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The Normalized Laplacian Spectrum of Pentagonal Graphs and Its Applications

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Abstract The eigenvalues of the normalized Laplacian of a graph provide information on its structural properties and also on some relevant dynamical aspects, in particular those related to random walks. In this paper, we give the spectra of the normalized Laplacian of iterated pentagonal of a simple connected graph. As an application, we also find the significant formulae for their multiplicative degree-Kirchhoff index, Kemeny's constant and number of spanning trees.

Keywords normalized Laplacian spectrum; multiplicative degree-Kirchhoff index; Kemeny's constant; the number of spanning trees

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1. Introduction

Spectral analysis of graphs has been the subject of considerable research effort in mathematics and computer science [1–3], due to its wide applications in this area and in general [4, 5]. The spectra of the adjacency, Laplacian and normalized Laplacian matrices of a graph provide information on the diameter, degree distribution, paths of a given length, total number of links, number of spanning trees and many more invariants.

In recent years, there has been an increasing interest in the study of the normalized Laplacian, as many measures for random walks on a network are linked to the eigenvalues and eigenvectors of normalized Laplacian of the associated graph. These include the hitting time, mixing time and Kemeny's constant which can be used as a measure of efficiency of navigation on the network [6–9].

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. An edge connecting two vertices $i, j \in V(G)$ is denoted by ij. If $ij \in E(G)$, we say i is a neighbor of j and write as $i \sim j$ or we say i and j are adjacent. The degree of a vertex i is denoted by d_i. Let A_G be the adjacency matrix of G and D_G be the diagonal matrix of vertex degree of G. The matrix $L_G = D_G - A_G$ called the Laplacian matrix of G.

The random walk is defined as the Markov chain $X_n (n \geq 0)$, that from its current vertex i jumps arbitrarily to its neighboring vertex j with probability $p_{ij} = \frac{1}{d_i}$. We denote by $M = (p_{ij})$

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the transition probabilities matrix for random walks on G. So

$$p_{ij} = \begin{cases} \frac{1}{d_i}, & \text{if } i \sim j, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $M = D_G^{-1} A_G$ is a stochastic matrix.

The normalized Laplacian is defined to be

$$\ell = I - D_G^{\frac{1}{2}} M D_G^{-\frac{1}{2}},$$

where I is the identity matrix with the same order as M.

Let δ_{ij} be the Kronecker delta. From the definition of ℓ, we have the following relationship easily:

$$\ell(i,j) = \delta_{ij} - \frac{A(i,j)}{\sqrt{d_i d_j}},$$

where $\ell(i,j)$ and $A(i,j)$ denote the (i,j)-entry of ℓ and A, respectively.

Since ℓ is Hermitian and similar to $I - M = D^{-1}L$, the eigenvalues of ℓ are non-negative. We label the eigenvalues of ℓ so that $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$, where n is the number of vertices of G. The spectrum on the normalized Laplacian matrix ℓ of the graph G is defined as $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, which also is called the normalized Laplacian spectrum of G.

Lemma 1.1 ([10]) *Let G be a simple connected graph with n vertices and ℓ be the normalized Laplacian of G. The normalized Laplacian spectrum of G is $\sigma = \{0 = \lambda_1, \lambda_2, \ldots, \lambda_n\}$. We have*

(i) For all $i \leq n$, we have $\frac{n}{n-1} \leq \lambda_i \leq 2$ with $\lambda_n = 2$ if and only if G is bipartite;

(ii) G is bipartite if and only if λ_i is an eigenvalue of ℓ, then the value $2 - \lambda_i$ is also an eigenvalue of ℓ and $m_\ell(\lambda_i) = m_\ell(2 - \lambda_i)$, where $m_\ell(\lambda_i)$ denotes the multiplicity of the eigenvalue λ_i of ℓ.

In terms of the spectrum on the normalized Laplacian of G, the special calculation formulae for the multiplicative degree-Kirchhoff index, the Kemeny's constant and the number of spanning trees of graph can be expressed as follows.

Lemma 1.2 *Let G be a simple connected graph with N_0 vertices and E_0 edges and $\sigma = \{0 = \lambda_1, \lambda_2, \ldots, \lambda_{N_0}\}$ be the spectrum on the normalized Laplacian ℓ of G. Then*

(i) ([11]) The multiplicative degree-Kirchhoff index of G is

$$Kf'(G) = 2E_0 \sum_{i=2}^{N_0} \frac{1}{\lambda_i}.$$

(ii) ([12]) The Kemeny's constant of G is

$$K(G) = \sum_{i=2}^{N_0} \frac{1}{\lambda_i}.$$

(iii) ([10]) The number $N_{st}(G)$ of spanning trees of G is

$$N_{st}(G) = \frac{\prod_{i=1}^{N_0} d_i \prod_{k=2}^{N_0} \lambda_k}{\sum_{i=1}^{N_0} d_i}.$$

Apparently, from Lemma 1.2 (i) and (ii) the relation between the multiplicative degree-Kirchhoff index and the Kemeny's constant is

$$Kf'(G) = 2E_0K(G). \tag{1.1}$$

Let G be a simple connected graph with N_0 vertices and E_0 edges. Replacing each edge of G with two parallel paths of lengths 1 and 4 results in a new graph $W(G)$, which is called the pentagonal graph of the graph G. Figure 1 gives an example of the pentagonal graph of the cycle C_5.

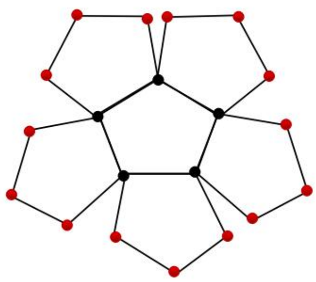

Figure 1 the pentagonal graph $W(C_5)$ of the cycle C_5

Let N and E denote the number of vertices and edges of $W(G)$. It is clear that

$$E = 5E_0, N = N_0 + 3E_0. \tag{1.2}$$

This work is motivated by [13–15], in which the researchers described the normalized Laplacian spectra of the quadrilateral graph, iterated triangulation and subdivisions of a graph and their applications are also described.

In Section 2, we give the spectra of the normalized Laplacian of pentagonal graphs. In Section 3, we determine the spectrum of the normalized Laplacian for $W_n(G)$ $(n > 0)$. Finally, the specific formulae to calculate three significant invariants, the multiplicative degree-Kirchhoff index, the Kemeny's constant and the number of spanning trees of $W(G)$ and $W_n(G)$ are derived.

2. The normalized Laplacian spectrum of the pentagonal graph $W(G)$

For the pentagonal graph $W(G)$ of G, the normalized Laplacian of it is written as ℓ_W. Let the degree of the vertex $i \in V(W(G))$ be $d_i^{'}$. A_W is the adjacency matrix of $W(G)$ and $P_W = D_W^{-\frac{1}{2}} A_W D_W^{-\frac{1}{2}}$, where D_W is the degree matrix of $W(G)$. In order to keep accordance, the normalized Laplacian of G is denoted by ℓ_G and let $P_G = D_G^{-\frac{1}{2}} A_G D_G^{-\frac{1}{2}}$.

The following lemma expresses the relationship of the normalized Laplacian eigenvalues of $W(G)$ and G.

Lemma 2.1 *Let λ be an eigenvalue of ℓ_W such that $\lambda \neq 1, \frac{1}{2}$. Then $\frac{4\lambda(1-\lambda)}{1-2\lambda}$ is an eigenvalue of ℓ_G with the same multiplicity as the eigenvalue λ of ℓ_W.*

Proof Let V_N be the set of all the newly added vertices in $W(G)$ and V_O be the set of the vertices inherited from G. That is, the vertex set $V(W(G))$ of $W(G)$ is the union of V_N and V_O.

Let $v=(v_1,v_2,\ldots,v_N)^T$ be an eigenvector with respect to the eigenvalue λ of ℓ_W, i.e.,

$$\ell_W v=(I-P_W)v=\lambda v. \tag{2.1}$$

For any vertex $u\in V(W(G))$, Eq. (2.1) indicates that

$$(1-\lambda)v_u=\sum_{k=1}^{N}P_W(u,k)v_k=\sum_{k=1}^{N}\frac{A_W(u,k)}{\sqrt{d'_ud'_k}}v_k. \tag{2.2}$$

For any vertex $i\in V_O$, let $N_N\subseteq V_N$ denote the set of the new neighbors of vertex i in $W(G)$ and $N_O\subseteq V_O$ denote the set of the neighbors of vertex i inherited from G. By the construction of $W(G)$ and Eq. (2.2), we have

$$(1-\lambda)v_i=\sum_{i'_1\in N_N}\frac{1}{\sqrt{d'_id'_{i'_1}}}v_{i'_1}+\sum_{j\in N_O}\frac{1}{\sqrt{d'_id'_j}}v_j=\sum_{i'_1\in N_N}\frac{1}{2\sqrt{d_i}}v_{i'_1}+\sum_{j\in N_O}\frac{1}{2\sqrt{d_id_j}}v_j, \tag{2.3}$$

where $i'_1\in V_N$ and $j\in V_O$ are neighbor vertices of i in $W(G)$.

Similarly, for any $i'_1\in N_N$, it follows

$$(1-\lambda)v_{i'_1}=\frac{1}{\sqrt{d'_{i'_1}d'_{i'_2}}}v_{i'_2}+\frac{1}{\sqrt{d'_{i'_1}d'_i}}v_i=\frac{1}{2}v_{i'_2}+\frac{1}{2\sqrt{d_i}}v_i, \tag{2.4}$$

where $i'_2\in V_N$ and $i\in V_O$ are neighbor vertices of i'_1 in $W(G)$.

Similarly, for the vertex $i'_3\in V_N$ which is adjacent to $i'_2\in N_N$ and $j\in N_O$, we obtain

$$(1-\lambda)v_{i'_3}=\frac{1}{2}v_{i'_2}+\frac{1}{2\sqrt{d_j}}v_j. \tag{2.5}$$

For the vertex $i'_2\in V_N$ which is adjacent to $i'_1\in N_N$ and $i'_3\in V_N$, we obtain

$$(1-\lambda)v_{i'_2}=\frac{1}{2}(v_{i'_1}+v_{i'_3}). \tag{2.6}$$

Combining Eqs. (2.4) and (2.6), we have

$$((1-\lambda)^2-\frac{1}{4})v_{i'_1}=\frac{1}{4}v_{i'_3}+\frac{(1-\lambda)}{2\sqrt{d_i}}v_i,\ \text{for } \lambda\neq 1. \tag{2.7}$$

Combining Eqs. (2.5) and (2.6), we have

$$((1-\lambda)^2-\frac{1}{4})v_{i'_3}=\frac{1}{4}v_{i'_1}+\frac{(1-\lambda)}{2\sqrt{d_j}}v_j,\ \text{for } \lambda\neq 1. \tag{2.8}$$

Combining Eqs. (2.7) and (2.8), we have

$$((1-\lambda)^2-\frac{1}{2})(1-\lambda)v_{i'_1}=\frac{1}{8\sqrt{d_j}}v_j+\frac{(1-\lambda)^2-\frac{1}{4}}{2\sqrt{d_i}}v_i,\ \text{for } \lambda\neq\frac{1}{2},\frac{3}{2} \text{ and } 1. \tag{2.9}$$

Again, combining Eqs. (2.3) and (2.9), for $\lambda\neq\frac{1}{2},1$ and $\frac{3}{2}$, it follows

$$((1-\lambda)^2-\frac{1}{2})(1-\lambda)^2v_{i_1}=\frac{(1-\lambda)^2-\frac{1}{4}}{4}v_i+\sum_{j\in N_O}\frac{1+8(1-\lambda)((1-\lambda)^2-\frac{1}{2})}{16\sqrt{d_id_j}}v_j. \tag{2.10}$$

Therefore, the equation

$$\frac{4\lambda^2-6\lambda+1}{1-2\lambda}v_i = \sum_{j\in N_O} \frac{1}{\sqrt{d_i d_j}} v_j \tag{2.11}$$

holds for $\lambda \neq \frac{1}{2}, 1$ and $\frac{3}{2}$.

From Eq. (2.11), it is obvious that $\frac{4\lambda^2-6\lambda+1}{1-2\lambda}$ is an eigenvalue of the matrix P_G for $\lambda \neq \frac{1}{2}, 1$ and $\frac{3}{2}$. So $\frac{4\lambda(1-\lambda)}{1-2\lambda}$ is an eigenvalue of ℓ_G and $v_0 = (v_i)^T_{i\in V_O}$ is the corresponding eigenvector. Moreover, the eigenvectors v with respect to the eigenvalue λ ($\lambda \neq \frac{1}{2}, 1, \frac{3}{2}$) of ℓ_W can be completely decided by v_0 and Eqs. (2.4), (2.5) and (2.9).

Since $\frac{4\lambda(1-\lambda)}{1-2\lambda}$ is the corresponding eigenvalue of ℓ_G for any eigenvalue λ ($\lambda \neq \frac{1}{2}, 1, \frac{3}{2}$) of ℓ_W, we have $m_{\ell_G}(\frac{4\lambda(1-\lambda)}{1-2\lambda}) \geq m_{\ell_W}(\lambda)$. In fact, $m_{\ell_G}(\frac{4\lambda(1-\lambda)}{1-2\lambda}) = m_{\ell_W}(\lambda)$. Otherwise there exists at least an extra eigenvector v_0' associated to $\frac{4\lambda(1-\lambda)}{1-2\lambda}$ without a corresponding eigenvector in ℓ_W. But Eq.(2.9) gives v_0' an associated eigenvector of ℓ_W when $\lambda \neq \frac{1}{2}, 1, \frac{3}{2}$, which is a contradiction with $m_{\ell_G}(\frac{4\lambda(1-\lambda)}{1-2\lambda}) > m_{\ell_W}(\lambda)$.

By verifying, $\lambda = \frac{3}{2}$ also satisfies Eq. (2.9).

The proof is completed. □

Now, we give a complete representation about the normalized Laplacian eigenvalues and corresponding eigenvectors of $W(G)$.

Theorem 2.2 *Let G be a simple connected graph with N_0 vertices and E_0 edges and $W(G)$ be the pentagonal graph of G. The normalized Laplacian spectrum of $W(G)$ can be obtained as follows:*

(i) 0 is the eigenvalue of ℓ_W with the multiplicity 1;

(ii) If λ is any eigenvalue of ℓ_G such that $\lambda \neq 0, 2$, then both $\frac{2+\lambda+\sqrt{4+\lambda^2}}{4}$ and $\frac{2+\lambda-\sqrt{4+\lambda^2}}{4}$ are eigenvalues of ℓ_W with $m_{\ell_W}(\frac{2+\lambda+\sqrt{4+\lambda^2}}{4}) = m_{\ell_W}(\frac{2+\lambda-\sqrt{4+\lambda^2}}{4}) = m_{\ell_G}(\lambda)$;

(iii) $\frac{5\pm\sqrt{5}}{4}$ is the eigenvalue of ℓ_W with the multiplicity N_0;

(iv) 1 is the eigenvalue of ℓ_W with the multiplicity $E_0 - N_0 + 1$;

(v) $\frac{2\pm\sqrt{2}}{2}$ is the eigenvalue of ℓ_W with the multiplicity $E_0 - N_0 + m_{\ell_G}(2)$.

Proof In this proof, we continue to use the representing approach and notations of Lemma 2.1 for convenience.

(i) It is obvious from Lemma 1.1.

(ii) Let x be an eigenvalue of ℓ_W such that $x \neq \frac{1}{2}, 1$. By Lemma 2.1, we have $\lambda = \frac{4x(1-x)}{1-2x}$, for $\lambda \neq 0, 2$. Thus $x = \frac{2+\lambda\pm\sqrt{4+\lambda^2}}{4}$.

(iii) Substituting $\lambda = \frac{5+\sqrt{5}}{4}$ into Eqs. (2.7) and (2.8), we get

$$(1+\sqrt{5})v_{i_1'} = 2v_{i_3'} + \frac{-1-\sqrt{5}}{\sqrt{d_i}}v_i, \tag{2.12}$$

$$(1+\sqrt{5})v_{i_3'} = 2v_{i_1'} + \frac{-1-\sqrt{5}}{\sqrt{d_j}}v_j. \tag{2.13}$$

After substituting Eq. (2.13) into Eq. (2.12) and eliminating $v_{i_3'}$, we get

$$(1+\sqrt{5})v_{i_1'}=\frac{-1-\sqrt{5}}{\sqrt{d_j}}v_j-\frac{3+\sqrt{5}}{\sqrt{d_i}}v_i. \tag{2.14}$$

After substituting Eq.(2.14) and $\lambda=\frac{5+\sqrt{5}}{4}$ into Eq. (2.3) again, we find

$$\begin{aligned}
&\sum_{i_1'\in N_N}\frac{1}{2\sqrt{d_i}}\times\frac{1}{1+\sqrt{5}}\Big(\frac{-1-\sqrt{5}}{\sqrt{d_j}}v_j-\frac{3+\sqrt{5}}{\sqrt{d_i}}v_i\Big)+\sum_{j\in N_O}\frac{1}{2\sqrt{d_id_j}}v_j\\
&=\sum_{i_1'\in N_N}\frac{1}{2\sqrt{d_i}}\Big(\frac{-1}{d_j}v_j-\frac{3+\sqrt{5}}{1+\sqrt{5}}\times\frac{1}{\sqrt{d_i}}v_i\Big)+\sum_{j\in N_O}\frac{1}{2\sqrt{d_id_j}}v_j\\
&=\sum_{j\in N_O}\Big(\frac{-1}{2\sqrt{d_id_j}}v_j-\frac{1+\sqrt{5}}{4}\times\frac{1}{d_i}v_i\Big)+\sum_{j\in N_O}\frac{1}{2\sqrt{d_id_j}}v_j\\
&=\sum_{j\in N_O}\frac{-1-\sqrt{5}}{4d_i}v_i=\frac{-1-\sqrt{5}}{4}v_i.
\end{aligned}$$

The above equality indicates Eq. (2.3) is an identical equation when $\lambda=\frac{5+\sqrt{5}}{4}$ and Eq. (2.14) hold. Therefore, the eigenvectors associated with $\lambda=\frac{5+\sqrt{5}}{4}$ are completely determined by any v_i and v_j. We get $m_{\ell_W}(\frac{5+\sqrt{5}}{4})=N_0$.

The same theory proves $m_{\ell_W}(\frac{5-\sqrt{5}}{4})=N_0$.

(iv) Substituting $\lambda=1$ into Eq.(2.9), we get $\frac{v_i}{\sqrt{d_i}}=\frac{v_j}{\sqrt{d_j}}$, for $i\sim j$ and $i,j\in V_O$. Set $\frac{v_i}{\sqrt{d_i}}=t, i\in V_O$ because of the connectivity of G. Substituting into (2.7) and (2.3), we have

$$v_{i_1'}+v_{i_3'}=0,\quad i_1'\nsim i_3',\quad i_1',i_3'\in V_N, \tag{2.15}$$

$$\sum_{i_1'\in N_N}v_{i_1'}=-td_i,\quad i\in V_O. \tag{2.16}$$

By Eq. (2.15), we have

$$\sum_{i\in V_O}\sum_{i_1'\in N_N}v_{i_1'}=\frac{1}{2}\sum_{i_1'\nsim i_3'}(v_{i_1'}+v_{i_3'})=0,\quad i_1',i_3'\in V_N. \tag{2.17}$$

On the other hand, using Eq. (2.16), we also have

$$\sum_{i\in V_O}\sum_{i_1'\in N_N}v_{i_1'}=-t\sum_{i\in V_O}d_i=-2tm. \tag{2.18}$$

Thus, $t=0$ from Eqs. (2.17) and (2.18), i.e., $v_i=0$, for any $i\in V_O$. Therefore, the eigenvectors $v=(v_1,v_2,\ldots,v_N)^T$ with respect to $\lambda=1$ can be completely obtained by equations below

$$v_i=0,\quad i\in V_O; \tag{2.19}$$

$$\sum_{i_1'\in N_N}v_{i_1'}=0; \tag{2.20}$$

and

$$v_{i_1'}+v_{i_3'}=0. \tag{2.21}$$

In addition, due to $\lambda \neq 1$, combining Eqs. (2.21) and (2.6), we have $v_{i_2'} = 0$.

(v) Theorem 2.2 (i)–(iv) show a large fraction of the eigenvalues of ℓ_W and we conclude that we have $(1+2(N_0 - m_{\ell_G}(2)-1)+2N_0+E_0-N_0+1)$ eigenvalues of ℓ_W. The rest of the spectrum consists of $\frac{2\pm\sqrt{2}}{2}$s. According to Eq. (1.2), the number of vertices of $W(G)$ is $N = N_0 + 3E_0$. Because the sum of the number of eigenvalues is equal to the number of vertices, we have

$$N_0 + 3E_0 = (1 + 2(N_0 - m_{\ell_G}(2) - 1) + 2N_0 + E_0 - N_0 + 1) + 2m_{\ell_W}(\frac{2 \pm \sqrt{2}}{2}).$$

Simplifying this equation, we have $m_{\ell_W}(\frac{2\pm\sqrt{2}}{2}) = E_0 - N_0 + m_{\ell_G}(2)$. □

3. Some applications of spectra of $W(G)$

Let $W_0(G) = G$, $W_1(G) = W(G)$, $W_n(G) = W(W_{n-1}(G))$. The graph $W_n(G)$ is called nth pentagonal iterative graph of the initial graph G. Figure 1 also provides an example of the first two generations of the pentagonal iterative graph whose initial graph is K_2.

The number of vertices and edges of $W_n(G)$ $(n > 0)$ are denoted by N_n and E_n. From the iterative method of the pentagonal iterative graph, we have $E_n = 5E_{n-1}$, $N_n = N_{n-1} + 3E_{n-1}$ and

$$E_n = 5^n E_0, \quad N_n = N_0 + \frac{3(5^n - 1)}{4} E_0. \tag{3.1}$$

Let $f_1(x) = \frac{2+x+\sqrt{4+x^2}}{4}$, $f_1(x) = \frac{2+x-\sqrt{4+x^2}}{4}$ and A be a finite multiset of real number. We define

$$f_1(A) = \{f_1(x)|\forall x \in A\} \text{ and } f_2(A) = \{f_2(x)|\forall x \in A\}. \tag{3.2}$$

The normalized Laplacian of $W_n(G)$ is denoted by ℓ_n. The normalized Laplacian spectrum σ_n of $W_n(G)$ can be characterized directly by Theorem 2.2.

The largest eigenvalue of ℓ_n is not equal to 2 if and only if random walks on $W_n(G)$ are aperiodic. Since each edge of $W_n(G)$,$n > 0$, belongs to an odd-length cycle(a pentagon), the graph is aperiodic [16, 17]. Thus $m_{\ell_n}(2) = 0$ holds for $n > 0$. Note, however, that the value of $m_{\ell_n}(2) = 0$ depends on the structure of the initial graph G, since it may be periodic [18].

Theorem 3.1 *Let G be a simple connected graph. The normalized Laplacian spectrum σ_n of the pentagonal iterative graph $W_n(G)(n \geq 1)$ is*

$$\sigma_n = \begin{cases} f_1(\sigma_0\ \{0,2\}) \cup f_2(\sigma_0\ \{0,2\}) \cup 0 \cup \underbrace{\{1,1,\ldots,1\}}_{E_0-N_0+1} \cup \\ \underbrace{\{\frac{5\pm\sqrt{5}}{4}, \frac{5\pm\sqrt{5}}{4}, \ldots, \frac{5\pm\sqrt{5}}{4}\}}_{N_0} \cup \underbrace{\{\frac{2\pm\sqrt{2}}{2}, \frac{2\pm\sqrt{2}}{2}, \ldots, \frac{2\pm\sqrt{2}}{2}\}}_{r_1+m_{\ell_0}(2)}, & n = 1; \\ f_1(\sigma_{n-1}\ \{0\}) \cup f_2(\sigma_{n-1}\ \{0\}) \cup 0 \cup \underbrace{\{1,1,\ldots,1\}}_{E_{n-1}-N_{n-1}+1} \cup \\ \underbrace{\{\frac{5\pm\sqrt{5}}{4}, \frac{5\pm\sqrt{5}}{4}, \ldots, \frac{5\pm\sqrt{5}}{4}\}}_{N_{n-1}} \cup \underbrace{\{\frac{2\pm\sqrt{2}}{2}, \frac{2\pm\sqrt{2}}{2}, \ldots, \frac{2\pm\sqrt{2}}{2}\}}_{r_n}, & n > 1, \end{cases}$$

where $r_n = \frac{5^{n-1}+3}{4}E_0 - N_0$.

Proof The result holds by Theorem 2.2. By iteration, we concluded $(2(N_{n-1} - m_{\ell_{n-1}}(2) - m_{\ell_{n-1}}(0)) + 1 + E_{n-1} - N_{n-1} + 1 + 2N_{n-1})$ eigenvalues of ℓ_n we knew. According to Eq. (3.1), the number of vertices of $W_n(G)$ is $N_n = N_0 + \frac{3(5^n-1)}{4}E_0$. Because the sum of the number of eigenvalues is equal to the number of vertices, the multiplicity of the eigenvalue $\frac{2+\sqrt{2}}{2}$ of ℓ_n can be determined indirectly:

$$\begin{aligned}m_{\ell_n}(\frac{2\pm\sqrt{2}}{2}) &= \frac{N_r - [2(N_{n-1} - m_{\ell_{n-1}}(2) - m_{\ell_{n-1}}(0)) + 1 + E_{n-1} - N_{n-1} + 1 + 2N_{n-1}]}{2}\\ &= \frac{5^{n-1}+3}{4}E_0 - N_0 + m_{\ell_{n-1}}(2).\end{aligned}$$

Since $m_{\ell_{n-1}}(2) = 0$ for $n > 1$, the theorem is true. □

Theorem 3.2 *Let G be a simple connected graph. The multiplicative degree-Kirchhoff index $W_n(G)$ and $W_{n-1}(G)$ ($n > 1$), are related as follows:*

$$Kf^*(W_n(G)) = 20Kf^*(W_{n-1}(G)) + \frac{17\cdot 5^{2n-1} + 3\cdot 5^n}{2}E_0^2 - 2\cdot 5^n(E_0 + E_0N_0). \tag{3.3}$$

Thus, the general expression for $Kf^*(W_n(G))$ is

$$Kf^*(W_n(G)) = 20^nKf^*(G) + \frac{17(5^{2n} - 20^n) + 20^n - 5^n}{2}E_0^2 - \frac{2\cdot 5^n(2^{2n}-1)}{3}(E_0 + E_0N_0).$$

In addition,

$$Kf^*(W(G)) = 20Kf^*(G) + 50E_0^2 - 10(E_0 + E_0N_0) + 40E_0m_{\ell_0}(2), \quad n = 1.$$

Proof We denote the spectrum of $W_n(G)$ by $\sigma_n = \{\lambda_1^{(n)}, \lambda_2^{(n)}, \dots, \lambda_{N_n}^{(n)}\}$, where $0 = \lambda_1^{(n)} < \lambda_2^{(n)} \le \cdots \le \lambda_{N_n-1}^{(n)} \le \lambda_{N_n}^{(n)} \le 2$. We use Lemma 1.2 (i) and Theorem 3.1:

When $n = 1$ and $m_{\ell_{n-1}}(2) \ne 0$, we obtain:

$$\begin{aligned}Kf^*(W(G)) =& 2E_1\Big[\sum_{i=2}^{N_0}(\frac{1}{f_1(\lambda_i)} + \frac{1}{f_2(\lambda_i)}) + (E_0 - N_0 + 1) + (\frac{4}{5+\sqrt{5}} + \frac{4}{5-\sqrt{5}})\cdot N_0 +\\ &(\frac{2}{2+\sqrt{2}} + \frac{2}{2-\sqrt{2}})\cdot(E_0 - N_0 + m_{\ell_0}(2))\Big]\\ =& 2E_1\Big[\sum_{i=2}^{N_0}(2 + \frac{4}{\lambda_i}) + (E_0 - N_0 + 1) + 2N_0 + 4(E_0 - N_0 + m_{\ell_0}(2))\Big]\\ =& 10E_0\sum_{i=2}^{N_0}(2 + \frac{4}{\lambda_i}) + 10E_0(5E_0 - 3N_0 + 1 + m_{\ell_0}(2))\\ =& 20Kf^*(G)) + 50E_0^2 - 10(E_0 + E_0N_0) + 40E_0m_{\ell_0}(2).\end{aligned}$$

When $n > 1$ and $m_{\ell_{n-1}}(2) = 0$, we obtain:

$$\begin{aligned}Kf^*(W_n(G)) =& 2E_n\Big[\sum_{i=2}^{N_{n-1}}(\frac{1}{f_1(\lambda_i^{(n-1)})} + \frac{1}{f_2(\lambda_i^{(n-1)})}) + (E_{n-1} - N_{n-1} + 1) +\\ &(\frac{4}{5+\sqrt{5}} + \frac{4}{5-\sqrt{5}})\cdot N_{n-1} + (\frac{2}{2+\sqrt{2}} + \frac{2}{2-\sqrt{2}})\cdot(\frac{5^{n-1}+3}{4}E_0 - N_0)\Big]\end{aligned}$$

$$=10E_{n-1}\Big[\sum_{i=2}^{N_{n-1}}(2+\frac{4}{\lambda_i^{(n-1)}})+(5^{n-1}E_0-N_0-\frac{3(5^{n-1}-1)}{4}E_0+1)+$$
$$2(N_0+\frac{3(5^{n-1}-1)}{4}E_0)+4(\frac{(5^{n-1}+3)}{4}E_0-N_0)\Big]$$
$$=20Kf^*(W_{n-1}(G))+\frac{17\cdot 5^{2n-1}+3\cdot 5^n}{2}E_0^2-2\cdot 5^n(E_0+E_0N_0).$$

From Eq. (3.3) and the definition of the pentagonal iterative graph we can get the recursive relation

$$Kf^*(W_n(G))=20^nKf^*(G)+\frac{17(5^{2n}-20^n)+20^n-5^n}{2}E_0^2-\frac{2\cdot 5^n(2^{2n}-1)}{3}(E_0+E_0N_0).\ \square$$

Theorem 3.3 *The Kemeny's constant for random walks on $W_n(G)$ can be obtained from $K(W_{n-1}(G))$ $(n>1)$ through*

$$K(W_n(G))=4K(W_{n-1}(G))+\frac{17\cdot 5^{n-1}+3}{4}E_0-N_0-1.$$

The general expression is

$$K(W_n(G))=4^nK(G)+\frac{17(5^n-4^n)+(4^n-1)}{4}E_0-\frac{4^n-1}{3}(N_0+1).$$

In addition, when $n=1$ and $m_{\ell_{n-1}}(2)\neq 0$, we obtain:

$$K(W(G))=4K(W(G))+5E_0-N_0-1+4m_{\ell_0}(2).$$

Proof This result is a direct consequence of Theorem 3.2 and Eq. (1.1). □

Theorem 3.4 *The number of spanning trees of $W(G)$ is*

$$N_{st}(W_n(G))=5^{N_{n-1}-1}\cdot 2^{N_{n-1}-3N_0+\frac{8-5^{n-1}}{4}E_0+2}N_{st}(W_{n-1}(G)).$$

In general, $N_{st}(W_n(G))$ can be expressed by

$$N_{st}(W_n(G))=5^{\sum_{i=0}^{n-1}N_i-n}\cdot 2^{\sum_{i=0}^{n-1}N_i-\frac{5^n-1}{16}E_0+n(2E_0-3N_0+2)}N_{st}(G).$$

Proof From Lemma 1.2 (iii) and the definition of pentagonal of a graph:

$$\frac{N_{st}(W_n(G))}{N_{st}(W_{n-1}(G))}=\frac{2^{N_n}}{5}\cdot\frac{\prod_{i=2}^{N_n}\lambda_i^{(n)}}{\prod_{i=2}^{N_{n-1}}\lambda_i^{(n-1)}},$$

where $\lambda_i^{(n)}$ are the eigenvalues of ℓ_n. We obtain, for $n>0$:

$$\prod_{i=2}^{N_n}\lambda_i^{(n)}=(\frac{5+\sqrt{5}}{4}\cdot\frac{5-\sqrt{5}}{4})^{N_{n-1}}\cdot(\frac{2+\sqrt{2}}{2}\cdot\frac{2-\sqrt{2}}{2})^{\frac{5^{n-1}+3}{4}E_0-N_0}\cdot$$
$$\prod_{i=2}^{N_{n-1}}\left(f_1(\lambda_i^{(n-1)})\cdot f_2(\lambda_i^{(n-1)})\right)$$
$$=(\frac{5}{4})^{N_{n-1}}\cdot(\frac{1}{2})^{\frac{5^{n-1}+3}{4}E_0-N_0}\cdot\prod_{i=2}^{N_{n-1}}\frac{\lambda_i^{(n-1)}}{4}.$$

Therefore, the following equality

$$\begin{aligned}N_{st}(W_n(G)) =&\frac{2^{N_n}}{5}\cdot\frac{(\frac{5}{4})^{N_{n-1}}\cdot(\frac{1}{2})^{\frac{5^{n-1}+3}{4}E_0-N_0}\cdot\prod_{i=2}^{N_{n-1}}\frac{\lambda_i^{(n-1)}}{4}}{\prod_{i=2}^{N_{n-1}}\lambda_i^{(n-1)}}N_{st}\big(W_{n-1}(G)\big)\\ =&5^{N_{n-1}-1}\cdot 2^{N_{n-1}-3N_0+\frac{8-5^{n-1}}{4}E_0+2}N_{st}\big(W_{n-1}(G)\big)\end{aligned}$$

holds for any $n>0$, and finally we have:

$$N_{st}\big(W_n(G)\big)=5^{\sum_{i=0}^{n-1}N_i-n}\cdot 2^{\sum_{i=0}^{n-1}N_i-\frac{5^n-1}{16}E_0+n(2E_0-3N_0+2)}N_{st}(G).\ \square$$

References

[1] A. DASGUPTA, J. E. HOPCROFT, F. MCSHERRY. *Spectral analysis of random graphs with skewed degree distributions*. in: Proceedings of the Forty Fifth Annual IEEE Symposium on Foundations of Computer Science, 2004, 602–610.

[2] D. A. SPIELMAN. *Spectral graph theory and its applications*. in: Proceedings of the Forty Eighth Annual IEEE Symposium on Foundations of Computer Science, 2007, 29–38.

[3] L. V. TRAN, V. H. VU, Ke WANG. *Sparse random graphs: Eigenvalues and eigenvectors*. Random Structures and Algorithms, 2012, **42**(1): 110–134.

[4] D.CVETKOVIĆ, S. SIMIĆ. *Graph spectra in computer science*. Linear Algebra Appl., 2011, **434**(6): 1545–1562.

[5] B. ARSIĆ, D. CVETKOVIĆ, S. K. SIMIĆ, et al. *Graph spectral techniques in computer sciences*. Appl. Anal. Discrete Math., 2012, **6**(1): 1–30.

[6] L. LOVÁSZ, L. LOV, O. P. ERDOS. *Random walks on graphs: a survey*. Combinatorics, 1993, **8**(4): 1–46.

[7] Zhongzhi ZHANG, Tong SHAN, Guanrong CHAN. *Random walks on weighted networks*. Physical Review E Statistical Nonlinear and Soft Matter Physics, 2013, **87**(1): 012112.

[8] J. G. KEMENY, J. L. SNELL. *Finite Markov Chains*. Springer, New York, 1976.

[9] M. LEVENE, G. LOIZOU. *Kemeny's constant and the random surfer*. Amer. Math. Monthly, 2002, **109**(8): 741–745.

[10] F. R. K. CHUNG. *Spectral Graph Theory*. American Mathematical Society, 1997.

[11] Haiyan CHEN, Fuji ZHANG. *Resistance distance and the normalized laplacian spectrum*. Discrete Appl. Math., 2007, **155**(5): 654–661.

[12] S. BUTLER. *Algebraic Aspects of the Normalized Laplacian*. Springer, 2016.

[13] Pinchen XIE, Zhongzhi ZHANG, F. COMELLAS. *On the spectrum of the normalized Laplacian of iterated triangulations of graphs*. Appl. Math. Comput., 2015, **273**: 1123–1129.

[14] Pinchen XIE, Zhongzhi ZHANG, F. COMELLAS. *The normalized Laplacian spectrum of subdivisions of a graph*. Appl. Math. Comput., 2016, **286**: 250–256.

[15] Deqiong LI, Yaoping HOU. *The normalized Laplacian spectrum of quadrilateral graphs and its applications*. Appl. Math. Comput., 2017, **297**: 180–188.

[16] P. DIACONIS, D. STROOCK. *Geometric bounds for eigenvalues of Markov chains*. Ann. Appl. Probab., 1991, **1**(1): 36–61.

[17] L. TIERNEY. *Introduction to General State-Space Markov Chain Theory*. Markov Chain Monte Carlo in Practice, 1996.

[18] Xiaodong ZHANG. *The smallest eigenvalue for reversible Markov chains*. Linear Algebra Appl., 2004, **383**: 175–186.