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Nonlocal Integral Boundary Value Problem of Bagley-Torvik Type Fractional Differential Equations and Inclusions

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Abstract In this article, we consider the Bagley-Torvik type fractional differential equation ${}^cD^{\nu_1}l(t) - a{}^cD^{\nu_2}l(t) = g(t,l(t))$ and differential inclusion ${}^cD^{\nu_1}l(t) - a{}^cD^{\nu_2}l(t) \in G(t,l(t))$, $t \in (0,1)$ subjecting to $l(0) = l_0$, and $l(1) = \lambda' \int_0^\omega \frac{(\omega-s)^{\chi-1}l(s)}{\Gamma(\chi)}\mathrm{d}s$, where $1 < \nu_1 \le 2$, $1 \le \nu_2 < \nu_1$, $0 < \omega \le 1$, $\chi = \nu_1 - \nu_2 > 0$, a, λ' are given constants. By using Leray-Schauder degree theory and fixed point theorems, we prove the existence of solutions. Our results extend the existence theorems for the classical Bagley-Torvik equation and some related models.

Keywords fractional differential equations and inclusions; integral boundary conditions; Leray-Schauder theory

MR(2010) Subject Classification 34A08; 34B15

1. Introduction

We are concerned with the following generalized Bagley-Torvik type fractional differential equation and inclusion:

$$\begin{cases} {}^cD^{\nu_1}l(t) - a{}^cD^{\nu_2}l(t) = g(t,l(t)), \quad t \in (0,1), \\ l(0) = l_0, \quad l(1) = \lambda' I_{0^+}^{\chi} l(\omega) = \lambda' \int_0^\omega \frac{(\omega-s)^{\chi-1}l(s)}{\Gamma(\chi)}\mathrm{d}s, \end{cases} \tag{1.1}$$

$$\begin{cases} {}^cD^{\nu_1}l(t) - a{}^cD^{\nu_2}l(t) \in G(t,l(t)), \quad t \in (0,1), \\ l(0) = l_0, \quad l(1) = \lambda' I_{0^+}^{\chi} l(\omega) = \lambda' \int_0^\omega \frac{(\omega-s)^{\chi-1}l(s)}{\Gamma(\chi)}\mathrm{d}s, \end{cases} \tag{1.2}$$

respectively, where ${}^cD^{\nu_1}$ and ${}^cD^{\nu_2}$ are Caputo fractional derivative with $1 < \nu_1 \le 2$, $1 \le \nu_2 < \nu_1$, $0 < \omega \le 1$, $\chi = \nu_1 - \nu_2 > 0$, a, λ' are given constants, $G : [0,1] \times \mathbb{R} \to P(\mathbb{R})$ is a multivalued map, $P(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

Many physical phenomena including abnormal diffusion and complex viscosity can be modeled as fractional differential equations, which become a key issue to investigate many physical

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phenomena and a number of results on this topic have emerged in the last decade. As a consequence there was an intensive development of the theory of fractional differential equations and differential inclusion, for example [1–4].

For the problem of fractional differential equations, multi-term fractional differential equation is a hot research direction owing to its wide use in practice and technique sciences, such as physics, mechanics, chemistry, etc. An important result on multi-term fractional calculus is formulated by Bagley and Torvik in [5], where the multi-term fractional differential equation $Ax''(t) + B^cD^{3/2}x(t) + Cx(t) = g(t)$ is used to describe the motion of thin plates in Newtonian fluids, which is called Bagley-Torvik equation [6]. Based on this model, the nonlinear multi-term fractional differential equations were rediscovered and popularized by Kaufmann and Yao in [7]. As far as the author knows, there are few papers on the existence of the generalized Bagley-Torvik type fractional differential inclusions (1.2) besides Hamza Eibadawi Ibrahim, Dong and Fan [8]. In [8], the authors studied the following equation

$$\begin{cases} {}^cD^\nu w(t) - a^cD^\omega w(t) + h(t, w(t)) = 0, \quad t \in (0,1), \\ w(0) = w_0, \quad w(1) = w_1, \end{cases}$$

where ${}^cD^\nu$ and ${}^cD^\omega$ are the Caputo fractional derivatives, $1 < \nu \le 2$, $1 \le \omega < \nu$.

Very recently, in [9], the authors considered the following equation

$$\begin{cases} D^\nu_{0+}v(t) + p(t)g(t, v(t)) = 0, \quad t \in (0,1), \\ v(0) = v'(0) = \cdots = v^{n-2}(0) = 0, \\ v(1) = \epsilon I^\sigma_{0+}v(\rho) = \epsilon \int_0^\rho \frac{(\rho-\tau)^{\sigma-1}v(\tau)}{\Gamma(\sigma)}\mathrm{d}\tau, \end{cases}$$

where $\nu \in (n-1, n]$ is a real number, $n > 2$, $0 < \rho \le 1$.

In this article, we shall be concerned with the Bagley-Torvik type nonlinear fractional differential equation (1.1) and differential inclusion (1.2) with nonlocal integral boundary conditions via Leray-Schauder degree theory and fixed point theorems. Our results extend the existence theorems for the classical Bagley-Torvik equation and some related models.

The structure of this article is as follows: some preliminary knowledge is introduced in Section 2; some existence criteria are derived for equation (1.1) in Section 3; some existence criteria are derived for equation (1.2) with convex valued and nonconvex valued mutifunctions in Section 4; In the end, we consider an application of our main work.

2. Preliminaries

Now, we outline some necessary definitions and lemmas of the fractional order differential and integral theory, which can be found in the literature [6].

Definition 2.1 *Suppose* $\eta \in L^1([0,1], \mathbb{R})$, $\iota > 0$. *If* $\int_0^t \frac{(t-\tau)^{\iota-1}}{\Gamma(\iota)}\eta(\tau)\mathrm{d}\tau < \infty$, *then*

$$I^\iota_{0+}\eta(t) = \int_0^t \frac{(t-\tau)^{\iota-1}}{\Gamma(\iota)}\eta(\tau)\mathrm{d}\tau, \quad t \in [0,1],$$

is called ι *order Riemann-Liouville fractional integral of a function* η, *where* $\Gamma(\cdot)$ *is the Euler's*

Gamma function defined by $\Gamma(\iota)=\int_0^\infty t^{\iota-1}e^{-\iota}\mathrm{d}t$.

Definition 2.2 *Suppose* $\eta\in L^1([0,1],\mathbb{R})$, $\eta^{(n)}\in L^1([0,1],\mathbb{R})$, $\iota>0$. *We define*

$$^cD^{\iota}_{0+}\eta(t)=\frac{1}{\Gamma(n-\iota)}\int_0^t\frac{\eta^{(n)}(s)}{(t-s)^{\iota-n+1}}\mathrm{d}s,\quad t\in[0,1],$$

as the ι *order Caputo fractional derivative of a function* η, *where* $n=[\iota]+1$, $[\iota]$ *denotes the integer part of the real number* ι.

Lemma 2.3 *Suppose* $\iota>0$ *and* $\eta\in L^1([0,1],\mathbb{R})$. *Consider the following differential equation*

$$^cD^{\iota}_{0+}\eta(t)=0,$$

then there exist some constants $d_k\in\mathbb{R}$, $k=0,1,2,\dots,n-1$ *such that*

$$\eta(t)=d_0+d_1t+d_2t^2+\cdots+d_{n-1}t^{n-1},$$

where $n=[\iota]+1$, $[\iota]$ *denotes the integer part of the real number* ι.

Lemma 2.4 *Suppose* $\eta\in L^1([0,1],\mathbb{R})$, *and* $\eta^{(n)}\in L^1([0,1],\mathbb{R})$. *Then there exist some constants* $d_k\in\mathbb{R}$, $k=0,1,2,\dots,n-1$ *satisfying*

$$I^{\iota c}_{0+}D^{\iota}_{0+}\eta(t)=\eta(t)+d_0+d_1t+d_2t^2+\cdots+d_{n-1}t^{n-1}.$$

For simplicity, we denote $^cD^{\iota}_{0+}$ and I^{ι}_{0+} by $^cD^{\iota}$ and I^{ι}, respectively.

Lemma 2.5 *Let* $a-\Gamma(\nu_1-\nu_2+2)\neq 0$, $h\in C([0,1],\mathbb{R})$, $1<\nu_1\le 2$, $1\le\nu_2<\nu_1$, $\chi=\nu_1-\nu_2$ *and* $0<\omega<1$. *Consider the following equation*

$$^cD^{\nu_1}l(t)-a^cD^{\nu_2}l(t)-h(t)=0,\quad t\in(0,1),\tag{2.1}$$

with

$$l(0)=l_0,\quad l(1)=\lambda' I^{\chi}l(\omega)=\lambda'\int_0^{\omega}\frac{(\omega-\xi)^{\chi-1}l(\xi)}{\Gamma(\chi)}\mathrm{d}\xi.\tag{2.2}$$

The solution of (2.1) with (2.2) is given by

$$l(t)=e(t)+\int_0^1T_1(t,\xi)l(\xi)\mathrm{d}\xi+\int_0^1T_2(t,\xi)h(\xi)\mathrm{d}\xi,\tag{2.3}$$

where

$$e(t)=l_0-\frac{al_0}{\Gamma(\chi-1)}t^{\chi}+\frac{[at^{\chi+1}-t\Gamma(\chi+2)]}{\Gamma(\chi+2)-a}\cdot\frac{(\Gamma(\chi+1)-a)l_0}{\Gamma(\chi+1)},$$

$$T_1(t,\xi)=\frac{1}{\Gamma(\chi)}\begin{cases}\frac{[at^{\chi+1}-t\Gamma'(\chi+2)]}{\Gamma(\chi+2)-a}(a(1-\xi)^{\chi-1}-\lambda'(\omega-\xi)^{\chi-1})+ & \\ \qquad a(t-\xi)^{\chi-1}, & 0\le\xi\le t\le 1,\xi\le\omega,\\ \frac{[at^{\chi+1}-t\Gamma(\chi+2)]}{\Gamma(\chi+2)-a}(a(1-\xi)^{\chi-1}-\lambda'(\omega-\xi)^{\chi-1}), & 0\le t\le\xi\le\omega\le 1,\\ \frac{[at^{\chi+1}-t\Gamma(\chi+2)]}{\Gamma(\chi+2)-a}a(1-\xi)^{\chi-1}+a(t-\xi)^{\chi-1}, & 0\le\omega\le\xi\le t\le 1,\\ \frac{[at^{\chi+1}-t\Gamma(\chi+2)]}{\Gamma(\chi+2)-a}a(1-\xi)^{\chi-1}, & 0\le t\le\xi\le 1,\xi\ge\omega,\end{cases}\tag{2.4}$$

and

$$T_2(t,\xi)=\frac{1}{\Gamma(\nu_1)}\begin{cases}\frac{[at^{\chi+1}-t\Gamma(\chi+2)]}{\Gamma(\chi+2)-a}(1-\xi)^{\nu_1-1}+(t-\xi)^{\nu_1-1}, & 0\le\xi\le t\le 1,\\ \frac{[at^{\chi+1}-t\Gamma(\chi+2)]}{\Gamma(\chi+2)-a}(1-\xi)^{\nu_1-1}, & 0\le t\le\xi\le 1.\end{cases}\tag{2.5}$$

Proof From Lemma 2.4, it follows that

$$I^{\nu_2 c}D^{\nu_2}l(t)=l(t)+c_1+c_2t,\quad t\in[0,1],$$

for some constants c_1, c_2. Applying the operator I^{ν_1} to both sides of (2.1), one obtains that

$$I^{\nu_1 c}D^{\nu_1}l(t)=aI^{\nu_1 c}D^{\nu_2}l(t)+I^{\nu_1}h(t),\quad t\in[0,1].$$

Due to the property of fractional integral,

$$\begin{aligned}I^{\nu_1 c}D^{\nu_2}l(t)=&I^{\chi}(I^{\nu_2 c}D^{\nu_2}l(t))=I^{\chi}(l(t)+c_1+c_2t)\\=&I^{\chi}l(t)+\frac{c_1t^{\chi}}{\Gamma(\chi+1)}+\frac{c_2t^{\chi+1}}{\Gamma(\chi+2)}.\end{aligned}$$

Then

$$l(t)+c_1+c_2t=aI^{\chi}l(t)+\frac{ac_1t^{\chi}}{\Gamma(\chi+1)}+\frac{ac_2t^{\chi+1}}{\Gamma(\chi+2)}+I^{\nu_1}h(t),\quad t\in[0,1].\tag{2.6}$$

By $l(0)=l_0$, we have $c_1=-l_0$. By $l(1)=\lambda' I^{\chi}l(\omega)$, we have

$$\lambda' I^{\chi}l(\omega)-l_0+c_2=aI^{\chi}l(1)-\frac{al_0}{\Gamma(\chi+1)}+\frac{ac_2}{\Gamma(\chi+2)}+I^{\nu_1}h(1),$$

when $1-\frac{a}{\Gamma(\chi+2)}\neq 0$, we obtain

$$c_2=\frac{1}{1-\frac{a}{\Gamma(\chi+2)}}(l_0-\frac{al_0}{\Gamma(\chi+1)}+aI^{\chi}l(1)-\lambda' I^{\chi}l(\omega)+I^{\nu_1}h(1)).$$

Hence, the solution of (2.1) and (2.2) is

$$\begin{aligned}l(t)=&l_0-\frac{al_0}{\Gamma(\chi+1)}t^{\chi}+(\frac{a}{\Gamma(\chi+2)}t^{\chi+1}-t)c_2+aI^{\chi}l(t)+I^{\nu_1}h(t)\\=&l_0-\frac{al_0}{\Gamma(\chi+1)}t^{\chi}+\frac{[at^{\chi+1}-t\Gamma(\chi+2)]}{\Gamma(\chi+2)-a}(l_0-\frac{al_0}{\Gamma(\chi+1)})+\\&\frac{[at^{\chi+1}-t\Gamma(\chi+2)]}{\Gamma(\chi+2)-a}(aI^{\chi}l(1)-\lambda' I^{\chi}l(\omega))+\\&aI^{\chi}l(t)+\frac{[at^{\chi+1}-t\Gamma(\chi+2)]}{\Gamma(\chi+2)-a}(I^{\nu_1}h(1))+I^{\nu_1}h(t).\end{aligned}$$

Let

$$e(t)=l_0-\frac{al_0}{\Gamma(\chi+1)}t^{\chi}+\frac{[at^{\chi+1}-t\Gamma(\chi+2)]}{\Gamma(\chi+2)-a}(l_0-\frac{al_0}{\Gamma(\chi+1)})$$

and

$$C(t)=\frac{at^{\chi+1}-t\Gamma(\chi+2)}{\Gamma(\chi+2)-a}.$$

Then, for $t\le\omega$, we obtain

$$l(t)=e(t)+\int_0^t\frac{[aC(t)(1-\xi)^{\chi-1}-\lambda' C(t)(\omega-\xi)^{\chi-1}+a(t-\xi)^{\chi-1}]}{\Gamma(\chi)}l(\xi)\mathrm{d}\xi+$$

$$\begin{aligned}&\int_t^{\omega}\frac{[aC(t)(1-\xi)^{\chi-1}-\lambda' C(t)(\omega-\xi)^{\chi-1}]}{\Gamma(\chi)}l(\xi)\mathrm{d}\xi+\int_{\omega}^1\frac{aC(t)(1-\xi)^{\chi-1}}{\Gamma(\chi)}l(\xi)\mathrm{d}\xi+\\&\int_0^t\frac{[C(t)(1-\xi)^{\nu_1-1}+(t-\xi)^{\nu_1-1}]}{\Gamma(\nu_1)}h(\xi)\mathrm{d}\xi+\int_t^1\frac{C(t)(1-\xi)^{\nu_1-1}}{\Gamma(\nu_1)}h(\xi)\mathrm{d}\xi\\=&e(t)+\int_0^1T_1(t,\xi)l(\xi)\mathrm{d}\xi+\int_0^1T_2(t,\xi)h(\xi)\mathrm{d}\xi.\end{aligned}$$

For $t\geq\omega$, one has

$$\begin{aligned}l(t)=&e(t)+\int_0^{\omega}\frac{[aC(t)(1-\xi)^{\chi-1}-\lambda' C(t)(\omega-\xi)^{\chi-1}+a(t-\xi)^{\chi-1}]}{\Gamma(\chi)}l(\xi)\mathrm{d}\xi+\\&\int_{\omega}^t\frac{[aC(t)(1-\xi)^{\chi-1}+a(t-\xi)^{\chi-1}]}{\Gamma(\chi)}l(\xi)\mathrm{d}\xi+\int_t^1\frac{aC(t)(1-\xi)^{\chi-1}}{\Gamma(\chi)}l(\xi)\mathrm{d}\xi+\\&\int_0^t\frac{[C(t)(1-\xi)^{\nu_1-1}+(t-\xi)^{\nu_1-1}]}{\Gamma(\nu_1)}h(\xi)\mathrm{d}\xi+\int_t^1\frac{C(t)(1-\xi)^{\nu_1-1}}{\Gamma(\nu_1)}h(\xi)\mathrm{d}\xi\\=&e(t)+\int_0^1T_1(t,\xi)l(\xi)\mathrm{d}\xi+\int_0^1T_2(t,\xi)h(\xi)\mathrm{d}\xi,\end{aligned}$$

where $T_1(t,\xi)$ and $T_2(t,\xi)$ are given as (2.4) and (2.5). □

Since $\nu_1-\nu_2-1<0$, T_1 is unbounded. However, $\int_0^1T_1(t,\xi)\mathrm{d}\xi$ is uniformly bounded for $t\in[0,1]$. This is because

$$\begin{aligned}\int_0^1|T_1(t,\xi)|\mathrm{d}\xi\leq&\frac{|a+|\lambda'|[|a|t^{\chi+1}+t\Gamma'(\chi+2)]}{\Gamma(\chi)|\Gamma(\chi+2)-a|}\int_0^1(1-\xi)^{\chi-1}\mathrm{d}\xi+\\&\frac{|a|}{\Gamma(\chi)}\int_0^t(t-\xi)^{\chi-1}\mathrm{d}\xi\\\leq&\frac{|a+\lambda'|[|a|+\Gamma(\chi+2)]}{\Gamma(\chi+1)|\Gamma(\chi+2)-a|}+\frac{|a|}{\Gamma(\chi+1)}\end{aligned}$$

for all $t\in[0,1]$. From the definition of T_2, it is easy to see that T_2 is continuous, therefore T_2 is bounded on $[0,1]\times[0,1]$. Since e is a polynomial type function, it is obviously continuous and bounded on $[0,1]$. So, we denote by $M_1=\max_{0\leq t\leq 1}\int_0^1|T_1(t,\xi)|\mathrm{d}\xi$, $M_2=\max\{|T_2(t,\xi)|,(t,\xi)\in[0,1]\times[0,1]\}$, $M_3=\max_{0\leq t\leq 1}|e(t)|$.

3. Existence of solutions for differential equations

In this section, we prove the existence result for the differential equation (1.1) with nonlocal integral boundary conditions by Leray-Schauder degree theory.

Definition 3.1 *A continuous function $l:[0,1]\to\mathbb{R}$ is said to be a solution to (1.1), if l satisfies*

$$l(t)=e(t)+\int_0^1T_1(t,\xi)l(\xi)\mathrm{d}\xi+\int_0^1T_2(t,\xi)g(\xi,l(\xi))\mathrm{d}\xi,\quad t\in[0,1].$$

Theorem 3.2 *Assume $a\neq\Gamma(\nu_1-\nu_2+2)$, $g\in C([0,1]\times\mathbb{R},\mathbb{R})$, and suppose there exist $0\leq\varepsilon<\frac{1-M_1}{M_2}$, $M>0$ such that $|g(t,l)|\leq\varepsilon|l|+M$, for all $t\in[0,1]$, $l\in C([0,1],\mathbb{R})$, then there exists at least one solution for (1.1).*

Proof We define integral operator $A : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ by

$$(Al)(t) = e(t) + \int_0^1 T_1(t,\xi)l(\xi)\mathrm{d}\xi + \int_0^1 T_2(t,\xi)g(\xi,l(\xi))\mathrm{d}\xi, \quad t \in [0,1].$$

Clearly, if l is a fixed point of A, l is also a solution to (1.1). Thus, all we have to do is to prove that the fixed point of A exists. For this purpose, we set $L > 0$, and

$$Q_L = \{l \in C([0,1],\mathbb{R}) : \max_{t\in[0,1]} |l(t)| < L\}.$$

According to the definition of M_1, M_2, M_3, we have $\|Al\| \le M_3 + M_1 L + M_2(\varepsilon L + M)$, $l \in \overline{Q}_L$, which means that $A(\overline{Q}_L)$ is uniformly bounded. And, let $l \in \overline{Q}_L$ be arbitrary and $s_2, s_1 \in [0,1]$ with $s_1 < s_2$. Then

$$\begin{aligned}
&|A(l)(s_2) - A(l)(s_1)| \\
&\le |e(s_2) - e(s_1)| + \Big| \int_0^1 (T_1(s_2,\xi) - T_1(s_1,\xi))l(\xi)\mathrm{d}\xi \Big| + \Big| \int_0^1 (T_2(s_2,\xi) - T_2(s_1,\xi))g(\xi,l(\xi))\mathrm{d}\xi \Big| \\
&\le |e(s_2) - e(s_1)| + \frac{|a+\lambda'||a(s_2^{\chi+1} - s_1^{\chi+1}) + \Gamma(\chi+2)(s_1 - s_2)|}{\Gamma(\chi)|\Gamma(\chi+2) - a|} \int_0^1 (1-\xi)^{\chi-1}|l(\xi)|\mathrm{d}\xi + \\
&\quad \frac{|a|}{\Gamma(\chi)} \Big| \int_0^{s_2} (s_2-\xi)^{\chi-1} l(\xi)\mathrm{d}\xi - \int_0^{s_1} (s_1-\xi)^{\chi-1} l(\xi)\mathrm{d}\xi \Big| + \\
&\quad \frac{|a(s_2^{\chi+1} - s_1^{\chi+1}) + \Gamma(\chi+2)(s_1 - s_2)|}{\Gamma(\nu_1)|\Gamma(\chi+2) - a|} \int_0^1 (1-\xi)^{\nu_1-1}|g(\xi,l(\xi))|\mathrm{d}\xi + \\
&\quad \frac{1}{\Gamma(\nu_1)} \Big| \int_0^{s_2} (s_2-\xi)^{\nu_1-1} g(\xi,l(\xi))\mathrm{d}\xi - \int_0^{s_1} (s_1-\xi)^{\nu_1-1} g(\xi,l(\xi))\mathrm{d}\xi \Big| \\
&\le |e(s_2) - e(s_1)| + \frac{L|a+\lambda'||a(s_2^{\chi+1} - s_1^{\chi+1}) + \Gamma(\chi+2)(s_1 - s_2)|}{\Gamma(\chi+1)|\Gamma(\chi+2) - a|} + \frac{L|a|}{\Gamma(\chi+1)}|s_2^{\chi} - s_1^{\chi}| + \\
&\quad \frac{(\epsilon L + M)|a(s_2^{\chi+1} - s_1^{\chi+1}) + \Gamma(\chi+2)(s_1 - s_2)|}{\Gamma(\nu_1+1)|\Gamma(\chi+2) - a|} + \frac{(\epsilon L + M)}{\Gamma(\nu_1+1)}[|s_2^{\nu_1} - s_1^{\nu_1}| + 2|s_2 - s_1|^{\nu_1}].
\end{aligned}$$

And e is a polynomial like function, then the right side of the above inequality tends to zero as $s_2 - s_1 \to 0$, which means $|A(l)(s_2) - A(l)(s_1)| \to 0$, and the convergence is dependent of $l \in \overline{Q}_L$, i.e., $A\overline{Q}_L$ is equicontinuous. By the Arzela-Ascoli theorem we know that $A\overline{Q}_L$ is compact. Therefore, $A : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous.

Define $\Sigma : [0,1] \times \overline{Q}_L \to \mathbb{R}$ as

$$\Sigma(\mu, l) = \mu A l, \ l \in \overline{Q}_L, \quad \mu \in [0,1].$$

Obviously, Σ is continuous. Indeed, set $\mu_1, \mu_2 \in [0,1]$, $l_1, l_2 \in \overline{Q}_L$, we have

$$|\Sigma(\mu_1, l_1) - \Sigma(\mu_2, l_2)| = |\mu_1 A l_1 - \mu_2 A l_2| \le \mu_1 |A l_1 - A l_2| + |\mu_1 - \mu_2| A l_2.$$

Since $A\overline{Q}_L$ is compact, we get $|\Sigma(\mu_1, l_1) - \Sigma(\mu_2, l_2)| \to 0$, when $|\mu_1 - \mu_2| \to 0$ and $|l_1 - l_2| \to 0$. Further, the $\Sigma : [0,1] \times \overline{Q}_L \to \mathbb{R}$ is completely continuous. In fact, according to the above inequality $\Sigma : [0,1] \times \overline{Q}_L \to \mathbb{R}$ is continuous. And for fixed $\mu \in [0,1]$, by the Arzela-Ascoli theorem, $\Sigma(\mu, \cdot) : \overline{Q}_L \to \mathbb{R}$ is compact. Moreover, for any fixed $\mu_0 \in [0,1]$, we have $|\Sigma(\mu, l_1) - \Sigma(\mu_0, l_2)| \le |\mu - \mu_0||Al|$, that is, the continuity of $\Sigma(\mu, l)$ at μ_0 is uniformly with respect to

$l \in \overline{Q}_L$. According to the reference [10], the Σ is completely continuous.

Also define

$$d_\mu(l) = l - \Sigma(\mu, l) = l - \mu A l,\ l \in \overline{Q}_L, \quad \mu \in [0,1].$$

According to the Leray-Schauder degree theory, we only need to prove $A : \overline{Q}_L \to C([0,1],\mathbb{R})$ satisfying

$$l \neq \mu A l,\ \forall l \in \partial Q_L, \quad \mu \in [0,1]. \tag{3.1}$$

Suppose $l(t) = \mu A l(t)$ for some $\mu \in [0,1]$, one has

$$\begin{aligned} |l(t)| = |\mu A l(t)| &\leq |e(t)| + \int_0^1 |T_1(t,\xi)||l(\xi)|\mathrm{d}\xi + \int_0^1 |T_2(t,\xi)|(\varepsilon|l(\xi)| + M)\mathrm{d}\xi \\ &\leq M_3 + M_1\|l\| + M_2(\varepsilon\|l\| + M), \end{aligned}$$

or

$$\|l\| = \max_{t\in[0,1]} |l(t)| \leq \frac{M_3 + M_2 M}{1 - M_1 - \varepsilon M_2}.$$

So, if we take

$$L = \frac{M_3 + M_2 M}{1 - M_1 - \varepsilon M_2} + 1,$$

then (3.1) holds. In terms of the homotopy invariance of topological degree, we obtain

$$\begin{aligned} \deg(d_\mu, Q_L, 0) &= \deg(I - \mu A, Q_L, 0) = \deg(d_1, Q_L, 0) \\ &= \deg(d_0, Q_L, 0) = \deg(I, Q_L, 0) = 1 \neq 0, \quad 0 \in Q_L. \end{aligned}$$

Therefore, there exists at least one $l \in Q_L$ satisfying $d_1(l) = l - Al = 0$, which is a solution of (1.1). □

4. Existence of solutions for differential inclusions

Next, we consider the fractional differential inclusion (1.2) with convex valued and nonconvex valued multifunctions respectively.

Let $(W, \|\cdot\|)$ be a Banach space. Let $K(W) = \{\Delta \in W : \Delta \text{ is nonempty}\}$; $K_b(W) = \{\Delta \in K(W) : \Delta \text{ is bounded}\}$; $K_f(W) = \{\Delta \in K(W) : \Delta \text{ is closed}\}$; $K_{bf}(W) = \{\Delta \in K(W) : \Delta \text{ is closed and bounded}\}$; $K_{pc}(W) = \{\Delta \in K(W) : \Delta \text{ is compact and convex}\}$; $K_{fc}(W) = \{\Delta \in K(W) : \Delta \text{ is closed and convex}\}$.

Let $Q_1, Q_2 \in K_{bf}(W)$, $q_1 \in Q_1$. Let (W, d) be a metric space induced from the normed space $(W, \|\cdot\|)$. Denote

$$D(q_1, Q_2) = \inf\{d(q_1, q_2) : q_2 \in Q_2\}, \quad \rho(Q_1, Q_2) = \sup\{D(q_1, Q_2) : q_1 \in Q_1\}.$$

A function $H : K_{bf}(W) \times K_{bf}(W) \to \mathbb{R}^+$ is called the Hausdorff metric on W, if

$$H(A_1, A_2) = \max\{\rho(Q_1, Q_2), \rho(Q_2, Q_1)\}.$$

A multi function $G : W \to K_f(W)$ is called contraction, if there exists $0 < \epsilon < 1$, satisfying

$$H(G(w_1), G(w_2)) \leq \epsilon d(w_1, w_2), \quad \forall w_1, w_2 \in W.$$

Definition 4.1 *A continuous function $l:[0,1]\to\mathbb{R}$ is said to be a solution to (1.2), if there exists a function $g\in L^1([0,1],\mathbb{R})$ with $g(t)\in G(t,l(t))$ a.e. $[0,1]$ such that*

$$l(t)=e(t)+\int_0^1 T_1(t,\xi)l(\xi)\mathrm{d}\xi+\int_0^1 T_2(t,\xi)g(\xi,l(\xi))\mathrm{d}\xi,\quad t\in[0,1].$$

In the after, we give the following assumptions.

(H_1) $G:[0,1]\times\mathbb{R}\to K_{pc}(\mathbb{R})$; $(t,l)\to G(t,l)$ meets the Caratheodory condition, and for fixed $l\in C([0,1],\mathbb{R})$,

$$S_{G,l}=\{g\in L^1([0,1],\mathbb{R}):g(t)\in G(t,l(t))\ \text{ for almost everywhere }\ t\in[0,1]\}\neq\emptyset;$$

(H_1') $G:[0,1]\times\mathbb{R}\to K_p(\mathbb{R})$ is measurable with respect to t for each $l\in\mathbb{R}$;

(H_2) $|G(t,l)|=\sup\{|k|:k\in G(t,l)\}\le\phi(t)\Psi(|l|)$, for almost everywhere $t\in[0,1]$, where $\Psi:\mathbb{R}^+\to(0,\infty)$ is increasing and continuous; $\phi\in L^1([0,1],\mathbb{R}^+)$;

(H_3) For almost everywhere $t\in[0,1]$, there exists $N(\cdot)\in L^1([0,1],\mathbb{R})$, the inequality

$$H(G(w_1),G(w_2))\le N(t)|w_1-w_2|,\quad\forall w_1,w_2\in\mathbb{R}$$

holds. Moreover, $H(0,G(0))\le N(t)$, a.e., $t\in[0,1]$;

(H_4) $M_1+M_2\|N\|_{L^1}-1<0$.

Lemma 4.2 ([11]) *Suppose (W,d) is a metric space. If $\Theta:W\to K_f(W)$ is a contraction, then the fixed points set $\operatorname{Fix}\Theta=\{w:w\in\Theta(w)\}\neq\emptyset$.*

Theorem 4.3 *Under assumptions (H_1), (H_2), if*

$$M_1+M_2\|\phi\|\limsup_{r\to\infty}\frac{\Psi(r)}{r}<1, \tag{4.1}$$

then there exists at least one solution for the differential inclusion (1.2).

Proof Define the following multivalued operator $\Theta:C([0,1],\mathbb{R})\to K(C([0,1],\mathbb{R}))$ by

$$\Theta(l)=\Big\{w\in C([0,1],\mathbb{R}):w(t)=e(t)+\int_0^1 T_1(t,\xi)l(\xi)\mathrm{d}\xi+\int_0^1 T_2(t,\xi)g(\xi)\mathrm{d}\xi,\quad g\in S_{G,l}\Big\}.$$

Clearly, the fixed point of Θ is a solution to (1.2).

Step 1. According to (H_1), for every $l\in C([0,1],\mathbb{R})$, $\Theta(l)$ is convex.

Step 2. Assume $w\in\Theta l$, there exists $g\in S_{G,l}$, satisfying

$$w(t)=e(t)+\int_0^1 T_1(t,\xi)l(\xi)\mathrm{d}\xi+\int_0^1 T_2(t,\xi)g(\xi)\mathrm{d}\xi,\quad t\in[0,1].$$

If we suppose $l\in B_\theta=\{l\in C([0,1],\mathbb{R}):\|l\|\le\theta\}$, applying condition (H_2), we obtain the estimate

$$\begin{aligned}|w(t)|\le&|e(t)|+\int_0^1|T_1(t,\xi)||l(\xi)|\mathrm{d}\xi+\int_0^1|T_2(t,\xi)||g(\xi)|\mathrm{d}\xi\\ \le&M_3+M_1\|l\|+M_2\int_0^1\phi(\xi)\Psi(\|l\|)\mathrm{d}\xi\\ \le&M_3+\theta M_1+M_2\|\phi\|_{L^1}\Psi(\theta).\end{aligned}$$

Then for each $w \in \Theta(B_\theta)$ we have

$$\|w\| \leq M_3 + \theta M_1 + M_2\|\phi\|_{L^1}\Psi(\theta) := R.$$

Step 3. Let $B_\theta = \{l \in C([0,1],E) : \|l\| \leq \theta\}$ be a bounded set of $C([0,1],\mathbb{R})$. we will show that for each $l \in B_\theta$, $\Theta(l)$ is equicontinuous on $[0,1]$. Indeed, let $s_1, s_2 \in [0,1]$, $s_1 < s_2$, $l \in B_\theta$ and $w \in \Theta(l)$. Then there exists $g \in S_{G,l}$ satisfying

$$\begin{aligned}
&|w(s_2) - w(s_1)| \\
&\leq |e(s_2) - e(s_1)| + \int_0^1 |(T_1(s_2,\xi) - T_1(s_1,\xi))l(\xi)|\mathrm{d}\xi + \int_0^1 |(T_2(s_2,\xi) - T_2(s_1,\xi))g(\xi)|\mathrm{d}\xi \\
&\leq |e(s_2) - e(s_1)| + \frac{|a||a(s_2^{\chi+1} - s_1^{\chi+1}) + \Gamma(\chi+2)(s_1 - s_2)|}{\Gamma(\nu_1 - \nu_2)|\Gamma(\chi+2) - a|}\int_0^1 (1-\xi)^{\chi-1}|l(\xi)|\mathrm{d}\xi + \\
&\quad \frac{|\lambda'||a(s_1^{\chi+1} - s_2^{\chi+1}) + \Gamma(\chi+2)(s_2 - s_1)|}{\Gamma(\chi)|\Gamma(\chi+2) - a|}\int_0^\omega (\omega-\xi)^{\chi-1}|l(\xi)|\mathrm{d}\xi + \\
&\quad \frac{|a|}{\Gamma(\chi)}\Big|\int_0^{s_2}(s_2-\xi)^{\chi-1}l(\xi)\mathrm{d}\xi - \int_0^{s_1}(s_1-\xi)^{\chi-1}l(\xi)\mathrm{d}\xi\Big| + \\
&\quad \frac{|a(s_2^{\chi+1} - s_1^{\nu_1-\nu_2+1}) + \Gamma(\chi+2)(s_1 - s_2)|}{\Gamma(\nu_1)|\Gamma(\chi+2) - a|}\int_0^1 (1-\xi)^{\nu_1-1}|g(\xi)|\mathrm{d}\xi + \\
&\quad \frac{1}{\Gamma(\nu_1)}\Big|\int_0^{s_2}(s_2-\xi)^{\nu_1-1}g(s)\mathrm{d}\xi - \int_0^{s_1}(s_1-\xi)^{\nu_1-1}g(\xi)\mathrm{d}\xi\Big| \\
&\leq \frac{\theta|a||a(s_2^{\chi+1} - s_1^{\chi+1}) + \Gamma(\chi+2)(s_1 - s_2)|}{\Gamma(\chi+1)|\Gamma(\chi+2) - a|} + \frac{\theta|\lambda'||a(s_1^{\chi+1} - s_2^{\chi+1}) + \Gamma(\chi+2)(s_2 - s_1)|}{\Gamma(\chi+1)|\Gamma(\chi+2) - a|} + \\
&\quad \frac{|a|\theta}{\Gamma(\nu_1 - \nu_2 + 1)}[|s_2^{\nu_1-\nu_2} - s_1^{\nu_1-\nu_2}| + 2(s_2 - s_1)^{\nu_1-\nu_2}] + \\
&\quad \frac{\|\phi\|\Psi(\theta)|a(s_2^{\chi+1} - s_1^{\chi+1}) + \Gamma(\chi+2)(s_1 - s_2)|}{\Gamma(\nu_1+1)|\Gamma(\chi+2) - a|} + \\
&\quad \frac{\|\phi\|\Psi(\theta)}{\Gamma(\nu_1+1)}[|s_2^{\nu_1} - s_1^{\nu_1}| + 2(s_2 - s_1)^{\nu_1}] + |e(s_2) - e(s_1)| \to 0, \quad s_1 \to s_2.
\end{aligned}$$

Taking into account the above discussion, Θ is completely continuous.

Step 4. Assume $\{l_n\}_{n=1}^\infty \subset C([0,1],\mathbb{R})$ with $l_n \to l$, and $w_n \in \Theta l_n$ with $w_n \to w$. Further, let $\{g_n\}_{n=1}^\infty \subset L^1([0,1],\mathbb{R})$ with $g_n \in S_{G,l_n}$,

$$w_n(t) = e(t) + \int_0^1 T_1(t,\xi)l_n(\xi)\mathrm{d}\xi + \int_0^1 T_2(t,\xi)g_n(\xi)\mathrm{d}\xi, \quad t \in [0,1]. \tag{4.2}$$

Next, we will prove that there is $g \in S_{G,l}$ satisfying

$$w(t) = e(t) + \int_0^1 T_1(t,\xi)l(\xi)\mathrm{d}\xi + \int_0^1 T_2(t,\xi)g(\xi)\mathrm{d}\xi, \quad t \in [0,1].$$

Since $l_n \to l$ and $w_n \to w$, one has

$$\Big\|\Big(w_n(t) - e(t) - \int_0^1 T_1(t,\xi)l_n(\xi)\mathrm{d}\xi\Big) - \Big(w(t) - e(t) - \int_0^1 T_1(t,\xi)l(\xi)\mathrm{d}\xi\Big)\Big\| \to 0, \quad \text{as } n \to \infty.$$

Define the operator

$$\Phi : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R}),$$

$$g \mapsto \Phi(g)(t) = \int_0^1 T_2(t,\xi)g(\xi)\mathrm{d}\xi.$$

Clearly $\|\Phi(g)\| \le M_2\|g\|_{L^1}$, that is, Φ is linear and continuous.

By (4.2), we obtain

$$w_n(t) - e(t) - \int_0^1 T_1(t,\xi)l_n(\xi)\mathrm{d}\xi \in \Phi(S_{G,l_n}).$$

And because $l_n \to l$, from [12], we can see that

$$w(t) - e(t) - \int_0^1 T_1(t,\xi)l(\xi)\mathrm{d}\xi \in \Phi(S_{G,l}),$$

or

$$w(t) - e(t) - \int_0^1 T_1(t,\xi)l(\xi)\mathrm{d}\xi = \int_0^1 T_2(t,\xi)g(\xi)\mathrm{d}\xi, \quad g \in S_{G,l}.$$

So, Θ has a closed graph.

Step 5. Combining (4.1), there exists $\overline{M} > 0$ satisfying

$$M_1\overline{M} + M_2\|\phi\|\Psi(\overline{M}) + M_3 < \overline{M}. \tag{4.3}$$

Let $V = \{l \in C([0,1],\mathbb{R}) : \|l\| < \overline{M}\}$. Then $\Theta : \overline{V} \to C([0,1],\mathbb{R})$ is completely continuous. If there are $l \in \overline{V}$ and $\gamma \in (0,1)$ such that $l = \gamma\Theta l$, then

$$\begin{aligned}|w(t)| =& |\gamma\Theta l(t)| \le |\Theta l(t)| \\ \le& |e(t)| + \Big|\int_0^1 T_1(t,\xi)l(\xi)\mathrm{d}\xi\Big| + \Big|\int_0^1 T_2(t,\xi)g(\xi,l(\xi))\mathrm{d}\xi\Big| \\ \le& M_3 + M_1\|l\| + M_2\|\phi\|_{L^1}\Psi(\|l\|),\end{aligned}$$

hence

$$\overline{M} = \|l\| \le M_3 + M_1\|l\| + M_2\|\phi\|_{L^1}\Psi(\|l\|) < \overline{M},$$

is a contradiction. Therefore, for any $l \in \overline{V}$ and $\gamma \in (0,1)$, $l \ne \gamma\Theta l$. Subsequently, Θ has at least one fixed point via Leray-Schauder alternative [13]. □

Theorem 4.4 *Under assumptions (H_1'), (H_3) and (H_4), then there exists at least one solution for the differential inclusion (1.2).*

Proof Similar to the above theorem, we only need to prove that $\operatorname{Fix}\Theta \ne \emptyset$. For this purpose, let us observe first that, by the measurable selection theorem [14], for every $l \in C([0,1],\mathbb{R})$, $S_{G,l} \ne \emptyset$. In addition, using similar arguments to those in Theorem 4.3, the multioperator Θ defined above has closed values.

Let $l_1, l_2 \in C([0,1],\mathbb{R})$, and $k_1 \in \Theta(l_1)$. Then there exists $g_1(t) \in G(t, l_1(t))$ such that k_1 is the solution of (1.2). By (H_3),

$$H(G(t,l_1), G(t,l_2)) \le N(t)\|l_1 - l_2\|_\infty.$$

So, there is $z \in G(t, l_2(t))$ satisfying

$$\|g_1(t) - z\| \le N(t)\|l_1 - l_2\|, \quad t \in [0,1].$$

In what follows, we define $\Omega_1 : [0,1] \to P(\mathbb{R})$ by

$$\Omega_1(t) = \{z \in \mathbb{R} : \|g_1(t) - z| \leq N(t)\|l_1 - l_2\|_\infty\}.$$

It follows from [14], $\Omega_1(t) \cap G(t, l_2(t))$ is measurable, there exists g_2 a measurable selection for $\Omega_1(t) \cap G(t, l_2(t))$. Therefore, $g_2(t) \in G(t, l_2(t))$ and

$$\|g_1(t) - g_2(t)| \leq N(t)\|l_1 - l_2\|_\infty, \quad t \in [0,1]. \tag{4.4}$$

Let Eq. (1.2) have another solution $k_2 \in C([0,1], \mathbb{R})$ with $g_2(t) \in G(t, l_2(t))$, i.e.,

$$k_2(t) = e(t) + \int_0^1 T_1(t,\xi) l_2(\xi) \mathrm{d}\xi + \int_0^1 T_2(t,\xi) g_2(\xi) \mathrm{d}\xi, \quad t \in [0,1].$$

Now, (4.4) yields

$$\begin{aligned} |k_1(t) - k_2(t)| \leq & \int_0^1 |T_1(t,\xi)||l_1(\xi) - l_2(\xi)|\mathrm{d}\xi + \int_0^1 |T_2(t,\xi)||g_1(\xi) - g_2(\xi)|\mathrm{d}\xi \\ \leq & M_1\|l_1 - l_2\|_\infty + M_2\|N\|_{L^1}\|l_1 - l_2\|_\infty. \end{aligned}$$

Then

$$\|k_1 - k_2\|_\infty \leq (M_1 + M_2\|N\|_{L^1})\|v_1 - v_2\|_\infty.$$

Replacing l_1 by l_2, we obtain

$$H(N(l_1), N(l_2)) \leq (M_1 + M_2\|N\|_{L^1})\|l_1 - l_2\|_\infty.$$

Invoking (H_4), and Lemma 4.2, the generalized Bagley-Torvik type differential inclusion (1.2) has at least one solution. □

5. Example

In this section, as an application of our main results, an example is presented.

Example 5.1 We consider the generalized Bagley-Torvik type differential inclusion

$$\begin{cases} {}^cD^2 l(t) - \frac{\sqrt{\pi}}{8} {}^cD^{3/2} l(t) \in G(t, l(t)), \quad t \in (0,1), \\ l(0) = 0, \quad l(1) = \frac{\sqrt{\pi}}{8} I_{0^+}^{1/2} l(\frac{1}{2}) = \frac{\sqrt{\pi}}{8} \int_0^{\frac{1}{2}} \frac{(\frac{1}{2} - s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} l(s) \mathrm{d}s, \end{cases} \tag{5.1}$$

where $\nu_1 = 2$, $\nu_2 = \frac{3}{2}$, $\chi = \frac{1}{2}$, $a = \frac{\sqrt{\pi}}{8}$, $\lambda' = \frac{\sqrt{\pi}}{8}$ and $G : [0,1] \times \mathbb{R} \to P(\mathbb{R})$ is a multivalued map given by

$$l \to G(t,l) = [\frac{e^l}{e^l + 1} + t^2 + 1, \quad \frac{e^{l+1}}{e^l + 1} + t + 3].$$

When $g \in G$,

$$|g| \leq \max\{y : y \in [\frac{e^l}{e^l + 1} + t^2 + 1, \ \frac{e^{l+1}}{e^l + 1} + t + 3], \quad l \in \mathbb{R}, \ t \in [0,1]\} \leq 5.$$

Thus, $\|G(t,l)\| \leq 5$, $l \in \mathbb{R}$, $t \in [0,1]$, with $\phi(t) \equiv 1$, $\Psi(\|l\|) \equiv 5$.

Obviously, (H_1), (H_2) hold. Further,

$$\int_0^1 |T_1(t,s)|\mathrm{d}s \leq \frac{|a|}{\Gamma(\nu_1 - \nu_2)} \cdot \frac{[|a|t^{\chi+1} + t\Gamma(\chi+2)]}{|\Gamma(\chi+2) - a|} \int_0^1 (1-s)^{\chi-1}\mathrm{d}s +$$

$$
\begin{aligned}
&\frac{|\lambda'|}{\Gamma(\chi)}\cdot\frac{[|a|t^{\chi+1}+t\Gamma(\chi+2)]}{|\Gamma(\chi+2)-a|}\int_0^\omega(\omega-s)^{\chi-1}\mathrm{d}s+\\
&\frac{|a|}{\Gamma(\chi)}\int_0^t(t-s)^{\chi-1}\mathrm{d}s\\
\le&\frac{|a|}{\Gamma(\chi+1)}\cdot\frac{[|a|+\Gamma(\chi+2)]}{|\Gamma(\chi+2)-a|}+\frac{|\lambda'|}{\Gamma(\chi+1)}\cdot\frac{[|a|+\Gamma(\chi+2)]}{|\Gamma(\chi+2)-a|}+\frac{|a|}{\Gamma(\chi+1)}\\
=&\frac{\sqrt{\pi}/8}{\Gamma(3/2)}\cdot\frac{\sqrt{\pi}/8+\Gamma(5/2)}{\Gamma(5/2)-\sqrt{\pi}/8}+\frac{\sqrt{\pi}/8}{\Gamma(3/2)}\cdot\frac{\sqrt{\pi}/8+\Gamma(5/2)}{\Gamma(5/2)-\sqrt{\pi}/8}+\frac{\sqrt{\pi}/8}{\Gamma(3/2)}.
\end{aligned}
$$

Therefore,

$$
M_1=\max_{0\le t\le 1}\int_0^1|T_1(t,s)|\mathrm{d}s\le\frac{1}{4}\times\frac{7}{5}+\frac{1}{4}\times\frac{7}{5}+\frac{1}{4}=\frac{19}{20}<1.
$$

Thus, there exists $\overline{M}$ sufficiently large such that the inequality

$$
M_1+M_2\|\phi\|\limsup_{\overline{M}\to\infty}\frac{\Psi(\overline{M})}{\overline{M}}<1,
$$

holds. By Theorem 4.3, Eq. (5.1) has at least one solution. □

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