Journal of Mathematical Research with Applications
Jul., 2019, Vol. 39, No. 4, pp. 361–377
DOI:10.3770/j.issn:2095-2651.2019.04.004
Http://jmre.dlut.edu.cn

Third Hankel Determinant for Ma-Minda Bi-univalent Functions

Haiyan ZHANG, Huo TANG*
School of Mathematics and Computer Science, Chifeng University, Inner Mongolia 024000*, P. R. China*

Abstract In this paper, we investigate the third Hankel determinant $H_3(1)$ for the class $H^{\mu}_{\sigma}(\lambda,\varphi)$ $(\lambda\geq 1,\ \mu\geq 1)$ of Ma-Minda bi-univalent functions in the open unit disk $\mathbb{D}=\{z:|z|<1\}$ and obtain the upper bound of the above determinant $H_3(1)$.

Keywords analytic function; Hankel determinant; Ma-Minda bi-univalent function; upper bound

MR(2010) Subject Classification 30C45; 30C50; 30C80

1. Introduction

Let $\mathbb{C}$ be a set of complex numbers and $\mathcal{A}$ be a class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$. Also, assume that $\mathcal{S}$ be a subclass of all functions in $\mathcal{A}$ which are univalent in $\mathbb{D}$ (see [1]).

Because univalent functions are one-to-one and invertible, and so they need not be defined on the entire unit disk $\mathbb{D}$. However, the famous Koebe one-quarter theorem [1] ensures that the image of the unit disk $\mathbb{D}$ under every function $f\in\mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus, every univalent function $f\in\mathcal{S}$ has an inverse f^{-1} satisfying

$$f^{-1}(f(z))=z,\quad z\in\mathbb{D}$$

and

$$f(f^{-1}(\omega))=\omega,\quad |\omega|<r_0(f);\ r_0(f)\geq\frac{1}{4},$$

where

$$f^{-1}(\omega)=\omega-a_2\omega^2+(2a_2^2-a_3)\omega^3-(5a_2^3-5a_2a_3+a_4)\omega^4+\cdots. \tag{1.2}$$

Received September 5, 2018; Accepted October 26, 2018
Supported by the National Natural Science Foundation of China (Grant Nos. 11561001; 11271045), the Program for Young Talents of Science and Technology in Universities of Inner Mongolia Autonomous Region (Grant No. NJYT-18-A14), the Natural Science Foundation of Inner Mongolia (Grant No. 2018MS01026), the Higher School Foundation of Inner Mongolia (Grant Nos. NJZY17300; NJZY17301) and the Natural Science Foundation of Chifeng of Inner Mongolia.
* Corresponding author
E-mail address: cfxyzhhy@163.com (Haiyan ZHANG); thth2009@163.com (Huo TANG)

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. Now let σ denote the class of bi-univalent functions defined in $\mathbb{D}$.

Again, let $\mathcal{P}$ denote the class of analytic functions p normalized by

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

and satisfying the condition $\operatorname{Re} p(z) > 0 \ \ (z \in \mathbb{D})$.

It is easy to see that, if $p(z) \in \mathcal{P}$, then there exists a Schwarz function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that [2]

$$p(z) = \frac{1 + w(z)}{1 - w(z)}, \quad z \in \mathbb{D}.$$

Now, we begin with recalling the definition of subordination.

Suppose that f and g are two analytic functions in $\mathbb{D}$. Then, we say that the function g is subordinate to the function f, and we write

$$g(z) \prec f(z), \quad z \in \mathbb{D},$$

if there exists a Schwarz function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that [3]

$$g(z) = f(\omega(z)), \quad z \in \mathbb{D}.$$

Recently, Tang et al. [4] introduced the following subclass $H_\sigma^\mu(\lambda, \varphi)$ of Ma-Minda bi-univalent functions.

Definition 1.1 *A function $f \in \sigma$ given by (1.1) is said to be in the class $H_\sigma^\mu(\lambda, \varphi)$, if it satisfies the following condition:*

$$(1-\lambda)(\frac{f(z)}{z})^\mu + \lambda f'(z)(\frac{f(z)}{z})^{\mu-1} \prec \varphi(z), \quad \lambda \ge 1, \ \mu \ge 1, \ z \in \mathbb{D} \tag{1.3}$$

and

$$(1-\lambda)(\frac{g(\omega)}{\omega})^\mu + \lambda g'(\omega)(\frac{g(\omega)}{\omega})^{\mu-1} \prec \varphi(\omega), \quad \lambda \ge 1, \ \mu \ge 1, \ \omega \in \mathbb{D}, \tag{1.4}$$

where the function g is given by

$$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots. \tag{1.5}$$

We assume that φ is an analytic univalent function with positive real part in $\mathbb{D}$, $\varphi(\mathbb{D})$ is symmetric with respect to the real axis and starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$. Such a function has series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad B_1 > 0. \tag{1.6}$$

Remark 1.2 We note that, for suitable choices λ, μ and φ, the class $H_\sigma^\mu(\lambda, \varphi)$ reduces to the following known classes, for instance,

(1) $H_\sigma^\mu(\lambda, (\frac{1+z}{1-z})^\alpha) = H_\sigma^\mu(\lambda, \alpha)$ $(\lambda \ge 1, 0 < \alpha \le 1, \mu \ge 0)$ (see [5, Definition 2.1]);
(2) $H_\sigma^\mu(\lambda, (\frac{1+(1-2\beta)z}{1-z})) = H_\sigma^\mu(\lambda, \beta)$ $(\lambda \ge 1, 0 \le \beta < 1, \mu \ge 0)$ (see [5, Definition 3.1]);
(3) $H_\sigma^\mu(1, \varphi) = H_\sigma^\mu(\varphi)$ $(\mu \ge 0)$ (see [6, Definition 2.1]);
(4) $H_\sigma^1(1, \varphi) = H_\sigma(\varphi)$ (see [7]);

(5) $H_\sigma^1(\lambda,(\frac{1+z}{1-z})^\alpha)=H_\sigma(\lambda,\alpha)$ $(\lambda\geq 1, 0<\alpha\leq 1)$ (see [8, Definition 2.1]);
(6) $H_\sigma^1(\lambda,(\frac{1+(1-2\beta)z}{1-z}))=H_\sigma(\lambda,\beta)$ $(\lambda\geq 1, 0\leq\beta<1)$ (see [8, Definition 3.1]);
(7) $H_\sigma^1(1,(\frac{1+z}{1-z})^\alpha)=H_\sigma(\alpha)$ $(0<\alpha\leq 1)$ (see [9, Definition 1]);
(8) $H_\sigma^1(1,(\frac{1+(1-2\beta)z}{1-z}))=H_\sigma(\beta)$ $(0\leq\beta<1)$ (see [9, Definition 2]).

Pommerenke [10] (see also Noonan and Thomas [11]) defined the q^{th} Hankel determinant for a function f as

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix},\quad a_1=1;\ q\geq 1, n\geq 1.$$

This determinant has been considered by several authors, for example, Noor [12] determined the rate of growth of $H_q(n)$ as $n\to\infty$ for functions $f(z)$ given by (1.1) with bounded boundary and Ehrenborg [13] studied the Hankel determinant of exponential polynomials.

In particular, we have

$$H_2(1)=\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}=a_3-a_2^2,\quad a_1=1,\ n=1,\ q=2,$$

which is the well-known Fekete-Szego functional [14–17] and

$$H_2(2)=\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}=a_2a_4-a_3^2,\quad n=2,\ q=2$$

and

$$H_3(1)=\begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix},\quad n=1,\ q=3.$$

Since $f\in\mathcal{A}$, $a_1=1$, $H_3(1)$ can be written as

$$H_3(1)=a_3(a_2a_4-a_3^2)-a_4(a_4-a_2a_3)+a_5(a_3-a_2^2).$$

In recent years, many authors focused on the investigating of the second Hankel determinant $H_2(2)$ and the third Hankel determinant $H_3(1)$ for various classes of functions in the open unit disk $\mathbb{D}$, the interested readers can see, for example, [18–27]. Only a few papers have been devoted to the second Hankel determinant $H_2(2)$ for bi-univalent functions [28–31]. So, inspired by the papers [28–31], we mainly investigate the third Hankel determinant $H_3(1)$ for the class $H_\sigma^\mu(\lambda,\varphi)$ of Ma-Minda bi-univalent functions, and obtain the upper bound of the above determinant $H_3(1)$.

2. Main results

To obtain our desired results, we need the following lemmas.

Lemma 2.1 ([32]) *If $p(z) \in \mathcal{P}$, then there exists some x, z with $|x| \le 1$, $|z| \le 1$, such that*

$$2p_2 = p_1^2 + x(4 - p_1^2),$$

$$4p_3 = p_1^3 + 2p_1 x(4 - p_1^2) - (4 - p_1^2)p_1 x^2 + 2(4 - p_1^2)(1 - |x|^2)z.$$

Lemma 2.2 ([33]) *If $p(z) \in \mathcal{P}$, then*

$$|c_n| \le 2, \quad n = 1, 2, \ldots.$$

We now state and prove the main results of our present investigation.

Theorem 2.3 *If the function $f(z) \in H_\sigma^\mu(\lambda, \varphi)$ $(\lambda \ge 1,\ \mu \ge 1)$ and is of the form (1.1), then*

$$\begin{aligned}
|a_2| \le & \frac{B_1}{\lambda + \mu}, \quad |a_3| \le \frac{B_1}{2\lambda + \mu} + \frac{2|B_2 - B_1|}{(1 + \mu)(2\lambda + \mu)}, \\
|a_4| \le & \frac{B_1 + 2|B_2| + |B_3|}{3\lambda + \mu} + \frac{(\mu - 1)(2\mu - 1)B_1^3}{6(\lambda + \mu)^3} + \frac{(B_1^2 + B_1|B_2|)(\mu - 1)}{(2\lambda + \mu)(\lambda + \mu)}, \\
|a_5| \le & \frac{B_1 + 3|B_2| + 3|B_3| + |B_4|}{\mu + 4\lambda} + \frac{(\mu - 1)B_1^4|23\mu - 17\mu^2 - 8|}{24(\lambda + \mu)^4} + \\
& \frac{2(\mu - 1)(B_1^2 + 2|B_2|B_1 + |B_3|B_1)}{(\mu + \lambda)(3\lambda + \mu)} + \frac{(\mu - 1)(B_1 + B_2)^2}{(2\lambda + \mu)^2} + \\
& \frac{(B_1^3 + B_1^2|B_2|)(\mu - 1)(5\mu - 4)}{2(\mu + 2\lambda)(\lambda + \mu)^2}.
\end{aligned} \tag{2.1}$$

Proof Let $f(z) \in H_\sigma^\mu(\lambda, \varphi)$. Then, by the definition of subordination and (1.3), we have

$$(1 - \lambda)(\frac{f(z)}{z})^\mu + \lambda f'(z)(\frac{f(z)}{z})^{\mu - 1} = \varphi(\omega(z)), \tag{2.2}$$

where $\omega(z)$ is Schwarz function with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{D})$.

Setting

$$\omega(z) = \sum_{n=1}^{\infty} C_n z^n,$$

then, from (1.6), we easily get

$$\begin{aligned}
\varphi(\omega(z)) =& 1 + B_1 C_1 z + (B_1 C_2 + B_2 C_1^2) z^2 + (B_1 C_3 + 2B_2 C_1 C_2 + B_3 C_1^3) z^3 + \\
& [B_1 C_4 + (2C_1 C_3 + C_2^2) B_2 + 3C_1^2 C_2 B_3 + B_4 C_1^4] z^4 + \cdots.
\end{aligned} \tag{2.3}$$

On the other hand,

$$\begin{aligned}
& (1 - \lambda)(\frac{f(z)}{z})^\mu + \lambda f'(z)(\frac{f(z)}{z})^{\mu - 1} \\
& = 1 + a_2(\lambda + \mu) z + [\frac{(\mu - 1)(2\lambda + \mu)a_2^2}{2} + (2\lambda + \mu) a_3] z^2 + \\
& \quad [\frac{(\mu - 1)(\mu - 2)(3\lambda + \mu)a_2^3}{6} + (\mu - 1)(3\lambda + \mu) a_2 a_3 + (3\lambda + \mu) a_4] z^3 + \\
& \quad [\frac{(\mu + 4\lambda)(\mu - 1)(\mu - 2)(\mu - 3)a_2^4}{24} + (\mu - 1)(\mu + 4\lambda)(a_3^2 + 2a_2 a_4) + \\
& \quad (\mu + 4\lambda) a_5 + \frac{a_2^2 a_3 (\mu - 1)(\mu - 2)(\mu + 4\lambda)}{2}] z^4 + \cdots.
\end{aligned} \tag{2.4}$$

So, from (2.2), (2.3) and (2.4), we show that

$$\begin{aligned}
&1+a_2(\lambda+\mu)z+[\frac{(\mu-1)(2\lambda+\mu)a_2^2}{2}+(2\lambda+\mu)a_3]z^2+\\
&\quad[\frac{(\mu-1)(\mu-2)(3\lambda+\mu)a_2^3}{6}+(\mu-1)(3\lambda+\mu)a_2a_3+(3\lambda+\mu)a_4]z^3+\\
&\quad[\frac{(\mu+4\lambda)(\mu-1)(\mu-2)(\mu-3)a_2^4}{24}+(\mu-1)(\mu+4\lambda)(a_3^2+2a_2a_4)+\\
&\quad(\mu+4\lambda)a_5+\frac{a_2^2a_3(\mu-1)(\mu-2)(\mu+4\lambda)}{2}]z^4+\cdots\\
&\quad=1+B_1C_1z+(B_1C_2+B_2C_1^2)z^2+(B_1C_3+2B_2C_1C_2+B_3C_1^3)z^3+\\
&\quad[B_1C_4+(2C_1C_3+C_2^2)B_2+3C_1^2C_2B_3+B_4C_1^4]z^4+\cdots. \qquad (2.5)
\end{aligned}$$

Next, comparing the coefficients of z, z^2, z^3, z^4 on both sides of the equation (2.5), we obtain

$$\begin{aligned}
&a_2(\lambda+\mu)=B_1C_1,\\
&\frac{(\mu-1)(2\lambda+\mu)a_2^2}{2}+(2\lambda+\mu)a_3=B_1C_2+B_2C_1^2,\\
&\frac{(\mu-1)(\mu-2)(3\lambda+\mu)a_2^3}{6}+(\mu-1)(3\lambda+\mu)a_2a_3+(3\lambda+\mu)a_4\\
&\quad=B_1C_3+2B_2C_1C_2+B_3C_1^3,\\
&\frac{(\mu+4\lambda)(\mu-1)(\mu-2)(\mu-3)a_2^4}{24}+(\mu-1)(\mu+4\lambda)(a_3^2+2a_2a_4)+(\mu+4\lambda)a_5+\\
&\quad\frac{a_2^2a_3(\mu-1)(\mu-2)(\mu+4\lambda)}{2}=B_1C_4+(2C_1C_3+C_2^2)B_2+3C_1^2C_2B_3+B_4C_1^4.
\end{aligned}$$

After some simple computations, we get

$$\begin{aligned}
a_2=&\frac{B_1C_1}{\lambda+\mu},\\
a_3=&\frac{B_1C_2+B_2C_1^2}{2\lambda+\mu}-\frac{(\mu-1)B_1^2C_1^2}{2(\lambda+\mu)^2},\\
a_4=&\frac{B_1C_3+2B_2C_1C_2+B_3C_1^3}{3\lambda+\mu}+\frac{(\mu-1)(2\mu-1)B_1^3C_1^3}{6(\lambda+\mu)^3}-\frac{(B_1^2C_1C_2+C_1^3B_1B_2)(\mu-1)}{(2\lambda+\mu)(\lambda+\mu)},\\
a_5=&\frac{B_1C_4+(2C_1C_3+C_2^2)B_2+3C_1^2C_2B_3+B_4C_1^4}{\mu+4\lambda}+\frac{(\mu-1)B_1^4C_1^4(23\mu-17\mu^2-8)}{24(\lambda+\mu)^4}-\\
&\frac{2(\mu-1)(B_1^2C_1C_3+2B_2B_1C_1^2C_2+B_3B_1C_1^4)}{(\mu+\lambda)(3\lambda+\mu)}-\frac{(\mu-1)(B_1C_2+B_2C_1^2)^2}{(2\lambda+\mu)^2}+\\
&\frac{(B_1^3C_1^2C_2+B_1^2B_2C_1^4)(\mu-1)(5\mu-4)}{2(\mu+2\lambda)(\lambda+\mu)^2}.
\end{aligned}$$

Since $|C_n|\leq 1$, we have $\sum_{n=1}^{\infty}|C_n|\leq 1$ and also

$$\begin{aligned}
|a_2|\leq&\frac{B_1}{\lambda+\mu},\quad |a_3|\leq\frac{B_1+|B_2|}{2\lambda+\mu}+\frac{(\mu-1)B_1^2}{2(\lambda+\mu)^2},\\
|a_4|\leq&\frac{B_1+2|B_2|+|B_3|}{3\lambda+\mu}+\frac{(\mu-1)(2\mu-1)B_1^3}{6(\lambda+\mu)^3}+\frac{(B_1^2+B_1|B_2|)(\mu-1)}{(2\lambda+\mu)(\lambda+\mu)},\\
|a_5|\leq&\frac{B_1+3|B_2|+3|B_3|+|B_4|}{\mu+4\lambda}+\frac{(\mu-1)B_1^4|23\mu-17\mu^2-8|}{24(\lambda+\mu)^4}+
\end{aligned}$$

$$\frac{2(\mu-1)(B_1^2+2|B_2|B_1+|B_3|B_1)}{(\mu+\lambda)(3\lambda+\mu)}+\frac{(\mu-1)(B_1+B_2)^2}{(2\lambda+\mu)^2}+$$
$$\frac{(B_1^3+B_1^2|B_2|)(\mu-1)(5\mu-4)}{2(\mu+2\lambda)(\lambda+\mu)^2},$$

which completes the proof of Theorem 2.3. □

Theorem 2.4 *If the function $f(z)\in H_\sigma^\mu(\lambda,\varphi)$ $(\lambda\geq 1,\ \mu\geq 1)$ and is of the form (1.1), then we have*

$$|a_2a_4-a_3^2|\leq\begin{cases}\frac{B_1^4(\mu^2+3\mu+2)}{6(\lambda+\mu)^4}+\frac{B_1|B_3|}{(3\lambda+\mu)(\lambda+\mu)}, & p^*<p\leq 2,\\ \frac{B_1^2}{(2\lambda+\mu)^2}, & 0\leq p\leq p^*,\end{cases}\tag{2.6}$$

where

$$p^*=\sqrt{\frac{2(2L_3-L_2)}{3(L_1+L_3-L_2)}}\tag{2.7}$$

with

$$L_3=\frac{B_1^3}{16(2\lambda+\mu)^2},\tag{2.8}$$

$$L_2=\frac{B_1^2}{16(\mu+\lambda)(\mu+3\lambda)}+\frac{(B_1^3+4|B_1B_2|)}{32(\mu+2\lambda)(\mu+\lambda)^2},\tag{2.9}$$

$$L_1=\frac{B_1^4(\mu^2+3\mu+2)}{96(\mu+\lambda)^4}+\frac{|B_1B_3|}{16(\mu+\lambda)(\mu+3\lambda)}.\tag{2.10}$$

Proof Since $f(z)\in H_\sigma^\mu(\lambda,\varphi)$, according to subordination relationship, there exist two analytic functions $u,v:\mathbb{D}\longrightarrow\mathbb{D}$, with $u(0)=v(0)=0$, such that

$$(1-\lambda)(\frac{f(z)}{z})^\mu+\lambda f'(z)(\frac{f(z)}{z})^{\mu-1}=\varphi(u(z))\tag{2.11}$$

and

$$(1-\lambda)(\frac{g(w)}{w})^\mu+\lambda g'(w)(\frac{g(w)}{w})^{\mu-1}=\varphi(v(w)).\tag{2.12}$$

Define the functions $p(z)$ and $q(z)$ by

$$p(z)=\frac{1+u(z)}{1-u(z)}=1+p_1z+p_2z^2+\cdots$$

and

$$q(z)=\frac{1+v(z)}{1-v(z)}=1+q_1z+q_2z^2+\cdots.$$

Then, we notice that $p(z)$, $q(z)\in\mathcal{P}$ and also have

$$\begin{aligned}u(z)=&\frac{p(z)-1}{1+p(z)}=\frac{1}{2}[p_1z+(p_2-\frac{p_1^2}{2})z^2+(\frac{p_1^3}{4}+\\&p_3-p_1p_2)z^3+(\frac{3p_1^2p_2}{4}-p_1p_3+p_4-\frac{p_2^2}{2}-\frac{p_1^4}{8})z^4+\cdots],\end{aligned}\tag{2.13}$$

$$\begin{aligned}v(z)=&\frac{q(z)-1}{1+q(z)}=\frac{1}{2}[q_1z+(q_2-\frac{q_1^2}{2})z^2+(\frac{q_1^3}{4}+\\&q_3-q_1q_2)z^3+(\frac{3q_1^2q_2}{4}-q_1q_3+q_4-\frac{q_2^2}{2}-\frac{q_1^4}{8})z^4+\cdots].\end{aligned}\tag{2.14}$$

By virtue of (2.11)–(2.14), we have

$$(1-\lambda)(\frac{f(z)}{z})^{\mu}+\lambda f'(z)(\frac{f(z)}{z})^{\mu-1}=\varphi(\frac{p(z)-1}{1+p(z)}) \tag{2.15}$$

and

$$(1-\lambda)(\frac{g(w)}{w})^{\mu}+\lambda g'(w)(\frac{g(w)}{w})^{\mu-1}=\varphi(\frac{q(w)-1}{1+q(w)}). \tag{2.16}$$

Also, using (2.13), (2.14) together with (1.6), we easily obtain

$$\begin{aligned}\varphi(\frac{p(z)-1}{1+p(z)}) =&1+\frac{B_1p_1z}{2}+[\frac{1}{2}B_1(p_2-\frac{1}{2}p_1^2)+\frac{B_2p_1^2}{4}]z^2+\\ &[\frac{B_1}{2}(p_3+\frac{1}{4}p_1^3-p_1p_2)+\frac{B_2}{2}(p_1p_2-\frac{p_1^3}{2})+\frac{B_3p_1^3}{8}]z^3+\\ &[\frac{B_1}{2}(p_4-\frac{1}{8}p_1^4+\frac{3}{4}p_1^2p_2-p_1p_3-\frac{p_2^2}{2})+B_2(\frac{3p_1^4}{16}-\frac{3}{4}p_1^2p_2+\frac{p_2^2}{4}+\frac{p_1p_3}{2})+\\ &\frac{B_3}{8}(3p_1^2p_2-\frac{3p_1^4}{2})+\frac{B_4p_1^4}{16}]z^4+\cdots\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}\varphi(\frac{q(w)-1}{1+q(w)}) =&1+\frac{B_1q_1w}{2}+[\frac{1}{2}B_1(q_2-\frac{1}{2}q_1^2)+\frac{B_2q_1^2}{4}]w^2+\\ &[\frac{B_1}{2}(q_3+\frac{1}{4}q_1^3-q_1q_2)+\frac{B_2}{2}(q_1q_2-\frac{q_1^3}{2})+\frac{B_3q_1^3}{8}]w^3+\\ &[\frac{B_1}{2}(q_4-\frac{1}{8}q_1^4+\frac{3}{4}q_1^2q_2-q_1q_3-\frac{q_2^2}{2})+B_2(\frac{3q_1^4}{16}-\frac{3}{4}q_1^2q_2+\frac{q_2^2}{4}+\frac{q_1q_3}{2})+\\ &\frac{B_3}{8}(3q_1^2q_2-\frac{3q_1^4}{2})+\frac{B_4q_1^4}{16}]w^4+\cdots.\end{aligned} \tag{2.18}$$

Since, from (1.1) and (1.5), we find that

$$f'(z)=1+2a_2z+3a_3z^2+4a_4z^3+5a_5z^4+\cdots$$

and

$$g'(w)=1-2a_2w+3(2a_2^2-a_3)w^2-4(5a_2^3-5a_2a_3+a_4)w^3+5(14a_2^4-21a_2^2a_3+6a_2a_4+3a_3^2-a_5)w^4+\cdots,$$

so we have

$$\begin{aligned}&(1-\lambda)(\frac{f(z)}{z})^{\mu}+\lambda f'(z)(\frac{f(z)}{z})^{\mu-1}\\ &=1+a_2(\lambda+\mu)z+[\frac{(\mu-1)(2\lambda+\mu)a_2^2}{2}+(2\lambda+\mu)a_3]z^2+\\ &[\frac{(\mu-1)(\mu-2)(3\lambda+\mu)a_2^3}{6}+(\mu-1)(3\lambda+\mu)a_2a_3+(3\lambda+\mu)a_4]z^3+\\ &[\frac{(\mu+4\lambda)(\mu-1)(\mu-2)(\mu-3)a_2^4}{24}+(\mu-1)(\mu+4\lambda)(a_3^2+2a_2a_4)+\\ &(\mu+4\lambda)a_5+\frac{a_2^2a_3(\mu-1)(\mu-2)(\mu+4\lambda)}{2}]z^4+\cdots\end{aligned} \tag{2.19}$$

and

$$(1-\lambda)(\frac{g(w)}{w})^{\mu}+\lambda g'(w)(\frac{g(w)}{w})^{\mu-1}$$

$$
\begin{aligned}
&= 1 - a_2(\lambda+\mu)w + [\frac{(\mu-1)(2\lambda+\mu)a_2^2}{2} + (2\lambda+\mu)(2a_2^2 - a_3)]w^2 - \\
&[\frac{(\mu-1)(\mu-2)(3\lambda+\mu)a_2^3}{6} + (\mu-1)(3\lambda+\mu)(2a_2^3 - a_2a_3) + \\
&(3\lambda+\mu)(5a_2^3 - 5a_2a_3 + a_4)]w^3 + \\
&[\frac{(\mu+4\lambda)(\mu-1)(\mu-2)(\mu-3)a_2^4}{24} + \frac{(\mu-1)(\mu-2)(\mu+4\lambda)(2a_2^4 - a_2^2a_3)}{2} + \\
&\frac{(\mu-1)(\mu+4\lambda)(2a_2^2 - a_3)^2}{2} + (\mu+4\lambda)(14a_2^4 - 21a_2^2a_3 + 6a_2a_4 + 3a_3^2 - a_5) + \\
&(\mu-1)(\mu+4\lambda)(5a_2^4 - 5a_2^2a_3 + a_2a_4)]w^4 + \cdots.
\end{aligned} \tag{2.20}
$$

Again, comparing the coefficients of $z,\ z^2,\ z^3$ between the equations (2.17) and (2.19), we obtain

$$a_2(\lambda+\mu) = \frac{1}{2}B_1p_1, \tag{2.21}$$

$$\frac{(\mu-1)(2\lambda+\mu)a_2^2}{2} + (2\lambda+\mu)a_3 = \frac{1}{2}B_1(p_2 - \frac{1}{2}p_1^2) + \frac{B_2p_1^2}{4}, \tag{2.22}$$

$$
\begin{aligned}
&\frac{(\mu-1)(\mu-2)(3\lambda+\mu)a_2^3}{6} + (\mu-1)(3\lambda+\mu)a_2a_3 + (3\lambda+\mu)a_4 \\
&= \frac{B_1}{2}(p_3 + \frac{1}{4}p_1^3 - p_1p_2) + \frac{B_2}{2}(p_1p_2 - \frac{p_1^3}{2}) + \frac{B_3p_1^3}{8}.
\end{aligned} \tag{2.23}
$$

Similarly, from the equations (2.18) and (2.20), we can get

$$-a_2(\lambda+\mu) = \frac{1}{2}B_1q_1, \tag{2.24}$$

$$\frac{(\mu-1)(2\lambda+\mu)a_2^2}{2} + (2\lambda+\mu)(2a_2^2 - a_3) = \frac{1}{2}B_1(q_2 - \frac{1}{2}q_1^2) + \frac{B_2q_1^2}{4} \tag{2.25}$$

and

$$
\begin{aligned}
&-\frac{(\mu-1)(\mu-2)(3\lambda+\mu)a_2^3}{6} - (\mu-1)(3\lambda+\mu)(2a_2^3 - a_2a_3) - (3\lambda+\mu)(5a_2^3 - 5a_2a_3 + a_4) \\
&= \frac{B_1}{2}(q_3 + \frac{1}{4}q_1^3 - q_1q_2) + \frac{B_2}{2}(q_1q_2 - \frac{q_1^3}{2}) + \frac{B_3q_1^3}{8}.
\end{aligned} \tag{2.26}
$$

In view of (2.21) and (2.24), we have

$$p_1 = -q_1 \tag{2.27}$$

and

$$a_2 = \frac{B_1p_1}{2(\lambda+\mu)}. \tag{2.28}$$

In addition, from (2.22), (2.23) and (2.25)–(2.27), we show that

$$a_3 = \frac{B_1(p_2 - q_2)}{4(2\lambda+\mu)} + \frac{B_1^2p_1^2}{4(\lambda+\mu)^2}, \tag{2.29}$$

$$
\begin{aligned}
a_4 = &\frac{5B_1^2p_1(p_2 - q_2)}{16(2\lambda+\mu)(\lambda+\mu)} + \frac{B_1^3p_1^3(4 - \mu^2 - 3\mu)}{48(\lambda+\mu)^3} + \\
&\frac{p_1^3(B_1 - 2B_2 + B_3) + 2B_1(p_3 - q_3) + 2(B_2 - B_1)p_1(p_2 + q_2)}{8(3\lambda+\mu)}.
\end{aligned} \tag{2.30}
$$

Combining (2.28)–(2.30), we establish that

$$|a_2a_4-a_3^2| = \Big|\frac{B_1^4p_1^4(-\mu^2-3\mu-2)}{96(\mu+\lambda)^4}-\frac{B_1^3p_1^2(p_2-q_2)}{32(2\lambda+\mu)(\mu+\lambda)^2}-\frac{B_1^2(p_2-q_2)^2}{16(\mu+2\lambda)^2}+ \frac{B_1p_1^4(B_1-2B_2+B_3)+2B_1p_1[B_1(p_3-q_3)+p_1(B_2-B_1)(p_2+q_2)}{16(\mu+\lambda)(\mu+3\lambda)}\Big|.$$

Next, applying Lemma 2.1 and (2.27), we write

$$p_2-q_2=\frac{(4-p_1^2)(x-y)}{2}, \tag{2.31}$$

$$p_2+q_2=\frac{(4-p_1^2)(x+y)}{2}+p_1^2, \tag{2.32}$$

$$p_3-q_3=\frac{p_1(4-p_1^2)(x+y)}{2}+\frac{p_1^3}{2}-\frac{p_1(4-p_1^2)(x^2+y^2)}{4}+ \frac{(4-p_1^2)[(1-|x|^2)z-(1-|y|^2)w]}{2} \tag{2.33}$$

for some $x,\ y,\ z$ and w with $|x|\le 1,\ |y|\le 1,\ |z|\le 1$ and $|w|\le 1$.

Let $p_1=p\in[0,2]$. Then, from (2.31)–(2.33), we have

$$\begin{aligned}|a_2a_4-a_3^2| &= \Big|\frac{B_1^4p^4(-\mu^2-3\mu-2)}{96(\mu+\lambda)^4}-\frac{B_1^3p^2(4-p^2)(x-y)}{64(2\lambda+\mu)(\mu+\lambda)^2}-\frac{B_1^2(4-p^2)^2(x-y)^2}{64(\mu+2\lambda)^2}+\\ &\quad \frac{2B_1B_3p^4+(4-p^2)B_1p[2B_2p(x+y)-B_1p(x^2+y^2)+2B_1[(1-|x|^2)z-(1-|y|^2)w]]}{32(\mu+\lambda)(\mu+3\lambda)}\Big|\\ &\le \frac{B_1^4p^4(\mu^2+3\mu+2)}{96(\mu+\lambda)^4}+\frac{p^4|B_1B_3|+2B_1^2p(4-p^2)}{16(\mu+\lambda)(\mu+3\lambda)}+\frac{B_1^3(4-p^2)^2(|x|+|y|)^2}{64(2\lambda+\mu)^2}+\\ &\quad \frac{B_1^2p(4-p^2)(p-2)(|x|^2+|y|^2)}{32(\mu+\lambda)(\mu+3\lambda)}+\frac{p^2(4-p^2)(B_1^3+4|B_1B_2|)(|x|+|y|)}{64(\mu+2\lambda)(\mu+\lambda)^2}.\end{aligned}$$

Taking

$$T_1=T_1(p)=\frac{B_1^4p^4(\mu^2+3\mu+2)}{96(\mu+\lambda)^4}+\frac{p^4|B_1B_3|+2B_1^2p(4-p^2)}{16(\mu+\lambda)(\mu+3\lambda)},$$

$$T_2=T_2(p)=\frac{p^2(4-p^2)(B_1^3+4|B_1B_2|)}{64(\mu+2\lambda)(\mu+\lambda)^2},$$

$$T_3=T_3(p)=\frac{B_1^2p(4-p^2)(p-2)}{32(\mu+\lambda)(\mu+3\lambda)},$$

$$T_4=T_4(p)=\frac{B_1^3(4-p^2)^2}{64(2\lambda+\mu)^2}$$

and also assuming without restriction that $p\in[0,2]$, thus, for $\eta_1=|x|\le 1$ and $\eta_2=|y|\le 1$, we obtain

$$|a_2a_4-a_3^2|\le T_1+T_2(\eta_1+\eta_2)+T_3(\eta_1^2+\eta_2^2)+T_4(\eta_1+\eta_2)^2 := F(\eta_1,\eta_2).$$

Now, we need to maximize $F(\eta_1,\eta_2)$ in the closed square $\mathbb{E}=\{(\eta_1,\eta_2):0\le\eta_1\le 1, 0\le\eta_2\le 1\}$ for $p\in[0,2]$. And so we must study the maximum of $F(\eta_1,\eta_2)$ according to $p\in[0,2],\ p=0$ and $p=2$, by taking into account the sign of $F_{\eta_1,\eta_1}F_{\eta_2,\eta_2}-(F_{\eta_1,\eta_2})^2$.

Firstly, let $p \in [0,2]$. Since $T_3 \leq 0$ and $T_3 + 2T_4 > 0$ for $p \in [0,2]$, we conclude that

$$F_{\eta_1,\eta_1} F_{\eta_2,\eta_2} - (F_{\eta_1,\eta_2})^2 < 0.$$

Thus, the function $F(\eta_1, \eta_2)$ cannot have a local maximum in the interior of the square $\mathbb{E}$. Next, we discuss the maximum of $F(\eta_1, \eta_2)$ on the boundary of the square $\mathbb{E}$.

For $\eta_1 = 0$ and $0 \leq \eta_2 \leq 1$ (similarly $\eta_2 = 0$ and $0 \leq \eta_1 \leq 1$), we obtain

$$F(0, \eta_2) := G(\eta_2) = T_1 + T_2\eta_2 + (T_3 + T_4)\eta_2^2.$$

We discuss the above $G(\eta_2)$ from the following two cases:

(i) The case $T_3 + T_4 \geq 0$: For $0 \leq \eta_2 \leq 1$ and any fixed p with $0 < p < 2$, it is clear that $G'(\eta_2) = T_2 + 2(T_3 + T_4)\eta_2 > 0$, that is, $G(\eta_2)$ is an increasing function. Hence, for fixed $0 < p < 2$, the maximum of $G(\eta_2)$ occurs at $\eta_2 = 1$ and

$$\max G(\eta_2) = T_1 + T_2 + T_3 + T_4.$$

(ii) The case $T_3 + T_4 < 0$: Since $T_2 + 2(T_3 + T_4) \geq 0$ for $0 < \eta_2 \leq 1$ and any fixed p with $0 < p < 2$, it is clear that $T_2 + 2(T_3 + T_4) < 2(T_3 + T_4)\eta_2 + T_2 < T_2$ and so $G'(\eta_2) > 0$. Hence, for fixed $0 < p < 2$, the maximum of $G(\eta_2)$ occurs at $\eta_2 = 1$ and also for $c = 2$, we obtain

$$F(\eta_1, \eta_2) = \frac{B_1^4(\mu^2 + 3\mu + 2)}{6(\lambda + \mu)^4}. \tag{2.34}$$

Taking into account the value of (2.31) and the cases (i) and (ii), for $0 \leq \eta_2 \leq 1$ and any fixed p with $0 < p < 2$, we have

$$\max G(\eta_2) = T_1 + T_2 + T_3 + T_4.$$

For $\eta_1 = 1$ and $0 \leq \eta_2 \leq 1$ (similarly $\eta_2 = 1$ and $0 \leq \eta_1 \leq 1$), we obtain

$$F(1, \eta_2) := H(\eta_2) = T_1 + T_2 + T_3 + T_4 + (T_2 + 2T_4)\eta_2 + (T_3 + T_4)\eta_2^2.$$

Similarly, to the above cases of $T_3 + T_4$, we get

$$\max G(\eta_2) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

Since $G(1) \leq H(1)$ for $p \in (0,2)$, $\max F(\eta_1, \eta_2) = F(1,1)$ on the boundary of the square $\mathbb{E}$. Thus the maximum of $F(\eta_1, \eta_2)$ occurs at $\eta_1 = 1$ and $\eta_2 = 1$ in the closed square $\mathbb{E}$. Let

$$\begin{aligned} K(p) =& \max F(\eta_1, \eta_2) = F(1,1) = T_1 + 2T_2 + 2T_3 + 4T_4 \\ =& \frac{B_1^4 p^4(\mu^2 + 3\mu + 2)}{96(\mu + \lambda)^4} + \frac{p^4|B_1B_3| + B_1^2 p^2(4 - p^2)}{16(\mu + \lambda)(\mu + 3\lambda)} + \frac{B_1^3(4 - p^2)^2}{16(2\lambda + \mu)^2} + \\ & \frac{p^2(4 - p^2)(B_1^3 + 4|B_1B_2|)}{32(\mu + 2\lambda)(\mu + \lambda)^2}. \end{aligned}$$

Further let

$$K(p) = L_3(4 - p^2)^2 + L_2 p^2(4 - p^2) + L_1 p^4,$$

where

$$L_3 = \frac{B_1^3}{16(2\lambda + \mu)^2},$$

$$L_2 = \frac{B_1^2}{16(\mu+\lambda)(\mu+3\lambda)} + \frac{(B_1^3 + 4|B_1B_2|)}{32(\mu+2\lambda)(\mu+\lambda)^2},$$

$$L_1 = \frac{B_1^4(\mu^2+3\mu+2)}{96(\mu+\lambda)^4} + \frac{|B_1B_3|}{16(\mu+\lambda)(\mu+3\lambda)}.$$

Differentiating $K(p)$, we get

$$K'(p) = -4L_3p(4-p^2) + 2L_2p(4-2p^2) + 4L_1p^3,$$

$$K''(p) = 12(L_3 - L_2 + L_1)p^2 + 8L_2 - 16L_3.$$

(i) If $K''(p) > 0$, that is $p^* < p \le 2$, where p^* is given by (2.7). This implies that $K'(p)$ is an increasing function on the closed interval [0,2] about p and so the function $K'(p) \ge K'(0) = 0$, thus the function $K(p)$ is an increasing function on the closed interval [0,2] about p, which implies that $K(p)$ gets the maximum value at the point $p = 2$. That is

$$|a_2a_4 - a_3^2| \le K(2) = \frac{B_1^4(\mu^2+3\mu+2)}{6(\mu+\lambda)^4} + \frac{|B_1B_3|}{(\mu+\lambda)(\mu+3\lambda)}.$$

(ii) If $K'(p) \le 0$, that is $0 \le p \le p^*$, where p^* is given by (2.7). This implies that $K'(p)$ is an decreasing function on the closed interval [0,2] about p, thus, we have $K'(p) \le K'(0) = 0$, which means the the function $K(p)$ is an decreasing function on the closed interval [0,2] about p, so $K(p)$ gets the maximum value at the point $p = 0$. Namely,

$$|a_2a_4 - a_3^2| \le K(0) = \frac{B_1^3}{(2\lambda+\mu)^2}.$$

The proof of Theorem 2.4 is completed. □

Theorem 2.5 *If the function $f(z) \in H_\sigma^\mu(\lambda,\varphi)$ $(\lambda \ge 1,\ \mu \ge 1)$ and is of the form (1.1), then we have*

$$|a_3 - a_2^2| \le \frac{4\sqrt{3}B_1^2}{9(\mu+2\lambda)}. \tag{2.35}$$

Proof Because $f(z) \in H_\sigma^\mu(\lambda,\varphi)$, from (2.28) and (2.29), we have

$$|a_3 - a_2^2| = |\frac{B_1(p_2-q_2)}{4(2\lambda+\mu)} + \frac{B_1^2p_1^2}{4(\lambda+\mu)^2} - \frac{B_1^2p_1^2}{4(\lambda+\mu)^2}| = |\frac{B_1(p_2-q_2)}{4(2\lambda+\mu)}|.$$

Assume that $p_1 = p \in [0,2]$, then, by (2.31), we get

$$|a_3 - a_2^2| = |\frac{B_1^2p(4-p^2)(x-y)}{8(2\lambda+\mu)}| \le \frac{B_1^2p(4-p^2)(|x|+|y|)}{8(2\lambda+\mu)}.$$

Set

$$F(|x|,|y|) = \frac{B_1^2p(4-p^2)(|x|+|y|)}{8(2\lambda+\mu)},$$

it is obvious that, when $|x| = |y| = 1$, the function $F(|x|,|y|)$ can get the maximum value

$$F(1,1) = \frac{B_1^2p(4-p^2)}{4(2\lambda+\mu)} := G(p).$$

Differentiating $G(p)$, we obtain

$$G'(p) = \frac{B_1^2}{(2\lambda+\mu)} - \frac{3B_1^2p^2}{4(2\lambda+\mu)}.$$

If $G'(p)=0$, then we get $p=\frac{2\sqrt{3}}{3}$.

Since $G''(\frac{2\sqrt{3}}{3})=-\frac{B_1^2\sqrt{3}}{(2\lambda+\mu)}<0$, the function $G(p)$ gets the maximum value at the point $p=\frac{2\sqrt{3}}{3}$. That is, that

$$|a_3-a_2^2|\le\frac{4\sqrt{3}B_1^2}{9(\mu+2\lambda)},$$

which completes the proof of Theorem 2.5. □

Theorem 2.6 *If the function $f(z)\in H_\sigma^\mu(\lambda,\varphi)$ $(\lambda\ge 1,\ \mu\ge 1)$ and is of the form (1.1), then we have*

$$|a_2a_3-a_4|\le\begin{cases}\frac{B_1^3(\mu^2+3\mu+2)}{6(\lambda+\mu)^3}+\frac{|B_3|}{2(3\lambda+\mu)}, & r^*<p\le 2,\\ \frac{B_1}{(3\lambda+\mu)}, & 0\le p\le r^*,\end{cases}\tag{2.36}$$

where

$$r^*=\frac{-2M_3-2\sqrt{M_3^2+3(M_2+M_3)(M_2+M_3-M_1)}}{3(M_1-M_2-M_3)}.\tag{2.37}$$

$$M_1=\frac{B_1^3(\mu^2+3\mu+2)}{48(\lambda+\mu)^3}+\frac{|B_3|}{16(3\lambda+\mu)},$$

$$M_2=\frac{3B_1^2}{16(\lambda+\mu)(2\lambda+\mu)}+\frac{|B_2|}{4(3\lambda+\mu)},$$

$$M_3=\frac{B_1}{8(3\lambda+\mu)}.$$

Proof By using (2.28)–(2.30), we have

$$|a_2a_3-a_4|=\Big|\frac{B_1^3p_1^3(\mu^2+3\mu+2)}{48(\lambda+\mu)^3}-\frac{3B_1^2p_1(p_2-q_2)}{16(\lambda+\mu)(2\lambda+\mu)}-\frac{[p_1^3(B_1-2B_2+B_3)+2B_1(p_3-q_3)+2(B_2-B_1)p_1(p_2+q_2)]}{8(3\lambda+\mu)}\Big|.$$

Then, applying the equations (2.31)–(2.33) and also putting $p_1=p\in[0,2]$, we get

$$\begin{aligned}&|a_2a_3-a_4|\\ &=\Big|\frac{B_1^3p^3(\mu^2+3\mu+2)}{48(\lambda+\mu)^3}-\frac{3B_1^2p(4-p^2)(x-y)}{32(\lambda+\mu)(2\lambda+\mu)}-\\ &\quad\frac{B_3p^3+(4-p^2)[B_1p(x^2+y^2)-2B_2p(x+y)-2B_1(|x|^2+|y|^2)]}{16(3\lambda+\mu)}\Big|\\ &\le\frac{B_1^3p^3(\mu^2+3\mu+2)}{48(\lambda+\mu)^3}+\frac{p^3|B_3|}{16(3\lambda+\mu)}+p(4-p^2)(|x|+|y|)\times\\ &\quad\Big[\frac{3B_1^2}{32(\lambda+\mu)(2\lambda+\mu)}+\frac{|B_2|}{8(3\lambda+\mu)}\Big]+\frac{B_1(4-p^2)(p-2)(|x|^2+|y|^2)}{16(3\lambda+\mu)}.\end{aligned}$$

Choosing

$$T_1=T_1(p)=\frac{B_1^3p^3(\mu^2+3\mu+2)}{48(\lambda+\mu)^3}+\frac{p^3|B_3|}{16(3\lambda+\mu)},$$

$$T_2 = T_2(p) = p(4-p^2)[\frac{3B_1^2}{32(\lambda+\mu)(2\lambda+\mu)} + \frac{|B_2|}{8(3\lambda+\mu)}],$$

$$T_3 = T_3(p) = \frac{B_1(4-p^2)(p-2)}{16(3\lambda+\mu)}$$

and assuming without restriction that $p \in [0,2]$, thus, for $\eta_1 = |x| \le 1$ and $\eta_2 = |y| \le 1$, we obtain

$$|a_2a_3 - a_4| \le T_1 + T_2(\eta_1+\eta_2) + T_3(\eta_1^2+\eta_2^2) := F(\eta_1,\eta_2).$$

Using the same method as Theorem 2.3, we deduce that the function $F(\eta_1,\eta_2)$ can get the maximum value at the points $\eta_1 = 1$ and $\eta_2 = 1$. That is

$$\begin{aligned}\max F(\eta_1,\eta_2) =& F(1,1) = [\frac{B_1^3(\mu^2+3\mu+2)}{48(\lambda+\mu)^3} + \frac{|B_3|}{16(3\lambda+\mu)}]p^3 + \\ & p(4-p^2)[\frac{3B_1^2}{16(\lambda+\mu)(2\lambda+\mu)} + \frac{|B_2|}{4(3\lambda+\mu)}] + \frac{B_1(4-p^2)(p-2)}{8(3\lambda+\mu)}.\end{aligned}$$

Also, assume that

$$G(p) = M_1p^3 + M_2p(4-p^2) + M_3(4-p^2)(p-2),$$

where

$$M_1 = \frac{B_1^3(\mu^2+3\mu+2)}{48(\lambda+\mu)^3} + \frac{|B_3|}{16(3\lambda+\mu)},$$

$$M_2 = \frac{3B_1^2}{16(\lambda+\mu)(2\lambda+\mu)} + \frac{|B_2|}{4(3\lambda+\mu)},$$

$$M_3 = \frac{B_1}{8(3\lambda+\mu)}.$$

Therefore, we have

$$G'(p) = 3(M_1 - M_2 - M_3)p^2 + 4M_3p + 4(M_2+M_3),$$

$$G''(p) = 6(M_1 - M_2 - M_3)p + 4M_3.$$

If $M_1 - M_2 - M_3 > 0$, that is $M_1 > M_2 + M_3$, then we get $G'(p) > 0$. Thus the function $G(p)$ is an increasing function on the closed interval [0,2] about p and so the function $G(p)$ gets the maximum value at the point $p = 2$. That is

$$|a_2a_3 - a_4| \le G(2) = \frac{B_1^3(\mu^2+3\mu+2)}{6(\lambda+\mu)^3} + \frac{|B_3|}{2(3\lambda+\mu)}.$$

If $M_1 - M_2 - M_3 < 0$, let $G'(p) = 0$. Then we get

$$p = r^* = \frac{-2M_3 - 2\sqrt{M_3^2 + 3(M_2+M_3)(M_2+M_3-M_1)}}{3(M_1-M_2-M_3)}.$$

When $r^* < p \le 2$, then we have $G'(p) > 0$, which means the function $G(p)$ is an increasing function on the closed interval [0,2] about p, thus, the function $G(p)$ gets the maximum value at the point $p = 2$. That is

$$|a_2a_3 - a_4| \le G(2) = \frac{B_1^3(\mu^2+3\mu+2)}{6(\lambda+\mu)^3} + \frac{|B_3|}{2(3\lambda+\mu)}.$$

When $0 \le p \le r^*$, then we have $G'(p) < 0$, which means the function $G(p)$ is an decreasing function on the closed interval [0,2] about p, thus, the function $G(p)$ gets the maximum value at the point $p = 0$. That is

$$|a_2a_3 - a_4| \le G(0) = \frac{B_1}{(3\lambda + \mu)}.$$

The proof of Theorem 2.6 is completed. □

Theorem 2.7 *If the function $f(z) \in H^{\mu}_{\sigma}(\lambda, \varphi)$ $(\lambda \ge 1,\ \mu \ge 1)$ and is of the form (1.1), then we have*

$$|H_3(1)| \le \begin{cases} A_3[\frac{B_1^4(\mu^2+3\mu+2)}{6(\lambda+\mu)^4} + \frac{B_1|B_3|}{(3\lambda+\mu)(\lambda+\mu)}] + 8A_4M_1 + \frac{A_5 4\sqrt{3}B_1^2}{9(\mu+2\lambda)}, & p^* < p \le 2 \ \ or\ r^* < p \le 2, \\ A_3[\frac{B_1^2}{(2\lambda+\mu)^2}] + 8A_4M_1 + \frac{A_5 4\sqrt{3}B_1^2}{9(\mu+2\lambda)}, & r^* < p < p^*, \\ A_3[\frac{B_1^4(\mu^2+3\mu+2)}{6(\lambda+\mu)^4} + \frac{B_1|B_3|}{(3\lambda+\mu)(\lambda+\mu)}] + A_4\frac{B_1}{(3\lambda+\mu)} + \frac{A_5 4\sqrt{3}B_1^2}{9(\mu+2\lambda)}, & p^* < p < r^*, \\ A_3[\frac{B_1^2}{(2\lambda+\mu)^2}] + A_4\frac{B_1}{(3\lambda+\mu)} + \frac{A_5 4\sqrt{3}B_1^2}{9(\mu+2\lambda)}, & 0 \le p \le r^* \ \ or\ 0 \le p \le p^*. \end{cases} \tag{2.38}$$

where

$$M_1 = \frac{B_1^3(\mu^2 + 3\mu + 2)}{48(\lambda + \mu)^3} + \frac{|B_3|}{16(3\lambda + \mu)},$$

$$A_3 = \frac{B_1 + |B_2|}{2\lambda + \mu} + \frac{(\mu - 1)B_1^2}{2(\lambda + \mu)^2},$$

$$A_4 = \frac{B_1 + 2|B_2| + |B_3|}{3\lambda + \mu} + \frac{(\mu - 1)(2\mu - 1)B_1^3}{6(\lambda + \mu)^3} + \frac{(B_1^2 + B_1|B_2|)(\mu - 1)}{(2\lambda + \mu)(\lambda + \mu)},$$

$$\begin{aligned} A_5 =& \frac{B_1 + 3|B_2| + 3|B_3| + |B_4|}{\mu + 4\lambda} + \frac{(\mu - 1)B_1^4|23\mu - 17\mu^2 - 8|}{24(\lambda + \mu)^4} + \\ & \frac{2(\mu - 1)(B_1^2 + 2|B_2|B_1 + |B_3|B_1)}{(\mu + \lambda)(3\lambda + \mu)} + \frac{(\mu - 1)(B_1 + B_2)^2}{(2\lambda + \mu)^2} + \frac{(B_1^3 + B_1^2|B_2|)(\mu - 1)(5\mu - 4)}{2(\mu + 2\lambda)(\lambda + \mu)^2} \end{aligned}$$

and p^*, r^* are given by (2.7) and (2.37), respectively.

Proof Because

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2),$$

by applying the triangle inequality, we obtain

$$|H_3(1)| \le |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|. \tag{2.39}$$

Next, substituting (2.1), (2.6), (2.35) and (2.36) into (2.39), we easily get the desired assertion (2.38). □

If we choose $\mu = 1, \lambda = 1$ and $\lambda = \mu = 1$ in Theorem 2.7, respectively, we can obtain the following corollaries.

Corollary 2.8 *If the function $f(z) \in H_\sigma^1(\lambda, \varphi)$ $(\lambda \geq 1)$ and is of the form (1.1), then we have*

$$|H_3(1)| \leq \begin{cases} A_3[\frac{B_1^4}{(\lambda+1)^4} + \frac{B_1|B_3|}{(3\lambda+1)(\lambda+1)}] + 8A_4M_1 + \frac{A_5 4\sqrt{3}B_1^2}{9(1+2\lambda)}, & p^* < p \leq 2 \quad or\ r^* < p \leq 2, \\ A_3[\frac{B_1^2}{(2\lambda+1)^2}] + 8A_4M_1 + \frac{A_5 4\sqrt{3}B_1^2}{9(1+2\lambda)}, & r^* < p < p^*, \\ A_3[\frac{B_1^4}{(\lambda+1)^4} + \frac{B_1|B_3|}{(3\lambda+1)(\lambda+1)}] + A_4\frac{B_1}{(3\lambda+1)} + \frac{A_5 4\sqrt{3}B_1^2}{9(1+2\lambda)}, & p^* < p < r^*, \\ A_3[\frac{B_1^2}{(2\lambda+1)^2}] + A_4\frac{B_1}{(3\lambda+1)} + \frac{A_5 4\sqrt{3}B_1^2}{9(1+2\lambda)}, & 0 \leq p \leq r^* \ or\ 0 \leq p \leq p^*, \end{cases} \tag{2.40}$$

where

$$M_1 = \frac{B_1^3}{8(\lambda+1)^3} + \frac{|B_3|}{16(3\lambda+1)},$$

$$A_3 = \frac{B_1 + |B_2|}{2\lambda+1},$$

$$A_4 = \frac{B_1 + 2|B_2| + |B_3|}{3\lambda+1},$$

$$A_5 = \frac{B_1 + 3|B_2| + 3|B_3| + |B_4|}{1+4\lambda}$$

and p^*, r^* are given by (2.7) and (2.37) with $\mu = 1$, respectively.

Corollary 2.9 *If the function $f(z) \in H_\sigma^\mu(1, \varphi)$ $(\mu \geq 1)$ and is of the form (1.1), then we have*

$$|H_3(1)| \leq \begin{cases} A_3[\frac{B_1^4(\mu^2+3\mu+2)}{6(1+\mu)^4} + \frac{B_1|B_3|}{(3+\mu)(1+\mu)}] + 8A_4M_1 + \frac{A_5 4\sqrt{3}B_1^2}{9(\mu+2)}, & p^* < p \leq 2 \text{ or } r^* < p \leq 2, \\ A_3[\frac{B_1^2}{(2+\mu)^2}] + 8A_4M_1 + \frac{A_5 4\sqrt{3}B_1^2}{9(\mu+2)}, & r^* < p < p^*, \\ A_3[\frac{B_1^4(\mu^2+3\mu+2)}{6(1+\mu)^4} + \frac{B_1|B_3|}{(3+\mu)(1+\mu)}] + A_4\frac{B_1}{(3+\mu)} + \frac{A_5 4\sqrt{3}B_1^2}{9(\mu+2)}, & p^* < p < r^*, \\ A_3[\frac{B_1^2}{(2+\mu)^2}] + A_4\frac{B_1}{(3+\mu)} + \frac{A_5 4\sqrt{3}B_1^2}{9(\mu+2)}, & 0 \leq p \leq r^* \text{ or } 0 \leq p \leq p^*, \end{cases} \tag{2.41}$$

where

$$M_1 = \frac{B_1^3(\mu^2+3\mu+2)}{48(1+\mu)^3} + \frac{|B_3|}{16(3+\mu)},$$

$$A_3 = \frac{B_1 + |B_2|}{2+\mu} + \frac{(\mu-1)B_1^2}{2(1+\mu)^2},$$

$$A_4 = \frac{B_1 + 2|B_2| + |B_3|}{3+\mu} + \frac{(\mu-1)(2\mu-1)B_1^3}{6(1+\mu)^3} + \frac{(B_1^2 + B_1|B_2|)(\mu-1)}{(2+\mu)(1+\mu)},$$

$$\begin{aligned} A_5 = & \frac{B_1 + 3|B_2| + 3|B_3| + |B_4|}{\mu+4} + \frac{(\mu-1)B_1^4|23\mu - 17\mu^2 - 8|}{24(1+\mu)^4} + \\ & \frac{2(\mu-1)(B_1^2 + 2|B_2|B_1 + |B_3|B_1)}{(\mu+1)(3+\mu)} + \frac{(\mu-1)(B_1+B_2)^2}{(2+\mu)^2} + \\ & \frac{(B_1^3 + B_1^2|B_2|)(\mu-1)(5\mu-4)}{2(\mu+2)(1+\mu)^2} \end{aligned}$$

and p^*, r^* are given by (2.7) and (2.37) with $\lambda = 1$, respectively.

Corollary 2.10 *If the function $f(z) \in H^1_\sigma(1,\varphi)$ and is of the form (1.1), then we have*

$$|H_3(1)| \leq \begin{cases} \frac{(B_1+|B_2|)(B_1^4+8B_1|B_3|)}{192} + \frac{(B_1+2|B_2|+|B_3|)(48B_1^3+|B_3|)}{32} + \\ \quad \frac{4\sqrt{3}B_1^2(B_1+3|B_2|+3|B_3|+|B_4|)}{135}, \quad p^* < p \leq 2, \\ \frac{(B_1+|B_2|)(B_1^4+8B_1|B_3|)}{192} + \frac{(B_1+2|B_2|+|B_3|)(48B_1^3+|B_3|)}{32} + \\ \quad \frac{4\sqrt{3}B_1^2(B_1+3|B_2|+3|B_3|+|B_4|)}{135}, \quad r^* < p \leq 2, \\ \frac{(B_1+|B_2|)B_1^2}{27} + \frac{(B_1+2|B_2|+|B_3|)(48B_1^3+|B_3|)}{32} + \\ \quad \frac{4\sqrt{3}B_1^2(B_1+3|B_2|+3|B_3|+|B_4|)}{135}, \quad r^* < p < p^*, \\ \frac{(B_1+|B_2|)(B_1^4+8B_1|B_3|)}{192} + \frac{B_1(B_1+2|B_2|+|B_3|)}{16} + \\ \quad \frac{4\sqrt{3}B_1^2(B_1+3|B_2|+3|B_3|+|B_4|)}{135}, \quad p^* < p < r^*, \\ \frac{(B_1+|B_2|)B_1^2}{27} + \frac{B_1(B_1+2|B_2|+|B_3|)}{16} + \\ \quad \frac{4\sqrt{3}B_1^2(B_1+3|B_2|+3|B_3|+|B_4|)}{135}, \quad 0 \leq p \leq r^* \text{ or } 0 \leq p \leq p^*, \end{cases} \tag{2.42}$$

where p^*, r^* are given by (2.7) and (2.37) with $\lambda = \mu = 1$, respectively.

Acknowledgements We would like to thank the referees for their valuable comments, suggestions and corrections.

References

[1] P. L. DUREN. *Univalent Functions*. Grundlehren der Mathematischen Wissenschaften, Band 259, Springer, Berlin, 1983.

[2] H. M. SRIVASTAVA, S. OWA. *Current Topics in Analytic Function Theory*. World Scientific, Singapore, 1992.

[3] S. S. MILLER, P. T. MOCANU. *Differential Subordinations: Theory and Applications*. Marcel Dekker Incorporated, New York and Basel, 2000.

[4] Huo TANG, Guantie DENG, Shuhai LI. *Coefficient estimates for new subclasses of Ma-Minda bi-univalent functions*. J. Inequal. Appl., 2013, **2013**(317): 1–10.

[5] M. CAGLAR, H. ORHAN, N. YAGMUR. *Coefficient bounds for new subclasses of bi-univalent functions*. Filomat, 2013, **27**(7): 1165–1171.

[6] S. S. KUMAR, V. KUMAR, V. RAVICHANDRAN. *Estimates for the initial coefficients of bi-univalent functions*. Tamsui Oxford Journal of Information and Mathematics, 2012, **29**(4): 487–504.

[7] R. M. ALI, S. K. LEE, V. RAVICHANDRAN, et al. *Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions*. Appl. Math. Lett., 2012, **25**(2): 344–351.

[8] B. A. FRASIN, M. K. AOUF. *New subclasses of bi-univalent functions*. Appl. Math. Lett., 2011, **24**(9): 1569–1573.

[9] H. M. SRIVASTAVA, A. K. MISHRA, P. GOCHHAYAT. *Certain subclasses of analytic and bi-univalent functions*. Appl. Math. Lett., 2010, **23**(10): 1188–1192.

[10] C. POMMERENKE. *On the coefficients and Hankel determinants of univalent functions*. J. Lond. Math. Soc., 1966, **41**(1): 111–122.

[11] J. W. NOONAN, D. K. THOMAS. *On the second Hankel determinant of areally mean p-valent functions*. Trans. Amer. Math. Soc., 1976, **223**(2): 337–346.

[12] K. I. NOOR. *Hankel determinant problem for the class of functions with bounded boundary rotation*. Rev. Roumaine Math. Pure Appl., 1983, **28**(8): 731–739.

[13] R. EHRENBORG. *The Hankel determinant of exponential polynomials*. Amer. Math. Monthly, 2000, **107**(6): 557–560.

[14] W. KOEPF. *On the Fekete-Szegö problem for close-to-convex functions II*. Arch. Math., 1987, **49**: 420–433.

[15] H. M. SRIVASTAVA, S. HUSSAIN, A. RAZIQ, et al. *The Fekete-Szegö functional for a subclass of analytic functions associated with quasi-subordination*. Carpathian J. Math., 2018, **34**: 103–113.

[16] H. M. SRIVASTAVA, A. K. MISHRA, M. K. DAS. *The Fekete-Szegö problem for a subclass of close-to-convex functions*. Complex Variables Theory Appl., 2001, **44**: 145–163.

[17] Huo TANG, H. M. SRIVASTAVA, S. SIVASUBRAMANIAN, et al. *The Fekete-Szegö functional problems for some classes of m-fold symmetric bi-univalent functions*. J. Math. Inequal., 2016, **10**: 1063–1092.

[18] W. K. HAYMAN. *On the second Hankel determinant of mean univalent functions*. Proc. Lond. Math. Soc., 1968, **18** (1): 77–94.

[19] S. K. LEE, V. RAVICHANDRAN, S. SUBRAMANIAM. *Bounds for the second Hankel determinant of certain univalent functions*. J. Inequal. Appl. 2013, **2013** Article 281, 17 pp.

[20] M. RAZA, S. N. MALIK. *Upper bound of the third Hankel determinant for a class of analytic functions related with lemniscate of bernoulli*. J. Inequal. Appl., 2013, **2013**(1): 1–8.

[21] K. O. BABALOLA. *On $H_3(1)$ Hankel Determinants for Some Classes of Univalent Functions*. Nova Science Publishers, New York, 2010.

[22] D. BANSAL, S. MAHARANA, J. K. PRAJAPAY. *Third order Hankel determinant for certain univalent functions*. J. Korean Math. Soc., 2015, **52**(6): 1139–1148.

[23] P. ZAPRAWA. *Third Hankel determinant for classes of univalent functions*. Mediterr. J. Math., 2017, **14**(1): 1–10.

[24] Haiyan ZHANG, Huo TANG, Lina MA. *Third Hankel determinant for a class of analytic functions with respect to conjugate points*. Acta Analysis Functional Applicata, 2017, **19**(3): 279–286. (in Chinese)

[25] Haiyan ZHANG, Huo TANG, Lina MA. *Upper bound of third Hankel determinant for a class of analytic functions*. Acta Analysis Functional Applicata, 2017, **33**(2): 211–220. (in Chinese)

[26] Haiyan ZHANG, Lina MA, Huan WANG. *The estimate of upper bound of third Hankel determinant for a class of reciprocal starlike functions with respect to symmetric conjugate points*. Pure and Appl. Math., 2017, **33**(5): 503–512. (in Chinese)

[27] H. ORHAN, P. ZAPRAWA. *Third Hankel determinants for starlike and convex functions of order α*. Bull. Korean Math. Soc., 2018, **55**(1): 165–173.

[28] H. ORHAN, N. MAGESH, V. K. BALAJI. *Second Hankel determinant for certain class of bi-univalent functions*. arXiv:1502.06407v2.

[29] N. MAGESH, J. YAMINI. *Fekete-szegö problem and second Hankel determinant for a class of bi-univalent functions*. arXiv:1508.07462v2.

[30] M. CAGLAR, E. DENIZ, H. M. SRIVASTAVA. *Second Hankel determinant for certain subclasses of bi-univalent functions*. Turkish J. Math., 2017, **41**: 694–706.

[31] H. M. SRIVASTAVA, S. ALTINKAYA, S. YALCIN. *Hankel determinant for a subclass of bi-univalent functions defined by using a symmetric q-derivative operator*. Filomat, 2018, **32**(2): 503–516.

[32] R. J. LIBERA, E. J. ZLOTKIEWICZ. *Coefficient bounds for the inverse of a function with derivative in P*. Proc. Amer. Math. Soc., 1983, **87**(2): 251–257.

[33] CH. POMMERENKE. *Univalent Functions*. Vandenhoeck and Ruprecht, Gottingen, 1975.