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Brauer Upper Bound for the Z-Spectral Radius of Nonnegative Tensors

Jun HE*, Hua KE, Yanmin LIU, Junkang TIAN
School of Mathematics, Zunyi Normal College, Guizhou 563006*, P. R. China*

Abstract In this paper, we have proposed an upper bound for the largest Z-eigenvalue of an irreducible weakly symmetric and nonnegative tensor, which is called the Brauer upper bound:

$$\rho_Z(\mathcal{A}) \le \frac{1}{2} \max_{\substack{i,j\in N\\ j\neq i}} \Big(a_{i\cdots i} + a_{j\cdots j} + \sqrt{(a_{i\cdots i} - a_{j\cdots j})^2 + 4r_i(\mathcal{A})r_j(\mathcal{A})} \Big),$$

where $r_i(\mathcal{A}) = \sum\limits_{ii_2\cdots i_m \neq ii\cdots i} a_{ii_2\cdots i_m}$, $i, i_2, \ldots, i_m \in N = \{1, 2, \ldots, n\}$. As applications, a bound on the Z-spectral radius of uniform hypergraphs is presented.

Keywords bound; nonnegative tensor; Z-eigenvalue; hypergraph

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1. Introduction

Let $\mathbb{R}$ be the real field. An m-th order n dimensional square tensor $\mathcal{A}$ consists of n^m entries in $\mathbb{R}$, which is defined as follows:

$$\mathcal{A} = (a_{i_1 i_2 \cdots i_m}),\ a_{i_1 i_2 \cdots i_m} \in \mathbb{R},\quad 1 \le i_1, i_2, \ldots, i_m \le n.$$

$\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is called nonnegative, denoted by $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$, if each of its entries $a_{i_1 i_2 \cdots i_m} \ge 0$. For an n-vector x, real or complex, we define the n-vector:

$$\mathcal{A}x^{m-1} = \Big(\sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} \Big)_{1\le i\le n}$$

and

$$x^{[m-1]} = (x_i^{m-1})_{1\le i\le n}.$$

If $\mathcal{A}x^{m-1} = \lambda x^{[m-1]}$, x and λ are all real, then λ is called an H-eigenvalue of $\mathcal{A}$ and x an H-eigenvector of $\mathcal{A}$ associated with λ. If $\mathcal{A}x^{m-1} = \lambda x$ and $x^T x = 1$, x and λ are all real, then λ is called a Z-eigenvalue of $\mathcal{A}$ and x a Z-eigenvector of $\mathcal{A}$ associated with λ (see [1]). See more

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* Corresponding author
E-mail address: hejunfan1@163.com (Jun HE)

about the eigenvalue problems of tensors in [2–9]. Let $N=\{1,2,\ldots,n\}$. A real tensor of order m dimension n is called the unit tensor, if its entries are $\delta_{i_1\cdots i_m}$ for $i_1,\ldots,i_m\in N$, where

$$\delta_{i_1\cdots i_m}=\begin{cases}1, & \text{if } i_1=\cdots=i_m,\\ 0, & \text{otherwise.}\end{cases}$$

Let $\mathcal{H}$ be a hypergraph with vertex set $V(\mathcal{H})=[n]:=\{1,2,\ldots,n\}$ and edge set $E(\mathcal{H})$. If $|e|=k$ for $e\in E(\mathcal{H})$, then we say that $\mathcal{H}$ is a k-uniform hypergraph. In this paper, we consider k-uniform hypergraphs on n vertices with $2\le k\le n$. For $i\in[n]$, E_i denotes the set of edges of $\mathcal{H}$ containing i. The degree of a vertex i in $\mathcal{H}$ is defined as $d_i=|E_i|$. If $d_i=d$ for $i\in V(\mathcal{H})$, then $\mathcal{H}$ is called a regular hypergraph (of degree d). For $i,j\in V(\mathcal{H})$, if there is a sequence of edges $e_1,\ldots,e_r$ such that $i\in e_1$, $j\in e_r$ and $e_s\cap e_{s+1}\neq\emptyset$ for all $s\in[r-1]$, then we say that i and j are connected. A hypergraph is connected if every pair of different vertices of $\mathcal{H}$ is connected.

The adjacency tensor of $\mathcal{H}$ is defined as the k-th order n-dimensional tensor $\mathcal{A}(\mathcal{H})$ whose $(i_1\cdots i_k)$-entry is:

$$(\mathcal{A}(\mathcal{H}))_{i_1i_2\cdots i_k}=\begin{cases}\frac{1}{(k-1)!}, & \text{if } \{i_1,\ldots,i_k\}\in E(\mathcal{H}),\\ 0, & \text{otherwise.}\end{cases}$$

When $m=2$, the well-known Frobenius upper bound for the Perron root $\rho(A)$ of a nonnegative $n\times n$ matrix $A=(a_{ij})$ is introduced in [10, 11]:

$$\rho(A)\le\max_{i\in N}\sum_{j=1}^{n}a_{ij}.$$

By Brauer's theorem [12], Brauer and Gentry [13] derived the following improved Brauer upper bound for the Perron root $\rho(A)$ of a nonnegative $n\times n$ matrix $A=(a_{ij})$:

$$\rho(A)\le\frac{1}{2}\max_{\substack{i,j\in N\\ j\neq i}}\Big(a_{ii}+a_{jj}+\sqrt{(a_{ii}-a_{jj})^2+4R_i(A)R_j(A)}\Big),$$

where $R_i(A)=\sum_{j=1}^{n}a_{ij}-a_{ii}$.

When $m>2$, the Frobenius upper bound can be extended to establish the largest H-eigenvalue or Z-eigenvalue of a nonnegative tensor $\mathcal{A}$ (see [3, 14]). Then, we ask that, whether the Brauer upper bound can be generalized to the largest H-eigenvalue or Z-eigenvalue of a nonnegative tensor $\mathcal{A}$? Unfortunately, the answer is negative for the largest H-eigenvalue. The following example is given to show that the Brauer upper bound cannot be generalized to the largest H-eigenvalue of a nonnegative tensor $\mathcal{A}$.

Example 1.1 Let $\mathcal{A}=(a_{ijkl})$ be an order 4 dimension 2 tensor with entries defined as follows:

$$a_{1111}=7,\quad a_{1112}=a_{1211}=a_{1121}=a_{2111}=10,$$

$$a_{2222}=6,\quad a_{2221}=a_{2212}=a_{2122}=a_{1222}=1,$$

other $a_{ijkl}=0$. Now, let

$$\tau(\mathcal{A})=\frac{1}{2}\max_{\substack{i,j\in N\\ j\neq i}}\Big(a_{i\cdots i}+a_{j\cdots j}+\sqrt{(a_{i\cdots i}-a_{j\cdots j})^2+4r_i(\mathcal{A})r_j(\mathcal{A})}\Big)=26.5811,$$

where $r_i(\mathcal{A}) = \sum_{\delta_{ii_2\cdots i_m}=0} a_{ii_2\cdots i_m}$. In fact, the largest H-eigenvalue $\rho_H(\mathcal{A}) = 30.8865 > \tau(\mathcal{A})$. Hence, the Brauer upper bound cannot be generalized to the largest H-eigenvalue of a nonnegative tensor $\mathcal{A}$.

Let $\mathcal{A}$ be an m-order and n-dimensional tensor. We define $\sigma(\mathcal{A})$ the Z-spectrum of $\mathcal{A}$ by the set of all Z-eigenvalues of $\mathcal{A}$. Assume $\sigma(\mathcal{A}) \neq \emptyset$, then the Z-spectral radius of $\mathcal{A}$ is denoted by

$$\rho_Z(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.$$

In this paper, we will show that the Brauer upper bound still holds true for the largest Z-eigenvalue of a nonnegative tensor $\mathcal{A}$, that is

$$\rho_Z(\mathcal{A}) \leq \frac{1}{2} \max_{\substack{i,j\in N \\ j\neq i}} \Big(a_{i\cdots i} + a_{j\cdots j} + \sqrt{(a_{i\cdots i} - a_{j\cdots j})^2 + 4r_i(\mathcal{A})r_j(\mathcal{A})} \Big).$$

As applications, a new bound on the Z-spectral radius of uniform hypergraphs is presented.

2. Preliminaries

The following definition for irreducibility has been introduced in [15].

Definition 2.1 *The squre tensor $\mathcal{A}$ is called reducible if there exists a nonempty proper index subset $\mathbb{J} \subset \{1, 2, \ldots, n\}$ such that $a_{i_1,i_2,\ldots,i_m} = 0$, $\forall i_1 \in \mathbb{J}$, $\forall i_2, \ldots, i_m \notin \mathbb{J}$. If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ irreducible.*

In [16], Chang, Pearson and Zhang gave the following bound for the Z-eigenvalues of an m-order n-dimensional tensor $\mathcal{A}$.

Theorem 2.2 *Let $\mathcal{A}$ be an m-order and n-dimensional tensor. Then*

$$\rho_Z(\mathcal{A}) \leq \sqrt{n} \max_{i\in N} \sum_{i_2,\ldots,i_m=1}^{n} |a_{ii_2\cdots i_m}|. \tag{2.1}$$

For the positively homogeneous operators, Song and Qi [14] studied the relationship between the Gelfand formula and the spectral radius as well as the upper bound of the spectral radius. From [14, Corollary 4.5], we can get the following result:

Theorem 2.3 *Let $\mathcal{A}$ be an m-order and n-dimensional tensor. Then*

$$\rho_Z(\mathcal{A}) \leq \max_{i\in N} \sum_{i_2,\ldots,i_m=1}^{n} |a_{ii_2\cdots i_m}|. \tag{2.2}$$

A tensor $\mathcal{A}$ is called weakly symmetric if the associated homogeneous polynomial $\mathcal{A}x^m$ satisfies

$$\nabla \mathcal{A}x^m = m\mathcal{A}x^{m-1}.$$

If the tensor is positive, He and Huang gave the following Z-eigenpair bound [17, Theorem 2.7]:

Theorem 2.4 *Suppose that $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}_+^{[m,n]}$ is an irreducible weakly symmetric tensor. Then*

$$\rho_Z(\mathcal{A}) \leq R - l(1-\theta), \tag{2.3}$$

where $R_i=\sum_{i_2,\ldots,i_m=1}^n |a_{ii_2\cdots i_m}|$,

$$R=\max_{i\in N} R_i,\ r=\min_{i\in N} R_i,\quad l=\min_{i_1,\ldots,i_m} a_{i_1\cdots i_m},\quad \theta=\{\frac{r}{R}\}^{\frac{1}{m}}.$$

Li, Liu and Vong obtained the following upper bound [18, Theorem 3.5]:

Theorem 2.5 *Suppose that* $\mathcal{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{R}_+^{[m,n]}$ *is an irreducible weakly symmetric tensor. Then*

$$\rho_Z(\mathcal{A})\le\max_{i,j}\{R_i+a_{ij\cdots j}(\delta^{-\frac{m-1}{m}}-1)\},\tag{2.4}$$

where

$$\delta=\frac{\min_{i,j} a_{ij\cdots j}}{r-\min_{i,j} a_{ij\cdots j}}(\gamma^{\frac{m-1}{m}}-\gamma^{\frac{1}{m}})+\gamma,\quad \gamma=\frac{R-\min_{i,j} a_{ij\cdots j}}{r-\min_{i,j} a_{ij\cdots j}}.$$

And we define

$$r_i(\mathcal{A})=\sum_{\delta_{ii_2\cdots i_m}=0}|a_{ii_2\cdots i_m}|,\quad r_i^j(\mathcal{A})=\sum_{\substack{\delta_{ii_2\cdots i_m}=0,\\ \delta_{ji_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|=r_i(\mathcal{A})-|a_{ij\cdots j}|.$$

The following upper bound was given in [19, Theorem 3.5]:

Theorem 2.6 *Suppose that* $\mathcal{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{R}_+^{[m,n]}$ *is an irreducible weakly symmetric tensor. Then*

$$\rho_Z(\mathcal{A})\le\max_{i,j\in N,j\ne i}\frac{1}{2}\{a_{i\cdots i}+a_{j\cdots j}+r_i^j(\mathcal{A})+\Theta_{i,j}^{\frac{1}{2}}(\mathcal{A})\},\tag{2.5}$$

where

$$\Theta_{i,j}(\mathcal{A})=(a_{i\cdots i}-a_{j\cdots j}+r_i^j(\mathcal{A}))^2+4a_{ij\cdots j}r_j(\mathcal{A}).$$

3. Main results

In this section, we consider a new upper bound for the largest Z-eigenvalue of a nonnegative tensor. In [16], Chang, Pearson and Zhang presented the following Perron-Frobenius Theorem for the Z-eigenvalue of nonnegative tensors.

Lemma 3.1 *Suppose that* $\mathcal{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{R}_+^{[m,n]}$ *is an irreducible weakly symmetric tensor. Then the spectral radius* $\rho_Z(\mathcal{A})$ *is a positive Z-eigenvalue with a positive Z-eigenvector.*

And a lower bound on $\rho_Z(\mathcal{A})$ is given as follows [16].

Lemma 3.2 *Suppose that* $\mathcal{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{R}_+^{[m,n]}$ *is an irreducible weakly symmetric tensor. Then* $\rho_Z(\mathcal{A})\ge a_{i\cdots i}$, *for any* $1\le i\le n$.

Based on the Lemmas, we give our main results as follows.

Theorem 3.3 (Brauer upper bound) *Suppose that* $\mathcal{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{R}_+^{[m,n]}$ *is an irreducible weakly symmetric tensor. Then*

$$\rho_Z(\mathcal{A})\le\omega=\frac{1}{2}\max_{\substack{i,j\in N\\ j\ne i}}\Big(a_{i\cdots i}+a_{j\cdots j}+\sqrt{(a_{i\cdots i}-a_{j\cdots j})^2+4r_i(\mathcal{A})r_j(\mathcal{A})}\Big).$$

Proof First, let $x=(x_1,\ldots,x_n)^T$ be a Z-eigenvector of $\mathcal{A}$ corresponding to $\rho_Z(\mathcal{A})$, that is,

$$\mathcal{A}x^{m-1}=\rho_Z(\mathcal{A})x,\quad x^Tx=1. \tag{3.1}$$

Assume $x_t=\max_{i\in N}x_i$, $x_s=\max_{i\in N,i\neq t}x_i$, then, $x_s^{m-1}\le x_s$, we can get

$$\rho_Z(\mathcal{A})x_t=a_{t\cdots t}x_t^{m-1}+\sum_{\delta_{ti_2\cdots i_m}=0}a_{ti_2\cdots i_m}x_{i_2}\cdots x_{i_m}. \tag{3.2}$$

By using $x_t^{m-1}\le x_t$, $x_t^{m-2}\le 1$, we can get,

$$\begin{aligned}\rho_Z(\mathcal{A})x_t&\le a_{t\cdots t}x_t^{m-1}+\sum_{\delta_{ti_2\cdots i_m}=0}a_{ti_2\cdots i_m}x_t^{m-2}x_s\\&\le a_{t\cdots t}x_t+\sum_{\delta_{ti_2\cdots i_m}=0}a_{ti_2\cdots i_m}x_s.\end{aligned} \tag{3.3}$$

Similarly, we can get

$$\begin{aligned}\rho_Z(\mathcal{A})x_s&=a_{s\cdots s}x_s^{m-1}+\sum_{\delta_{si_2\cdots i_m}=0}a_{si_2\cdots i_m}x_{i_2}\cdots x_{i_m}\\&\le a_{s\cdots s}x_s^{m-1}+\sum_{\delta_{si_2\cdots i_m}=0}a_{si_2\cdots i_m}x_t^{m-1}\\&\le a_{s\cdots s}x_s+\sum_{\delta_{si_2\cdots i_m}=0}a_{si_2\cdots i_m}x_t.\end{aligned} \tag{3.4}$$

From Lemma 3.2, we have

$$\rho_Z(\mathcal{A})-a_{i\cdots i}\ge 0,\quad i=1,\ldots,n.$$

Then, by (3.3) and (3.4), we obtain

$$(\rho_Z(\mathcal{A})-a_{t\cdots t})(\rho_Z(\mathcal{A})-a_{s\cdots s})\le r_t(\mathcal{A})r_s(\mathcal{A}). \tag{3.5}$$

Therefore,

$$\rho_Z(\mathcal{A})\le\frac{1}{2}\Big(a_{t\cdots t}+a_{s\cdots s}+\sqrt{(a_{t\cdots t}-a_{s\cdots s})^2+4r_t(\mathcal{A})r_s(\mathcal{A})}\Big).$$

Then,

$$\rho_Z(\mathcal{A})\le\frac{1}{2}\max_{\substack{i,j\in N\\ j\neq i}}\Big(a_{i\cdots i}+a_{j\cdots j}+\sqrt{(a_{i\cdots i}-a_{j\cdots j})^2+4r_i(\mathcal{A})r_j(\mathcal{A})}\Big).$$

Thus, we complete the proof. □

We now compare the upper bound in Theorems 3.3 with that in Theorem 2.3.

Theorem 3.4 *Suppose that $\mathcal{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{R}_+^{[m,n]}$ is an irreducible weakly symmetric tensor. Then*

$$\omega\le\max_{i\in N}\sum_{i_2,\ldots,i_m=1}^{n}a_{ii_2\cdots i_m}.$$

Proof For any $i,j\in N$, $j\neq i$, assume that

$$\sum_{i_2,\ldots,i_m=1}^{n}a_{ii_2\cdots i_m}\le\sum_{i_2,\ldots,i_m=1}^{n}a_{ji_2\cdots i_m}.$$

Then

$$0 \le r_i(\mathcal{A}) \le r_j(\mathcal{A}) + a_{j\cdots j} - a_{i\cdots i}.$$

Hence,

$$\begin{aligned}(a_{j\cdots j} - a_{i\cdots i})^2 + 4r_i(\mathcal{A})r_j(\mathcal{A}) &\le (a_{j\cdots j} - a_{i\cdots i})^2 + 4(r_j(\mathcal{A}) - a_{j\cdots j} - a_{i\cdots i})r_j(\mathcal{A})\\ &= (a_{j\cdots j} - a_{i\cdots i} + 2r_j(\mathcal{A}))^2.\end{aligned}$$

Furthermore,

$$\begin{aligned}&a_{j\cdots j} + a_{i\cdots i} + \sqrt{(a_{j\cdots j} - a_{i\cdots i})^2 + 4r_i(\mathcal{A})r_j(\mathcal{A})} \le a_{j\cdots j} + a_{i\cdots i} + a_{j\cdots j} - a_{i\cdots i} + 2r_j(\mathcal{A})\\ &= 2a_{j\cdots j} + 2r_j(\mathcal{A}) = \sum_{i_2,\ldots,i_m=1}^{n} a_{ji_2\cdots i_m}\end{aligned}$$

which implies

$$\omega = \frac{1}{2}\max_{\substack{i,j\in N\\ j\neq i}} \Big(a_{i\cdots i} + a_{j\cdots j} + \sqrt{(a_{i\cdots i} - a_{j\cdots j})^2 + 4r_i(\mathcal{A})r_j(\mathcal{A})}\Big) \le \max_{i\in N} \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m}.$$

Thus, we complete the proof. □

Remark 3.5 From Theorem 3.4, we know that, the upper bound ω is tight and sharper than those in Theorems 2.2 and 2.3. And it is difficult to compare the upper bound ω with the results in Theorems 2.4–2.6. We will research this problem in the future. But, if $a_{ij\cdots j} = 0$ for all $i \in N$, then the upper bounds in Theorems 2.4–2.6 reduce to $\max_{i\in N} \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m}$, which means that, the upper bound ω is sharper than the results in Theorems 2.4–2.6 in some cases.

Example 3.6 We now show the efficiency of the new upper bound in Theorem 3.3 by the following example. Consider the tensor $\mathcal{A} = (a_{ijk})$ of order 3 dimension 3 with entries defined as follows:

$$a_{111} = \frac{1}{2},\quad a_{222} = 1,\quad a_{333} = 3,\quad \text{and}\quad a_{ijk} = \frac{1}{3}\quad \text{elsewhere}.$$

By Theorem 2.2, we have $\rho_Z(\mathcal{A}) \le 9.8150$.

By Theorem 2.3, we have $\rho_Z(\mathcal{A}) \le 5.6667$.

By Theorem 2.4, we have $\rho_Z(\mathcal{A}) \le 5.6079$.

By Theorem 2.5, we have $\rho_Z(\mathcal{A}) \le 5.5494$.

By Theorem 2.6, we have $\rho_Z(\mathcal{A}) \le 5.5296$.

By Theorem 3.3, we have $\rho_Z(\mathcal{A}) \le 4.8480$.

In fact $\rho_Z(\mathcal{A}) = 3.1970$. This example shows that the bound in Theorem 3.3 is the best.

4. Application to uniform hypergraphs

Let $|\mathcal{A}|$ mean that $(|\mathcal{A}|)_{i_1\cdots i_k} = |a_{i_1\cdots i_k}|$. We need the following Lemmas.

Lemma 4.1 ([20]) *Let $\mathcal{A}$ and $\mathcal{B}$ be two weakly symmetric and irreducible tensors of order m and dimension n. If $\mathcal{B}$ and $\mathcal{B} - |\mathcal{A}|$ are nonnegative, then $\rho_Z(\mathcal{B}) \ge \rho_Z(\mathcal{A})$.*

Lemma 4.2 ([20]) *Let $\{\mathcal{A}_k\}$ be a sequence of nonnegative, weakly symmetric and irreducible tensors of order m and dimension n, and $\mathcal{A}_k - \mathcal{A}_{k+1}$ is nonnegative for each positive integer k. Then*

$$\lim_{k\to\infty} \rho_Z(\mathcal{A}_k) = \rho_Z(\lim_{k\to\infty} \mathcal{A}_k).$$

Now we give a new upper bound for the largest Z-eigenvalues $\rho_Z(\mathcal{H})$ of the adjacency tensors for uniform hypergraphs.

Theorem 4.3 *Let $\mathcal{H}$ be a k-uniform hypergraph on n vertices. Then*

$$\rho_Z(\mathcal{H}) \le \max_{e\in E(H)} \max_{\{i,j\}\in e} \sqrt{d_i d_j}. \tag{4.1}$$

Proof Case I. $\mathcal{A}(\mathcal{H})$ is irreducible. In this case, by Lemma 3.1, there exists a positive eigenvector corresponding to the spectral radius $\rho_Z(\mathcal{H})$. Then, by Theorem 3.3, we have

$$\rho_Z(\mathcal{H}) \le \max_{e\in E(H)} \max_{\{i,j\}\in e} \sqrt{d_i d_j}.$$

Case II. $\mathcal{A}(\mathcal{H})$ is reducible. Let $\mathcal{A}_k(\mathcal{H}) = \mathcal{A}(\mathcal{H}) + \frac{1}{k}\mathcal{T}$, where $\mathcal{T}$ is an irreducible tensor whose diagonal entries are zero. By Lemmas 3.1 and 4.2, the inequality (4.1) also holds. □

For a k-uniform hypergraph $\mathcal{H}$, let $\Delta = d_1 \ge \cdots \ge d_n = \delta$ be the degree sequence of $\mathcal{H}$. In 2013, Xie and Chang [21] presented the following upper bound for the largest Z-eigenvalues $\rho_Z(\mathcal{H})$ of the adjacency tensors:

$$\rho_Z(\mathcal{H}) \le \Delta. \tag{4.2}$$

We now show the efficiency of the new upper bound in Theorem 4.3 by the following examples.

Example 4.4 Consider 3-uniform hypergraph $\mathcal{G}_1$ with vertex set $V(\mathcal{G}_1) = \{1,2,3,4,5,6\}$ and edge set $E(\mathcal{G}_1) = \{e_1, e_2, e_3\}$, where $e_1 = \{1,2,3\}$, $e_2 = \{1,2,4\}$, $e_3 = \{1,5,6\}$.

Example 4.5 Consider 3-uniform hypergraph $\mathcal{G}_2$ with vertex set $V(\mathcal{G}_2) = \{1,2,3,4,5,6,7\}$ and edge set $E(\mathcal{G}_2) = \{e_1, e_2, e_3, e_4\}$, where $e_1 = \{1,6,7\}$, $e_2 = \{2,6,7\}$, $e_3 = \{3,6,7\}$, $e_4 = \{4,5,7\}$.

	(11)	(12)
G_1	$\sqrt{6}$	3
G_2	$\sqrt{12}$	4

Table 1 Upper bounds for the hypergraphs $\mathcal{G}_1$ and $\mathcal{G}_2$

From Table 1, we can find that, the bound (4.1) is always better than (4.2).

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