_1(B_R)^{\frac{p+1}{pq-1}},$$

we can get that (u, v) blows up in finite time with R large enough.

Next we will concern the critical case $pq = (pq)_c = 1 + \max\{p + 1, q + 1\}$, namely, we will prove that every nontrivial solution of (1.1) must blow-up. Suppose for a contradiction that there exists a nonnegative nontrivial global solution (u, v) to (1.1), then

$$\Big(\frac{a(\delta)}{2}\int_{-\delta}^{\delta} R\phi_R \mathrm{d}x\Big)^{\frac{p+1}{pq-1}} \int_{-R}^{R} Ru(t)\phi_R \mathrm{d}x \leq (\lambda_1(B_R))^{\frac{p+1}{pq-1}} R^{\frac{pq+p}{pq-1}}$$

for any $t > 0$ via (2.7). If $R_0 \in (0, R)$, we can get that

$$\begin{aligned} &\Big(\frac{a(\delta)}{2}\int_{-\delta}^{\delta} R\phi_R \mathrm{d}x\Big)^{\frac{p+1}{pq-1}} \int_{-R}^{R} Ru(t)\phi_R \mathrm{d}x \\ &\geq \Big(\frac{a(\delta)}{2}\int_{-\delta}^{\delta} R\phi_R(\frac{Rx}{R_0})\mathrm{d}x\Big)^{\frac{p+1}{pq-1}} \int_{-R}^{R} Ru(t)\phi_R(\frac{Rx}{R_0})\mathrm{d}x. \end{aligned}$$

Letting $R\to\infty$ first, and then $R_0\to\infty$, by Fatou's lemma and (2.4), we can achieve that

$$\liminf_{R\to\infty}\Big(\frac{a(\delta)}{2}\int_{-\delta}^{\delta}R\phi_R\mathrm{d}x\Big)^{\frac{p+1}{pq-1}}\int_{-R}^{R}Ru(t)\phi_R\mathrm{d}x\ge(\frac{\delta a(\delta)}{2})^{\frac{p+1}{pq-1}}\Gamma^{\frac{pq+p}{pq-1}}(0)\int_{\mathbb{R}}u(t)\mathrm{d}x.$$

Noticing that

$$\lim_{R\to\infty}(\lambda_1(B_R))^{\frac{p+1}{pq-1}}R^{\frac{pq+p}{pq-1}}=\lim_{R\to\infty}(R^2\lambda_1)^{\frac{p+1}{pq-1}}R^{\frac{pq-p-2}{pq-1}}=C>0,$$

we can get

$$\sup_{t>0}\int_{\mathbb{R}}u(\cdot,t)\mathrm{d}x\le C. \tag{2.8}$$

Take $\zeta(x)\in\mathcal{D}(B_1)$ and $\eta(t)\in\mathcal{D}((-1,1))$ satisfying $\zeta\equiv1$ in $B_{1/2}$, $\eta\equiv1$ in [0,1/2). Denote

$$\zeta_R(x)=\zeta(\frac{x}{R}),\ x\in\mathbb{R},\ \psi_R(t)=\eta^m(\frac{t}{R^2}),\quad t\ge0.$$

From [11], we can achieve that

$$|\psi_R'(t)|=|mR^{-2}\eta^{m-1}(\frac{t}{R^2})\eta'(\frac{t}{R^2})|\le CR^{-2}\psi_R^{\frac{1}{q}}(t)\mathcal{X}_{\{\frac12R^2\le t\le R^2\}}. \tag{2.9}$$

And then we can get that

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}}u_t(x,t)\zeta_R(x)\psi_R(t)\mathrm{d}x\mathrm{d}t&=-\int_{\mathbb{R}}u_0(x)\zeta_R(x)dx-\int_0^\infty\int_{\mathbb{R}}u(x,t)\zeta_R(x)\psi_R'(t)\mathrm{d}x\mathrm{d}t\\&\le CR^{-2}\int_{\frac12R^2}^{R^2}\int_{\mathbb{R}}u(x,t)\zeta_R(x)\psi_R^{\frac1q}(t)\mathrm{d}x\mathrm{d}t-\int_{\mathbb{R}}u_0(x)\zeta_R(x)\mathrm{d}x.\end{aligned}$$

Multiplying the first equation of (1.1) by $\zeta_R(x)\psi_R(t)$ and integrating in Q, then we have

$$\begin{aligned}K_p(R)&:=\int_0^\infty\int_{\mathbb{R}}a(x)v^p(x,t)\zeta_R(x)\psi_R(t)\mathrm{d}x\mathrm{d}t\\&=\int_0^\infty\int_{\mathbb{R}}u_t(x,t)\zeta_R(x)\psi_R(t)\mathrm{d}x\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}}(J*u-u)(x,t)\zeta_R(x)\psi_R(t)\mathrm{d}x\mathrm{d}t\\&\le-\int_0^\infty\int_{\mathbb{R}}u(x,t)\zeta_R(x)\psi_R'(t)\mathrm{d}x\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}}(J*\zeta_R-\zeta_R)(x)u(x,t)\psi_R(t)\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{2.10}$$

We can also have from (2.9) that

$$-\int_0^\infty\int_{\mathbb{R}}u(x,t)\zeta_R(x)\psi_R'(t)\mathrm{d}x\mathrm{d}t\le CR^{-2}\int_{\frac12R^2}^{R^2}\int_{|x|\le R}u(x,t)\mathrm{d}x\mathrm{d}t. \tag{2.11}$$

Furthermore, by Taylor's formula,

$$\begin{aligned}(J*\zeta_R-\zeta_R)(x)&=\int_{\mathbb{R}}J(z)(\zeta(\frac{x+z}{R})-\zeta(\frac{x}{R}))\mathrm{d}z\\&=\int_{\mathbb{R}}J(z)(\frac{z}{R}\zeta_x(\frac{x}{R})+\frac{z^2}{2R^2}\zeta_{xx}(\frac{x+\theta z}{R}))\mathrm{d}z,\end{aligned}$$

where $0\le\theta\le1$. According to the symmetry of J, it follows that

$$\int_{\mathbb{R}}J(z)z\mathrm{d}z=0,\quad A(J):=\int_{\mathbb{R}}J(z)|z|^2\mathrm{d}z>0.$$

We can have that

$$|(J*\zeta_R-\zeta_R)(x)|=\frac{1}{2R^2}\Big|\int_{B_1}J(z)z^2\zeta_{xx}(\frac{x+\theta z}{R})\mathrm{d}z\Big|\le CR^{-2}\mathcal{X}_{\{\frac12R-1\le|x|\le R+1\}}.$$

Consequently, let R large enough such that

$$-\int_0^\infty \int_{\mathbb{R}} (J * \zeta_R - \zeta_R)(x)u(x,t)\psi_R(t)\mathrm{d}x\mathrm{d}t \le CR^{-2}\int_0^{R^2}\int_{\frac{1}{2}R-1\le|x|\le R+1} u(x,t)\mathrm{d}x\mathrm{d}t$$
$$\le CR^{-2}\int_0^{R^2}\int_{\frac{1}{4}R\le|x|\le 2R} u(x,t)\mathrm{d}x\mathrm{d}t. \qquad (2.12)$$

It follows from (2.10)–(2.12) and Hölder's inequality that

$$K_p(R) \le CR^{-2}\Big(\int_{\frac{1}{2}R^2}^{R^2}\int_{|x|\le R} u(x,t)\mathrm{d}x\mathrm{d}t + \int_0^{R^2}\int_{\frac{1}{4}R\le|x|\le 2R} u(x,t)\mathrm{d}x\mathrm{d}t\Big)$$
$$\le CR^{2-\frac{4}{q}}\Big[\Big(\int_{\frac{1}{2}R^2}^{R^2}\int_{|x|\le R} u^q(x,t)\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{q}} + \Big(\int_0^{R^2}\int_{\frac{1}{4}R\le|x|\le 2R} u^q(x,t)\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{q}}\Big]$$
$$\le CR^{2-\frac{4}{q}}\Big(\int_{\frac{1}{2}R^2}^{R^2}\int_{|x|\le R} u^q(x,t)\mathrm{d}x\mathrm{d}t + \int_0^{R^2}\int_{\frac{1}{4}R\le|x|\le 2R} u^q(x,t)\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{q}}. \qquad (2.13)$$

Similarly, we can also derive that

$$L_q(R) := \int_0^\infty\int_{\mathbb{R}} a(x)u^q(x,t)\zeta_R(x)\psi_R(t)\mathrm{d}x\mathrm{d}t$$
$$\le CR^{2-\frac{4}{p}}\Big(\int_{\frac{1}{2}R^2}^{R^2}\int_{|x|\le R} v^p(x,t)\mathrm{d}x\mathrm{d}t + \int_0^{R^2}\int_{\frac{1}{4}R\le|x|\le 2R} v^p(x,t)\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p}}. \qquad (2.14)$$

Thanks to that $a(x)$ is a compact support containing the origin, combining (2.13) and (2.14), we can achieve that

$$L_q(R/4) \le CR^{2-\frac{4}{p}}(K_p(R))^{\frac{1}{p}}$$
$$\le CR^{2-\frac{4}{p}}\Big(CR^{2-\frac{4}{q}}\Big(\int_{\frac{1}{2}R^2}^{R^2}\int_{|x|\le R} u^q(x,t)\mathrm{d}x\mathrm{d}t + \int_0^{R^2}\int_{\frac{1}{4}R\le|x|\le 2R} u^q(x,t)\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{q}}\Big)^{\frac{1}{p}}$$
$$= C\Big(\int_{\frac{1}{2}R^2}^{R^2}\int_{|x|\le R} u^q(x,t)\mathrm{d}x\mathrm{d}t + \int_0^{R^2}\int_{\frac{1}{4}R\le|x|\le 2R} u^q(x,t)\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{pq}},$$

this with (2.8) and letting R$\to\infty$, we can have

$$\int_0^\infty\int_{\mathbb{R}} u^q(x,t)\mathrm{d}x = 0,$$

a contradiction. □

The global existence in the case $pq > (pq)_c$ can be achieved in Theorem 1.2, we will study it in the next section.

3. Second critical curve

In this section, we will prove that the secondary critical curve of system (1.1) is $\min\{a - a_0,\ b - b_0\} = 0$.

Proof of Theorem 1.2 (i) Assume $0 < a < a_0$ or $0 < b < b_0$, we will prove that the solutions of the system (1.1) blow up in finite time under large initial data. From (2.3)–(2.5), we can also have that

$$\begin{aligned}&\Big(\frac{a(\delta)}{2}\int_{-\delta}^{\delta}R\phi_R\mathrm{d}x\Big)^{\frac{p+1}{pq-1}}\int_{-R}^{R}R\phi_R R^{a-1}u_0\mathrm{d}x\\ &\geq\Big(\frac{a(\delta)}{2}\int_{-\delta}^{\delta}R\phi_R(\frac{Rx}{R_0})\mathrm{d}x\Big)^{\frac{p+1}{pq-1}}\int_{-R}^{R}Ru_0\phi_R(\frac{Rx}{R_0})R^{a-1}\mathrm{d}x\\ &\geq(\frac{\delta a(\delta)}{2})^{\frac{p+1}{pq-1}}\Gamma^{\frac{pq+p}{pq-1}}(0)\int_{-R_1}^{R_1}u_0R^{a-1}\mathrm{d}x\geq C.\end{aligned}$$

On the other hand, we have with (2.6) that

$$\lim_{R\to\infty}(\lambda_1(B_R))^{\frac{p+1}{pq-1}}R^{\frac{pq+p}{pq-1}+a-1}=\lim_{R\to\infty}(R^2\lambda_1(B_R))^{\frac{p+1}{pq-1}}R^{a-\frac{p+1}{pq-1}}=0,$$

Namely,

$$\Big(\frac{a(\delta)}{2}\int_{-\delta}^{\delta}\phi_R\mathrm{d}x\Big)^{\frac{p+1}{pq-1}}\Phi(0)>A\lambda_1(B_R)^{\frac{p+1}{pq-1}},$$

this with (2.2) can get that the solution of system (1.1) blows up in finite time.

(ii) Next, assume $a_0 < a < 1$ and $b_0 < b < 1$ with $u_0 = \lambda\psi \in \mathbb{I}^a$, $v_0 = \mu\varphi \in \mathbb{I}^b$. We will achieve global mild solutions if λ and μ are small enough. Choose $\bar{a} \in (a_0, a)$, $\bar{b} \in (b_0, b)$ satisfying

$$\bar{a}q-2>\bar{b},\ \ \bar{b}p-2>\bar{a},$$

and set the Banach space

$$X=\{(u,v)|\ \|\ (u,v)\ \|_X\leq\varepsilon\},$$

with

$$\|\ (u,v)\ \|_X:=\sup_{t\in(0,\infty)}\{(1+t)^{\frac{\bar{a}}{2}}\|u(t)\|_{L^\infty(\mathbb{R})}+(1+t)^{\frac{\bar{b}}{2}}\|v(t)\|_{L^\infty(\mathbb{R})}\}.$$

Then set the operator $\mathcal{M}[(u,v)] := (\mathcal{M}_1[(u,v)], \mathcal{M}_2[(u,v)])$ with

$$\begin{cases}\mathcal{M}_1[(u,v)(x,t)]:=e^{-t}u_0+W(x,t)*u_0+\int_0^t e^{-(t-s)}av^p\mathrm{d}s+\int_0^t W(x,t-s)*av^p\mathrm{d}s,\\ \mathcal{M}_2[(u,v)(x,t)]:=e^{-t}v_0+W(x,t)*v_0+\int_0^t e^{-(t-s)}au^q\mathrm{d}s+\int_0^t W(x,t-s)*au^q\mathrm{d}s,\end{cases}$$

where

$$W(x,t)=e^{-t}\sum_{n=1}^{\infty}\frac{t^n}{n!}J^{*n}(x).$$

Obviously, we can achieve for $(u,v)\in X$ that

$$\begin{aligned}\|\mathcal{M}_1[(u,v)](x,t)\|\leq&e^{-t}\|u_0\|_\infty+\Big\|\int_{\mathbb{R}}W(x-y,t)u_0\mathrm{d}y\Big\|_\infty+\\ &\int_0^t e^{-(t-s)}\|av^p(s)\|_\infty\mathrm{d}s+\int_0^t\Big\|\int_{\mathbb{R}}W(x-y,t-s)av^p\mathrm{d}y\Big\|_\infty\mathrm{d}s.\end{aligned}\tag{3.1}$$

By [16, Proposition 1],

$$\Big\|\int_{\mathbb{R}}W(x-y,t)u_0\mathrm{d}y\Big\|_\infty\leq C(1+t)^{-\frac{\bar{a}}{2}}(\|u_0\|_\infty+\|u_0\|_{\bar{a},\infty})\tag{3.2}$$

with $\|u_0\|_{\bar{a},\infty} = \limsup_{|x|\to\infty} |x|^{\bar{a}} u_0(x)$. Consequently,

$$\int_0^t e^{-(t-s)}\|av(s)\|_\infty^p \mathrm{d}s \le Ce^{-\frac{t}{2}}\|(u,v)\|_X^p + C(1+t)^{-\frac{\bar{a}p}{2}} \int_{\frac{t}{2}}^t e^{-(t-s)}\mathrm{d}s\|(u,v)\|_X^p. \tag{3.3}$$

Noticing $\|W\|_1 \le 1,\ \|W\|_\infty \le Ct^{-\frac{1}{2}}$ (see [17]), by Young's inequality with convolution, we have that

$$\begin{aligned}\int_0^t \Big\|\int_{\mathbb{R}} W(x-y,t-s)av^p\mathrm{d}y\Big\|_\infty \mathrm{d}s &\le \int_0^t \|v^p\|_\infty \Big\|\int_{\mathbb{R}} W(x-y,t-s)a(y)\mathrm{d}y\Big\|_\infty \mathrm{d}s\\ &\le \Big(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\Big)(1+s)^{-\frac{\bar{a}p}{2}}(1+t-s)^{-\frac{1}{2}}\mathrm{d}s\|(u,v)\|_X^p\\ &\le C((1+t)^{-\frac{1}{2}} + (1+t)^{-\frac{\bar{a}p+1}{2}})\|(u,v)\|_X^p\\ &\le C((1+t)^{-\frac{1}{2}} + (1+t)^{-\frac{\bar{a}p+1}{2}})\varepsilon^p. \end{aligned} \tag{3.4}$$

It follows from (3.1)–(3.4) with $\bar{a} > a_0 = \frac{p+1}{pq-1}$ that

$$\|\mathcal{M}_1(u,v)\|_\infty \le C(1+t)^{-\frac{\bar{a}}{2}}(\|u_0\|_\infty + \|u_0\|_{\bar{a},\infty} + \varepsilon^p),$$

similarly, we can also derive that

$$\|\mathcal{M}_2(u,v)\|_\infty \le C(1+t)^{-\frac{\bar{b}}{2}}(\|v_0\|_\infty + \|v_0\|_{\bar{b},\infty} + \varepsilon^q).$$

Consequently,

$$\|\mathcal{M}(u,v)\|_\infty \le C(\|u_0\|_\infty + \|u_0\|_{\bar{a},\infty} + \|v_0\|_\infty + \|v_0\|_{\bar{b},\infty} + \varepsilon^p + \varepsilon^q).$$

Take λ, τ and ε small enough such that $\mathcal{M}$ maps X into itself. Furthermore, by standard method, it can be achieved that $\mathcal{M}$ is a strict contraction with ε very small, which ensures (u,v) is a mild solution to the Cauchy problem (1.1). $\square$

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