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# On Power Finite Rank Operators

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**Abstract** An operator $F \in \mathcal{B}(X)$ is called power finite rank if $F^n$ is of finite rank for some $n \in \mathbb{N}$. In this note, we provide several interesting characterizations of power finite rank operators. In particular, we show that the class of power finite rank operators is the intersection of the class of Riesz operators and the class of operators with eventual topological uniform descent.

**Keywords** power finite rank operator; Drazin invertible; eventual topological uniform descent; Riesz operator

**MR(2010) Subject Classification** 47A05; 47B06

## 1. Introduction

Power finite rank operators $F$ (that is, $F^n$ is of finite rank for some $n \in \mathbb{N}$) were first investigated by Kaashoek and Lay [1]: they showed that the descent spectrum (resp., the ascent spectrum) is invariant under any commuting power finite rank perturbation $F$, and they conjectured that this perturbation property characterizes such operators $F$. In 2006, Burgos, Kaidi, Mbekhta and Oudghiri [2] confirmed this conjecture for the descent spectrum. Later, several authors [3, 4] extended this result to the essential descent spectrum, the left Drazin spectrum and the left essentially Drazin spectrum. In [5], using the theory of operators with eventual topological uniform descent and the technique used by Burgos et al., the authors generalized these results to various spectra originating from semi-B-Fredholm theory. But we remark here that the conjecture of Kaashoek and Lay for the ascent spectrum (or the topological uniform descent spectrum) is still unsolved. In this note, we give several interesting characterizations of power finite rank operators from another perspective. In particular, we prove that the class of power finite rank operators is the intersection of the class of Riesz operators and the class of operators with eventual topological uniform descent.

We first fix some notations in spectral theory. Throughout this note, let $\mathcal{B}(X)$ denote the Banach algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space $X$, $\mathcal{F}(X)$ denote its ideal of finite rank operators and $\mathcal{K}(X)$ denote its closed ideal of

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compact operators. An operator $T \in \mathcal{B}(X)$ is called *Riesz* if the coset $T+\mathcal{K}(X)$ is quasinilpotent in the Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$. For an operator $T \in \mathcal{B}(X)$, let $T^*$ denote its dual, $\mathcal{N}(T)$ its kernel, $\mathcal{R}(T)$ its range, $\alpha(T) = \dim \mathcal{N}(T)$ and $\beta(T) = \dim X/\mathcal{R}(T)$. If the range $\mathcal{R}(T)$ is closed and $\alpha(T) < \infty$ (resp., $\beta(T) < \infty$), then $T \in \mathcal{B}(X)$ is said to be upper semi-Fredholm (resp., lower semi-Fredholm). If $T \in \mathcal{B}(X)$ is both upper and lower Fredholm opertor, then $T$ is said to be Fredholm. For $T \in \mathcal{B}(X)$, the essential spectrum of $T$ is defined by

$$\sigma_e(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}.$$

From Atkinson characterization of Fredholm operators ([6, Theorem 1.53]), it follows that $T \in \mathcal{B}(X)$ is Riesz if and only if $\sigma_e(T) = \{0\}$.

For each $n \in \mathbb{N} := \{0, 1, 2, \ldots\}$, we set

$$c_n(T) = \dim \mathcal{R}(T^n)/\mathcal{R}(T^{n+1}) \text{ and } c'_n(T) = \dim \mathcal{N}(T^{n+1})/\mathcal{N}(T^n).$$

It follows from [7, Lemmas 3.1 and 3.2] that, for every $n \in \mathbb{N}$,

$$c_n(T) = \dim X/(\mathcal{R}(T) + \mathcal{N}(T^n)), \quad c'_n(T) = \dim \mathcal{N}(T) \cap \mathcal{R}(T^n).$$

Hence, it is easy to see that the sequences $\{c_n(T)\}_{n=0}^{\infty}$ and $\{c'_n(T)\}_{n=0}^{\infty}$ are decreasing. Recall that the descent and the ascent of $T \in \mathcal{B}(X)$ are $\mathrm{dsc}(T) = \inf\{n \in \mathbb{N} : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}$ and $\mathrm{asc}(T) = \inf\{n \in \mathbb{N} : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}$, respectively (the infimum of an empty set is defined to be $\infty$). That is,

$$\mathrm{dsc}(T) = \inf\{n \in \mathbb{N} : c_n(T) = 0\}$$

and

$$\mathrm{asc}(T) = \inf\{n \in \mathbb{N} : c'_n(T) = 0\}.$$

Similarly, the essential descent and the essential ascent of $T \in \mathcal{B}(X)$ are

$$\mathrm{dsc}_e(T) = \inf\{n \in \mathbb{N} : c_n(T) < \infty\}$$

and

$$\mathrm{asc}_e(T) = \inf\{n \in \mathbb{N} : c'_n(T) < \infty\}.$$

If $\mathrm{asc}(T) < \infty$ and $\mathcal{R}(T^{\mathrm{asc}(T)+1})$ is closed, then $T$ is said to be left Drazin invertible. If $\mathrm{dsc}(T) < \infty$ and $\mathcal{R}(T^{\mathrm{dsc}(T)})$ is closed, then $T$ is said to be right Drazin invertible. If $\mathrm{asc}(T) = \mathrm{dsc}(T) < \infty$, then $T$ is said to be Drazin invertible. Clearly, $T \in \mathcal{B}(X)$ is both left and right Drazin invertible if and only if $T$ is Drazin invertible. If $\mathrm{asc}_e(T) < \infty$ and $\mathcal{R}(T^{\mathrm{asc}_e(T)+1})$ is closed, then $T$ is said to be left essentially Drazin invertible. If $\mathrm{dsc}_e(T) < \infty$ and $\mathcal{R}(T^{\mathrm{dsc}_e(T)})$ is closed, then $T$ is said to be right essentially Drazin invertible.

Let $T \in \mathcal{B}(X)$ and $d \in \mathbb{N}$. The operator range topology on $\mathcal{R}(T^d)$ is defined by the norm $\|\cdot\|_{\mathcal{R}(T^d)}$ such that for all $y \in \mathcal{R}(T^d)$,

$$\|y\|_{\mathcal{R}(T^d)} = \inf\{\|x\| : x \in X, y = T^d x\}.$$

We say that $T$ has uniform descent for $n \geq d$ if $\mathcal{R}(T) + \mathcal{N}(T^{n+1}) = \mathcal{R}(T) + \mathcal{N}(T^n)$ for all $n \geq d$. If in addition $\mathcal{R}(T^n)$ is closed in the operator range topology of $\mathcal{R}(T^d)$ for all $n \geq d$, then we say that $T$ has eventual topological uniform descent, and more precisely, that $T$ has topological uniform descent for $n \geq d$.

## 2. Main result

The main result of this note is the following theorem, which provides several interesting characterizations of power finite rank operators.

**Theorem 2.1** *Let $F \in \mathcal{B}(X)$. The following statements are equivalent:*

*(1) $F^n \in \mathcal{F}(X)$ for some $n \in \mathbb{N}$;*

*(2) $F$ is Riesz and Drazin invertible;*

*(3) $F$ is Riesz and left Drazin invertible;*

*(4) $F$ is Riesz and right Drazin invertible;*

*(5) $F$ is Riesz and left essentially Drazin invertible;*

*(6) $F$ is Riesz and right essentially Drazin invertible;*

*(7) $F$ is Riesz and $\mathrm{dsc}(F) < \infty$;*

*(8) $F$ is Riesz and $\mathrm{dsc}_e(F) < \infty$;*

*(9) $F$ is a Riesz operator with eventual topological uniform descent.*

**Proof** (1) $\Rightarrow$ (2). Since $F^n \in \mathcal{F}(X)$ for some $n \in \mathbb{N}$, $F^n$ is Riesz, that is, $\sigma_e(F^n) = \{0\}$. By the spectral mapping theorem for the essential spectrum ([6, Corollary 3.61]), we get that $\sigma_e(F) = \{0\}$, so $F$ is Riesz.

From the fact that the range spaces are decreasing

$$\mathcal{R}(F) \supseteq \mathcal{R}(F^2) \supseteq \cdots \supseteq \mathcal{R}(F^n) \supseteq \cdots,$$

and the fact that $\mathcal{R}(F^n)$ is finite-dimensional, it follows that $\mathrm{dsc}(F) < \infty$. Since $F^n \in \mathcal{F}(X)$, $\mathcal{R}(F^n)$ is a closed and finite-dimensional subspace, and hence $\dim X/\mathcal{N}(F^n) = \dim \mathcal{R}(F^n) < \infty$. That is, $\mathcal{N}(F^n)$ has finite-codimension in $X$. This together with the fact that the null spaces are increasing

$$\mathcal{N}(F) \subseteq \mathcal{N}(F^2) \subseteq \cdots \subseteq \mathcal{N}(F^n) \subseteq \cdots \subseteq X,$$

implies that $\mathrm{asc}(F) < \infty$. Thus $F$ is Drazin invertible.

(2) $\Rightarrow$ (3) $\Rightarrow$ (5). Clear.

(5) $\Rightarrow$ (1). Suppose that $F$ is Riesz and left essentially Drazin invertible. Then there exists $n \in \mathbb{N}$ such that $\mathrm{asc}_e(F) = n < \infty$ and $\mathcal{R}(F^{n+1})$ is closed. By [8, Lemma 7], it follows that $\mathcal{R}(F^n)$ is closed. Therefore, the restriction $F|_{\mathcal{R}(F^n)} : \mathcal{R}(F^n) \longrightarrow \mathcal{R}(F^n)$ is an upper semi-Fredholm operator. Since $F$ is Riesz, by [6, Theorem 3.113], the restriction $F|_{\mathcal{R}(F^n)}$ of $F$ to $\mathcal{R}(F^n)$ is also Riesz. Hence $F|_{\mathcal{R}(F^n)} - \lambda$ is upper semi-Fredholm for all non-zero $\lambda \in \mathbb{C}$. Thus $F|_{\mathcal{R}(F^n)} - \lambda$ is upper semi-Fredholm for all $\lambda \in \mathbb{C}$. That is, the upper semi-Fredholm spectrum

$$\sigma_{\mathrm{USF}}(F|_{\mathcal{R}(F^n)}) := \{\lambda \in \mathbb{C} : F|_{\mathcal{R}(F^n)} - \lambda \text{ is not upper semi-Fredholm}\}$$

of $F|_{\mathcal{R}(F^n)}$ is empty. Also, note that the upper semi-Fredholm spectrum of an operator acting on an infinite-dimensional complex Banach space is non-empty [9]. Therefore, $\mathcal{R}(F^n)$ is finite-dimensional.

(2) $\Rightarrow$ (4) $\Rightarrow$ (6) $\Rightarrow$ (8). Clear.

(2) $\Rightarrow$ (4) $\Rightarrow$ (7) $\Rightarrow$ (8). Clear.

(8) $\Rightarrow$ (9). Suppose that $F$ is Riesz and $\mathrm{dsc}_e(F) < \infty$. Then there exists $m \in \mathbb{N}$ such that $\mathrm{des}_e(F) = m < \infty$. That is, $\mathcal{R}(F) + \mathcal{N}(F^m)$ has finite-codimension in $X$. This together with the fact that the spaces $\{\mathcal{R}(F) + \mathcal{N}(F^n)\}_{n=0}^{\infty}$ are increasing

$$\mathcal{R}(F) + \mathcal{N}(F^0) \subseteq \mathcal{R}(F) + \mathcal{N}(F) \subseteq \mathcal{R}(F) + \mathcal{N}(F^2) \subseteq \cdots \subseteq \mathcal{R}(F) + \mathcal{N}(F^n) \subseteq \cdots \subseteq X,$$

we infer that there exists $d \geq m$ such that $\mathcal{R}(F)+\mathcal{N}(F^{n+1}) = \mathcal{R}(F)+\mathcal{N}(F^n)$ for all $n \geq d$. Note that $\dim \mathcal{R}(F^d)/\mathcal{R}(F^n) = \dim \mathcal{R}(F^d)/\mathcal{R}(F^{d+1}) + \dim \mathcal{R}(F^{d+1})/\mathcal{R}(F^{d+2}) + \cdots + \dim \mathcal{R}(F^{n-1})/\mathcal{R}(F^n)$ is finite for all $n > d$. Because $\mathcal{R}(F^n)$ can be viewed as the operator range of the restriction

$$F^{n-d}|_{\mathcal{R}(F^d)} : (\mathcal{R}(F^d), ||\cdot||_{\mathcal{R}(F^d)}) \longrightarrow (\mathcal{R}(F^d), ||\cdot||_{\mathcal{R}(F^d)})$$

of $F^{n-d}$ to $\mathcal{R}(F^d)$, by [6, Corollary 1.15] it follows that $\mathcal{R}(F^n)$ is closed in the operator range topology of $\mathcal{R}(F^d)$ for all $n > d$. Therefore, $F$ has eventual topological uniform descent.

(9) $\Rightarrow$ (1). Suppose that $F$ is a Riesz operator with eventual topological uniform descent. Then $F^*$ also is a Riesz operator ([6, Corollary 3.114]). Hence $F^*$ has single valued extension property at every $\lambda \in \mathbb{C}$ ([10, Theorem 0.3]): here we say that an operator $T \in \mathcal{B}(X)$ has the single-valued extension property at $\lambda_0 \in \mathbb{C}$, if for every open neighbourhood $U_{\lambda_0} \subseteq \mathbb{C}$ of $\lambda_0$, the only analytic solution $f : U_{\lambda_0} \to X$ of the equation $(T - \lambda)f(\lambda) = 0$ for all $\lambda \in U_{\lambda_0}$ is the zero function on $U_{\lambda_0}$. In particular, $F^*$ has single valued extension property at 0. Because $F$ has eventual topological uniform descent, by [11, Theorem 3.4] it follows that $\mathrm{des}(F) < \infty$. Then there exists $n \in \mathbb{N}$ such that $\mathrm{des}(F) = n < \infty$. Therefore, the dimension of the quotient space $X/(\mathcal{R}(F) + \mathcal{N}(F^n))$ is zero. So the induced operator $F_{\mathcal{N}(F^n)}$ defined on $X/\mathcal{N}(F^n)$ by

$$F_{\mathcal{N}(F^n)}(x + \mathcal{N}(F^n)) = Fx + \mathcal{N}(F^n)$$

is surjective, and hence is lower semi-Fredholm. Since $F$ is Riesz, by [6, Theorem 3.115], the induced operator $F_{\mathcal{N}(F^n)}$ is also Riesz. Hence $F_{\mathcal{N}(F^n)} - \lambda$ is lower semi-Fredholm for all non-zero $\lambda \in \mathbb{C}$. Thus $F_{\mathcal{N}(F^n)} - \lambda$ is lower semi-Fredholm for all $\lambda \in \mathbb{C}$. That is, the lower semi-Fredholm spectrum

$$\sigma_{\mathrm{LSF}}(F_{\mathcal{N}(F^n)}) := \{\lambda \in \mathbb{C} : F_{\mathcal{N}(F^n)} - \lambda \text{ is not upper semi-Fredholm}\}$$

of $F_{\mathcal{N}(F^n)}$ is empty. Also, note that the lower semi-Fredholm spectrum of an operator acting on an infinite-dimensional complex Banach space is non-empty [9]. Therefore, $X/\mathcal{N}(F^n)$ is finite-dimensional, so is $\mathcal{R}(F^n)$. $\square$

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