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A New Matrix Inversion for Bell Polynomials and Its Applications

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Abstract The present paper gives a new Bell matrix inversion which arises from the classical Lagrange inversion formula. Some new relations for the Bell polynomials are obtained, including a Bell matrix inversion in closed form and an inverse form of the classical Faà di Bruno formula.

Keywords Bell polynomials; Bell matrix; matrix inversion; Lagrange inversion formula; binomial-type sequence; Wronski determinant; Faà di Bruno formula

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1. Preliminary and main result

Throughout this paper, we will adopt the same notation of Henrici [1]. For instance, we will use $\mathbb{C}[[x]]$ to denote the ring of formal power series (in short, fps) over the complex number field $\mathbb{C}$ and for any $f(x)=\sum_{n\geq 0}a_nx^n\in\mathbb{C}[[x]]$, the coefficient functional

$$[x^n]f(x)=a_n,\quad n=0,1,2,\ldots.$$

For our convenience, define

$$\mathcal{L}_0=\Big\{\sum_{n=0}^{\infty}a_nx^n|a_0\neq 0\Big\},\quad \mathcal{L}_1=\Big\{\sum_{n=0}^{\infty}a_nx^n|a_0=0,a_1\neq 0\Big\}.$$

Moreover, for $f(x),g(x)\in\mathbb{C}[[x]]$, $g(x)$ is said to be the multiplicative inverse of $f(x)$ if $f(x)g(x)=1$ while $g(x)$ is said to be the composite inverse of $f(x)$ if $(f\circ g)(x)=(g\circ f)(x)=x$, here $\circ$ is the usual composite operation. Hereafter, we employ $f^{\langle -1\rangle}(x)$ to denote the composite inverse of $f(x)$. Obviously, if $f(x)=\sum_{n\geq 0}a_nx^n$, then its multiplicative inverse $g(x)$ is denoted by $\sum_{n\geq 0}\bar{a}_nx^n$. As such, we have $\bar{\bar{a}}_n=a_n$.

Proposition 1.1 *Given $f(x)\in\mathbb{C}[[x]]$, $f(x)$ has the multiplicative (resp., composite) inverse if and only if $f(x)\in\mathcal{L}_0$ (resp., $\mathcal{L}_1$).*

It is well known to us that the Bell polynomials play a very important role in Analysis, Combinatorics, and Number theory. In this paper, we will focus on the following (refined) Bell polynomials (Comtet [2, p.136, Remark] or Riordan [3]).

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Definition 1.2 (Refined Bell polynomials) *For integers $n \geq k \geq 0$ and infinite many variables $\{x_n\}_{n\geq 1}$, the sum*

$$\sum_{\sigma_k(n)} \frac{k!}{i_1!i_2!\cdots i_{n-k+1}!} x_1^{i_1} x_2^{i_2} \cdots x_{n-k+1}^{i_{n-k+1}} \tag{1.1}$$

is called the refined Bell polynomial in $x_1, x_2, \ldots, x_{n-k+1}$, where $\sigma_k(n)$ denotes the set of partition of n with k part, namely, all nonnegative integers $i_1, i_2, \ldots, i_{n-k+1}$ subject to

$$\begin{cases} i_1 + i_2 + \cdots + i_{n-k+1} = k, \\ i_1 + 2i_2 + \cdots + (n-k+1)i_{n-k+1} = n. \end{cases} \tag{1.2}$$

We denote (1.1) by $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$.

It should be pointed out that the above Bell polynomials are in agreement with the classical Bell polynomials [2, p. 133, Definition] in the case that $x_n \to x_n/n!$ and then multiplied by $n!/k!$. As a basic property, the refined Bell polynomials also satisfy

Proposition 1.3 ([2]) *For any fps $f(x) = \sum_{n\geq 1} a_n x^n$, it always holds*

$$f^k(x) = \sum_{n\geq k} B_{n,k}(a_1, a_2, \ldots, a_{n-k+1}) x^n. \tag{1.3}$$

Based on this fact, it is reasonable to introduce

Definition 1.4 (Bell matrix) *For any fps $f(x) = \sum_{n\geq 1} a_n x^n$, then the infinite-dimensional lower-triangular matrix with the (n, k)th entry $B_{n,k}(a_1, a_2, \ldots, a_{n-k+1})$ is called the Bell matrix given by $f(x)$. As a custom, we denote such matrix by*

$$B(f) = (B_{n,k}(a_1, a_2, \ldots, a_{n-k+1}))_{n\geq k\geq 1}. \tag{1.4}$$

In the study of the Bell polynomials, the inverse relation of the Bell polynomials is one of the most interesting problems first posed and solved by J. Riordan in his book [3], afterward re-investigated by L. C. Hsu et al. in [4] (see Lemma 3.4 and the comment afterwards). We would like to refer the reader to [3, Section 5.3] for further details. The aim of the present paper is to attack a similar problem, that is to find the matrix inverse of $B(f)$. As usual, the matrix inverse in the context of Combinatorial Analysis can be defined as follows.

Definition 1.5 ([2]) *Let $F = (A_{n,k})_{n\geq k\geq 1}$ and $G = (B_{n,k})_{n\geq k\geq 1}$ be two infinite-dimensional lower-triangular matrices over $\mathbb{C}$, which means $A_{n,k} = B_{n,k} = 0$ unless $n \geq k$. If there holds*

$$\sum_{n\geq i\geq k} A_{n,i} B_{i,k} = \sum_{n\geq i\geq k} B_{n,i} A_{i,k} = \delta_{n,k} \quad \text{for all} \quad n \geq k \geq 1, \tag{1.5}$$

where $\delta_{n,k}$ denotes the usual Kronecker symbol, then F together with G is called a matrix inversion.

As usual, the inverse matrix G is denoted by F^{-1}. Sometimes, F and G is called a pair of inverse (reciprocal) series relations, because it can be applied to summation and transformation of hypergeometric series.

In the context of Definition 1.5, we now present the main result of this paper.

Theorem 1.6 *For any sequences* $\{a_n\}_{n\geq 1}$ *with* $a_1 \neq 0$,

$$(B_{n,k}(a_1, a_2, \ldots, a_{n-k+1}))^{-1}_{n\geq k\geq 1} = (\frac{k}{n} B_{2n-k,n}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{n-k+1}))_{n\geq k\geq 1}, \tag{1.6}$$

where $\bar{a}_1 = 1/a_1$ *and for* $n \geq 2$,

$$\bar{a}_n = \sum_{k=1}^{n-1} (-1)^k a_1^{-k-1} B_{n-1,k}(a_2, a_3, \ldots, a_{n-k+1}). \tag{1.7}$$

Before proceeding further, let us give some comments on Theorem 1.6.

Remark 1.7 It should be mentioned that in a series of papers [5–8], M. Mihoubi and R. Mahdid investigated the role of binomial-type sequences in establishment of combinatorial identities and inverse relations with Bell polynomials involved. To our best knowledge, the matrix inversion given by Theorem 1.6 is different from [7, 8] in that all inverse relations given by Mihoubi and Mahdid are nonlinear connecting the coefficients of the inverse functions

$$f(t) = \sum_{n=1}^{\infty} a_n t^n \quad \text{and} \quad f^{\langle -1\rangle}(t) = \sum_{n=1}^{\infty} b_n t^n.$$

By contrast, Theorem 1.6 can be regarded as a general solution to the functional equation

$$f^k(t) = \sum_{n=k}^{\infty} A(n,k) t^n, \ (f^{\langle -1\rangle}(t))^k = \sum_{n=k}^{\infty} B(n,k) t^n, \quad k \geq 1.$$

Obviously, the matrix inverse relation $(A(n,k))^{-1} = (B(n,k))$ is essentially two-dimensional and linear, and it usually plays a role in the following linear transformation formula:

$$x_n = \sum_{k=1}^{n} A(n,k) y_k \Longleftrightarrow y_n = \sum_{k=1}^{n} B(n,k) x_k.$$

Our paper is organized as follows. The full proof of Theorem 1.6 will be given in Section 2. Some applications including a closed-form inverse of arbitrary Bell matrix associated with binomial-type sequence and an inverse form of the Faà di Bruno formula will be presented in Section 3.

2. Proof of Theorem 1.6

Instead of Riordan's umbral method used in [3], our main argument is based on the following well-known Lagrange inversion formula. As of today, the Lagrange inversion formula is a basic but useful tool of finding of the composite inverse of fps.

Lemma 2.1 ([2, p.150, Theorem C, Theorem D]) *Let* $\phi(x) \in \mathcal{L}_0$. *Then for any fps* $F(x)$, *it always holds that*

$$F(x) = \sum_{n=0}^{\infty} a_n (\frac{x}{\phi(x)})^n, \tag{2.1}$$

where

$$a_n = \frac{1}{n}[x^{n-1}]F'(x)\phi^n(x). \tag{2.2}$$

Now we are in a good position to show Theorem 1.6.

Proof It suffices to define the fps $f(x) = \sum_{n\geq 1} a_n x^n \in \mathcal{L}_1$. In light of Definition 1.4, it is easily verified that the $B(f)$ is invertible. Thus we assume temporarily

$$(B_{n,k}(a_1, a_2, \ldots, a_{n-k+1}))^{-1} = (A_{n,k}).$$

Then by Proposition 1.3, we see

$$f^k(x) = \sum_{n\geq k} B_{n,k}(a_1, a_2, \ldots, a_{n-k+1})x^n. \tag{2.3}$$

Recall that $(A_{n,k})$ and $(B_{n,k})$ are inverses of each other. Upon inverting (2.3), we immediately obtain

$$x^k = \sum_{n\geq k} A_{n,k} f^n(x). \tag{2.4}$$

In view of the definition before Proposition 1.1, we reformulate $f(x)$ by

$$f(x) = \frac{x}{\phi(x)},$$

where

$$\begin{aligned} \phi(x) &= \frac{1}{a_1 + a_2 x + \cdots + a_n x^{n-1} + \cdots} &(2.5)\\ &= \bar{a}_1 + \bar{a}_2 x + \cdots + \bar{a}_n x^{n-1} + \cdots \in \mathcal{L}_0. &(2.6)\end{aligned}$$

Then (2.4) turns out to

$$x^k = \sum_{n\geq k} A_{n,k}\left(\frac{x}{\phi(x)}\right)^n.$$

In this form, we are able to apply Lemma 2.1 to find

$$\begin{aligned} A_{n,k} &= \frac{1}{n}[x^{n-1}](x^k)'\phi^n(x) \\ &= \frac{k}{n}[x^{n-k}]\phi^n(x). &(2.7)\end{aligned}$$

Recalling the definition of the Bell polynomials and (2.6), we have

$$\begin{aligned} (x\phi(x))^n &= \sum_{m\geq n} B_{m,n}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{m-n+1})x^m, \\ \phi^n(x) &= \sum_{m\geq n} B_{m,n}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{m-n+1})x^{m-n}. &(2.8)\end{aligned}$$

Inserting (2.8) into (2.7) leads to

$$A_{n,k} = \frac{k}{n}[x^{n-k}]\sum_{m\geq n} B_{m,n}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{m-n+1})x^{m-n}$$

$$= \frac{k}{n} B_{2n-k,n}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{n-k+1}).$$

All remains to find $\bar{a}_n$ subject to (2.5)/(2.6). For this, it is clear that

$$\text{RHS of (2.5)} = \frac{1}{a_1(1+K(x))} = \frac{1}{a_1}\sum_{i=0}^{\infty}(-1)^i K^i(x),$$

where

$$K(x) = \sum_{n=1}^{\infty} \frac{a_{n+1}}{a_1} x^n.$$

By the definition of the Bell polynomials and Proposition 1.3, we have

$$\begin{aligned}
\text{RHS of (2.5)} &= \frac{1}{a_1}\sum_{i=0}^{\infty}(-1)^i \sum_{j=i}^{\infty} B_{j,i}\left(\frac{a_2}{a_1}, \frac{a_3}{a_1}, \ldots, \frac{a_{j-i+2}}{a_1}\right)x^j \\
&= \frac{1}{a_1} + \frac{1}{a_1}\sum_{j=1}^{\infty}\Big(\sum_{i=1}^{j}(-1)^i B_{j,i}\left(\frac{a_2}{a_1}, \frac{a_3}{a_1}, \ldots, \frac{a_{j-i+2}}{a_1}\right)\Big)x^j \\
&= \frac{1}{a_1} + \sum_{j=1}^{\infty}\Big(\sum_{i=1}^{j}\frac{(-1)^i}{a_1^{i+1}} B_{j,i}(a_2, a_3, \ldots, a_{j-i+2})\Big)x^j.
\end{aligned}$$

Thus (1.7) follows by comparing coefficients of x^{n-1} in the last expansion. The theorem is therefore proved. □

Regarding the computation of $\bar{a}_n$, besides (1.7), we further have

Corollary 2.2 *With the same notation as Theorem 1.6. Then*

$$\bar{a}_n = \frac{(-1)^{n+1}}{a_1^n}\begin{vmatrix} a_2 & a_3 & \cdots & a_{n-1} & a_n \\ a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ 0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_1 & a_2 \end{vmatrix}. \tag{2.9}$$

Here, $|\cdot|$ denotes the usual determinant.

Proof It is clear that (2.9) is just a direct application of the Wronski formula [1, Theorem 1.3] to (2.5)/(2.6). □

More interestingly, once further applying Theorem 1.6 to (1.7), we immediately obtain an expression for the coefficients of the multiplicative inverse.

Corollary 2.3 *With the same notation as Theorem 1.6. Suppose further $a_2 \neq 0$, $\dot{a}_2 = 1/a_2$, and for $n \geq 3$,*

$$\dot{a}_n = \frac{(-1)^n}{a_2^{n-1}}\begin{vmatrix} a_3 & a_4 & \cdots & a_{n-1} & a_n \\ a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} \\ 0 & a_2 & \cdots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_2 & a_3 \end{vmatrix}. \tag{2.10}$$

Then, for $n \geq 1$,

$$(-1)^n n\bar{a}_1^{n+1} = \sum_{k=1}^{n} k\bar{a}_{k+1} B_{2n-k,n}(\dot{a}_2, \dot{a}_3, \ldots, \dot{a}_{n-k+2}). \tag{2.11}$$

Proof It only needs to change the index n to $n+1$ in (1.7). We have

$$\bar{a}_{n+1} = \sum_{k=1}^{n} \frac{(-1)^k}{a_1^{k+1}} B_{n,k}(a_2, a_3, \ldots, a_{n-k+2}). \tag{2.12}$$

Note that $a_2 \neq 0$. It allows us to apply Theorem 1.6 to (2.12), yielding

$$\frac{(-1)^n}{a_1^{n+1}} = \sum_{k=1}^{n} \frac{k}{n} \bar{a}_{k+1} B_{2n-k,n}(\dot{a}_2, \dot{a}_3, \ldots, \dot{a}_{n-k+2}),$$

where, analogous to the above derivations of Theorem 1.6, we may write

$$\frac{1}{a_2 + a_3 x + a_4 x^2 + \cdots} = \sum_{n=2}^{\infty} \dot{a}_n x^{n-2}$$

and see that $\dot{a}_n$ are given by (2.10). Finally we have

$$(-1)^n n\bar{a}_1^{n+1} = \sum_{k=1}^{n} k\bar{a}_{k+1} B_{2n-k,n}(\dot{a}_2, \dot{a}_3, \ldots, \dot{a}_{n-k+2}).$$

The corollary is proved. $\square$

Remark 2.4 It should be pointed out that although formulas (1.7) and (2.9) are two ways of calculating the multiplicative inverse $g(x) = \sum_{n\geq 1} \bar{a}_n x^{n-1}$ of $f(x) = \sum_{n\geq 1} a_n x^{n-1}$, we think that in practice it seems more convenient to do so by use of the recurrence relation directly

$$\begin{cases} \bar{a}_1 = 1/a_1, \\ \bar{a}_k = -\bar{a}_1(a_2\bar{a}_{k-1} + a_3\bar{a}_{k-2} + \cdots + a_k\bar{a}_1). \end{cases} \tag{2.13}$$

3. Applications

In this section, we proceed to examine some specific applications of Theorem 1.6.

3.1. Bell matrix inversion in closed form

Evidently, the requirement that $f(x)g(x) = 1$ is key to Theorem 1.6. One of the simplest cases occurs when $f(x) = F^a(x)$, $g(x) = F^{-a}(x)$. In other words, $f(x)$ and $g(x)$ are generating functions of certain binomial-type sequences. In this situation, Theorem 1.6 takes a closed form as follows

Corollary 3.1 *For any* binomial-type *sequence* $\{x_m(a)\}_{m\geq 1}$ *with* $x_1(a) \neq 0$, *namely, there exists a fps* $F(x)$ *such that*

$$F^a(x) = \sum_{m=1}^{\infty} x_m(a) x^{m-1}, \tag{3.1}$$

we have

$$(x_{n-k+1}(ak))^{-1}_{n\geq k\geq 1} = (\frac{k}{n} x_{n-k+1}(-an))_{n\geq k\geq 1}. \tag{3.2}$$

Proof By Proposition 1.3 on the Bell polynomials, it is obvious that

$$(xF^a(x))^k = \sum_{n=k}^{\infty} B_{n,k}(x_1(a), x_2(a), \ldots, x_{n-k+1}(a))x^n.$$

In view of the known condition (3.1), it follows that

$$(xF^a(x))^k = \sum_{m=1}^{\infty} x_m(ak)x^{m+k-1}.$$

A comparison of both identities yields

$$B_{n,k}(x_1(a), x_2(a), \ldots, x_{n-k+1}(a)) = x_{n-k+1}(ak).$$

A similar derivation gives

$$B_{2n-k,n}(x_1(-a), x_2(-a), \ldots, x_{n-k+1}(-a)) = x_{n-k+1}(-an).$$

Taking these into (1.6) leads us to the claimed. $\square$

This result brings us back to Ma's paper [9], wherein he considered the expansion

$$h(x)(x\,\psi(x))^k = \sum_{n=k}^{\infty} A_{n,k}(\frac{x}{\phi(x)})^n \tag{3.3}$$

and showed by using the Lagrange inversion formula that

Lemma 3.2 ([9, Theorem 4.1]) *Let $A_{n,k}$ be given by (3.3). Then*

$$(A_{n,k})_{n\geq k\geq 1} = ([x^{n-k}](1-\Delta(\phi(x)))\phi^n(x)\psi^k(x)h(x))_{n\geq k\geq 1}, \tag{3.4}$$

$$(A_{n,k})^{-1}_{n\geq k\geq 1} = ([x^{n-k}](1+\Delta(\psi(x)))\phi^{-k}(x)\psi^{-n}(x)h^{-1}(x))_{n\geq k\geq 1}, \tag{3.5}$$

where $\Delta(\phi(x)) = x\phi'(x)/\phi(x)$.

Taking all these into account, we immediately extend Corollary 3.1 to the following

Corollary 3.3 *With the same notation as in Corollary 3.1. Then*

$$\begin{aligned}(&\frac{ak+bk+c}{ak+bn+c}x_{n-k+1}(ak+bn+c))^{-1}_{n\geq k\geq 1}\\ &=(\frac{ak+bk+c}{an+bk+c}x_{n-k+1}(-an-bk-c))_{n\geq k\geq 1}.\end{aligned} \tag{3.6}$$

Proof According to Lemma 3.2, we set $\phi(x) = F^b(x)$, $\psi(x) = F^a(x)$, $h(x) = F^c(x)$ and then compute directly

$$\begin{aligned}A_{n,k} &= [x^{n-k}](1-bx\frac{F'(x)}{F(x)})F^{ak+bn+c}(x)\\ &= \frac{ak+bk+c}{ak+bn+c}[x^{n-k}]F^{ak+bn+c}(x) = \frac{ak+bk+c}{ak+bn+c}x_{n-k+1}(ak+bn+c).\end{aligned}$$

Note that the last equality results from (3.1). Accordingly, the right-hand side of (3.5) turns out to be

$$[x^{n-k}](1+ax\frac{F'(x)}{F(x)})F^{-an-bk-c}(x)$$

$$= \frac{ak+bk+c}{an+bk+c}[x^{n-k}]F^{-an-bk-c}(x) = \frac{ak+bk+c}{an+bk+c}x_{n-k+1}(-an-bk-c).$$

This completes the proof of the theorem. □

3.2. Inverse form of the Faà di Bruno formula

It is well known that the classical Faà di Bruno formula is a powerful tool to calculate the high derivatives of composite fps. We refer the reader to [2, p. 137, Section 3.4, Theorem A] for further details.

Lemma 3.4 (The Faà di Bruno formula) *For any two fps*

$$f(x) = \sum_{n\geq 0} f_n x^n, \; g(x) = \sum_{n\geq 1} g_n x^n, \tag{3.7a}$$

assume that

$$h(x) = \sum_{n\geq 0} h_n x^n = (f\circ g)(x). \tag{3.7b}$$

Then the coefficient $h_0 = f_0$ and for $n \geq 1$,

$$h_n = \sum_{k=1}^{n} f_k B_{n,k}(g_1, g_2, \ldots, g_{n-k+1}). \tag{3.7c}$$

As already mentioned before, inverse relations related to the Bell polynomials in [3, Eqs. (6)/(7)] discovered by Riordan and those given by Hsu et al. are immediate consequences of applying the Faà di Bruno formula to the composite relation $h(x) = (f\circ g)(x)$ and its equivalent form $f(x) = (h\circ g^{\langle -1\rangle})(x)$ if $g(x)$ is invertible or $g(x) = (f^{\langle -1\rangle}\circ h)(x)$ if $f(x)$ is invertible. A full study on this topic can be found in two papers [7, 8] by Mihoubi and Mahdid. Being different from them, we now use $B^{-1}(g)$ to set up.

Theorem 3.5 (Inverse form of the Faà di Bruno formula) *With the same assumption as in Lemma 3.4. Then, for $g_1 \neq 0$,*

$$f_n = \frac{1}{n}\sum_{k=1}^{n} k h_k B_{2n-k,n}(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_{n-k+1}), \tag{3.8}$$

where $\bar{g}_n$ is given by (1.7) or (2.9).

Proof It is obvious that (3.7c) in Lemma 3.4 can be reformulated as

$$\alpha = \big(B_{n,k}(g_1, g_2, \ldots, g_{n-k+1})\big)\beta, \tag{3.9}$$

where α and β are two infinite column vectors

$$\alpha = (h_1, h_2, \ldots, h_n, \ldots)^T, \beta = (f_1, f_2, \ldots, f_n, \ldots)^T.$$

Here the superscript T denotes the transpose of vectors. Multiply both sides of (3.9) by $(B_{n,k}(g_1, g_2, \ldots, g_{n-k+1}))^{-1}$. Thus we have

$$\beta = (B_{n,k}(g_1, g_2, \ldots, g_{n-k+1}))^{-1}\alpha.$$

Equating the nth components on both sides gives rise to

$$f_n = \frac{1}{n}\sum_{k=1}^{n} kh_k B_{2n-k,n}(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_{n-k+1})$$

with each $\bar{g}_n$ being given by (1.7) or (2.9). The theorem is proved. □

Remark 3.6 It is worth pointing out that Henrici observed in [1, Chapter 1]: the Lagrange inversion formula boils down to a pair of matrix inversions. Indeed, suppose we have the expansion

$$F(x) = \sum_{n=0}^{\infty} a_n \left(\frac{x}{\phi(x)}\right)^n = (f \circ g)(x)$$

where

$$g(x) = x/\phi(x), \phi(x) = \sum_{n=1}^{\infty} g_n x^{n-1}, f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then, by the inverse form of the Faà di Bruno formula, i.e., Theorem 3.5, we at once obtain

$$\begin{aligned} a_n &= \frac{1}{n}\sum_{k=1}^{n} kB_{2n-k,n}(g_1, g_2, \ldots, g_{n-k+1})[x^k]F(x) \\ &= \frac{1}{n}[x^{n-1}]F'(x)\phi^n(x). \end{aligned}$$

In this sense, we believe in that the Faà di Bruno formula is equivalent to the Lagrange inversion formula, i.e., Lemma 2.1.

3.3. More relations among the Bell polynomials

Actually, there exists an anti-isomorphic mapping changing the composite of two fps into the product of two Bell matrices.

Lemma 3.7 *For any $f(x), g(x) \in \mathbb{C}[[x]]$ with $f(0) = g(0) = 0$, it holds*

$$B(f \circ g) = B(g)\, B(f), \tag{3.10}$$

where $B(f)$ is given by Definition 1.4.

Proof It can be verified directly by the definition of $B(f)$. □

Torrian [10, Theorem 1] described a constructive method for the composite inverse of a given fps. Using the inverses of the Bell matrices and the above anti-isomorphic relation, we are able to express the composite inverse in a more clear way.

Theorem 3.8 *Let $f(x) = \sum_{n\geq 1} f_n x^n$ be the composite inverse of $g(x) = \sum_{n\geq 1} g_n x^n$. Then we have*

$$B_{n,k}(f_1, f_2, \ldots, f_{n-k+1}) = \frac{k}{n} B_{2n-k,n}(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_{n-k+1}). \tag{3.11}$$

In particular, for $k = 1$,

$$f_n = \frac{1}{n} B_{2n-1,n}(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n). \tag{3.12}$$

Proof We only need to apply Lemma 3.7 to $(f\circ g)(x)=x$, obtaining $B(g)B(f)=(\delta_{n,k})$. In view of the definition of $B(f)$, we see

$$(B_{n,k}(g_1,g_2,\ldots,g_{n-k+1}))(B_{n,k}(f_1,f_2,\ldots,f_{n-k+1}))=(\delta_{n,k}).$$

Hence,

$$\begin{aligned}(B_{n,k}(f_1,f_2,\ldots,f_{n-k+1}))&=(B_{n,k}(g_1,g_2,\ldots,g_{n-k+1}))^{-1}\\&=(\frac{k}{n}B_{2n-k,n}(\bar g_1,\bar g_2,\ldots,\bar g_{n-k+1})),\end{aligned}$$

where $\sum_{n\geq 1}\bar g_n x^{n-1}$ is the multiplicative inverse of $\sum_{n\geq 1} g_n x^{n-1}$, each $\bar g_n$ being given by (1.7) or (2.9). □

A bit of experimental computation via (3.12) shows that each $\bar g_n$ can also be expressed in terms of f_n or $\bar f_n$, where $\sum_{n\geq 1}\bar f_n x^{n-1}$ denotes the multiplicative inverse of $f_0(x)=\sum_{n\geq 1} f_n x^{n-1}$.

Theorem 3.9 *With the same assumption as in Theorem 3.8, for $k>n/2$,*

$$B_{n,k}(\bar g_1,\bar g_2,\ldots,\bar g_{n-k+1})=\frac{k}{2k-n}B_{k,2k-n}(f_1,f_2,\ldots,f_{n-k+1}) \tag{3.13}$$

and for $k<n/2$,

$$B_{n,k}(\bar g_1,\bar g_2,\ldots,\bar g_{n-k+1})=\frac{k}{2k-n}B_{2n-3k,n-2k}(\bar f_1,\bar f_2,\ldots,\bar f_{n-k+1}). \tag{3.14}$$

In particular, when $k=1$ and $n\geq 3$,

$$\bar g_n=\frac{1}{2-n}B_{2n-3,n-2}(\bar f_1,\bar f_2,\ldots,\bar f_n). \tag{3.15}$$

Proof As previously, we consider the series expansion

$$(x\,\psi(x))^k=\sum_{n\geq k}B_{n,k}(\bar g_1,\bar g_2,\ldots,\bar g_{n-k+1})x^n, \tag{3.16}$$

where

$$\psi(x)=\sum_{n=1}^{\infty}\bar g_n x^{n-1}=\frac{1}{g_1+g_2x+\cdots+g_nx^{n-1}+\cdots}.$$

Replace x with $f(x)=xf_0(x)$ in (3.16). Then we get

$$(f_0(x)\,x\,(\psi\circ f)(x))^k=\sum_{n\geq k}B_{n,k}(\bar g_1,\bar g_2,\ldots,\bar g_{n-k+1})(x\,f_0(x))^n. \tag{3.17}$$

Since $(g\circ f)(x)=x$, from $g(x)=x/\psi(x)$, it is clear that

$$x\,(\psi\circ f)(x)=xf_0(x).$$

Inserting this relation into (3.17) we immediately obtain

$$(x\,f_0^2(x))^k=\sum_{n\geq k}B_{n,k}(\bar g_1,\bar g_2,\ldots,\bar g_{n-k+1})(x\,f_0(x))^n. \tag{3.18}$$

At this stage, we apply Lemma 2.1 to this expansion, thereby obtaining

$$B_{n,k}(\bar g_1,\bar g_2,\ldots,\bar g_{n-k+1})=\frac{1}{n}[x^{n-1}]((x\,f_0^2(x))^k)'f_0^{-n}(x)$$

$$
\begin{aligned}
&= \frac{1}{n}[x^{n-1}](kx^{k-1}f_0^{2k}(x) + 2kx^k f_0^{2k-1}(x)f_0^{'}(x))f_0^{-n}(x) \\
&= \frac{k}{n}[x^{n-k}]\, f_0^{2k-n}(x) + \frac{2k}{n(2k-n)}[x^{n-k-1}]\,(f_0^{2k-n}(x))^{'} \\
&= (\frac{k}{n} + \frac{2k(n-k)}{n(2k-n)})[x^{n-k}]\, f_0^{2k-n}(x) \\
&= \frac{k}{2k-n}[x^{n-k}]\, f_0^{2k-n}(x).
\end{aligned}
$$

All remains to compute all coefficients in the last identity according as two cases $2k-n>0$ or <0. If $2k-n>0$, then we find

$$
B_{n,k}(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_{n-k+1}) = \frac{k}{2k-n}B_{k,2k-n}(f_1, f_2, \ldots, f_{n-k+1}).
$$

If $2k-n<0$, then we find

$$
B_{n,k}(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_{n-k+1}) = \frac{k}{2k-n}B_{2n-3k,n-2k}(\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_{n-k+1}).
$$

This completes the proof of the theorem. $\square$

We end this paper by pointing out some algebraic properties of Theorem 3.9.

Remark 3.10 Note that (3.13) may also be derived from (3.11) directly by a linear transformation. More precisely, for the case $2k-n>0$, we are able to make the replacement

$$
\begin{pmatrix} N \\ K \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} n \\ k \end{pmatrix}
$$

in (3.11). Equivalently,

$$
\begin{pmatrix} n \\ k \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} N \\ K \end{pmatrix}.
$$

Under this replacement, (3.11) becomes

$$
B_{K,2K-N}(f_1, f_2, \ldots, f_{N-K+1}) = \frac{2K-N}{K}B_{N,K}(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_{N-K+1}).
$$

So the desired identity follows by relabelling $(N,K) \to (n,k)$.

In the meantime, we note that (3.14) is symmetric in two sets of variables $\{\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_{n-k+1}\}$ and $\{\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_{n-k+1}\}$ under the linear transformation

$$
\begin{pmatrix} n \\ k \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} N \\ K \end{pmatrix},
$$

namely, for $k < n/2$, we have

$$
B_{n,k}(\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_{n-k+1}) = \frac{k}{2k-n}B_{2n-3k,n-2k}(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_{n-k+1}). \tag{3.19}
$$

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