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Complete Convergence for Sung's Type Weighted Sums of Dependent Random Variables with General Moment Conditions

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Abstract In this paper, the complete convergence theorems for Sung's type weighted sums of END random variables and PNQD random variables with general moment conditions are obtained. The theorems extend the related known works in the literature.

Keywords complete convergence; general moment condition; END random variables; PNQD random variables

MR(2010) Subject Classification 60F15

1. Introduction and main results

Many statistical problems based on a random sample are weighted sums of random variables. Some limiting properties of weighted sums of random variables have been studied. We refer to [1–5] and so on.

The concept of complete convergence was first introduced by Hsu and Robbins [6]. After that, many scholars have made a deep and extensive study of complete convergence. Recently, Sung [1] obtained the following result (we call these Sung's type weighted sums).

Theorem 1.1 *Let $r > 1$ and $1 \le p < 2$. Let $\{X, X_n, n \ge 1\}$ be a sequence of identically distributed ρ^*-mixing random variables with $EX = 0$. Assume that $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^q = O(n) \tag{1.1}$$

for some $q > rp$. If $E|X|^{rp} < \infty$, then

$$\sum_{n=1}^{\infty} n^{r-2} P\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| > \varepsilon n^{1/p}\Big) < \infty, \quad \forall\, \varepsilon > 0. \tag{1.2}$$

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Conversely, if (1.2) holds for any array $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ satisfying (1.1) for some $q > rp$, then $E|X|^{rp} < \infty$.

The Sung's type weights are very meaningful. Zhang [3] investigated Theorem 1.1 for END random variables and Li et al. [4] investigated complete moment convergence for Banach-valued random elements.

Lehmann [7] introduced the concept of negatively quadrant dependent (NQD) and Liu [8] introduced the concept of extended negatively dependent (END). Obviously, $\{\{X_n\} : \{X_n\}$ is an NA sequence$\} \subsetneqq \{\{X_n\} : \{X_n\}$ is an NOD sequence$\} \subsetneqq \{\{X_n\} : \{X_n\}$ is an END sequence$\}$ (see [9, 10]). For the application and the limiting behavior of END random variables, we refer to [3] and [11–19] and so on.

It is of significance to find more generalized moment conditions such that the complete convergence holds. Under higher order moment conditions, Lanzinger [20], Gut and Stadtmüller [21], Chen and Sung [22] extended the Baum-Katz theorem; Under some generalized moment conditions, Sung [23] obtained the complete convergence for pairwise independent random variables and Chen et al. [24] extended the Baum-Katz theorem to i.i.d. random variables. Qiu et al. [25] extended the Baum-Katz theorem with general moment conditions. In particular, Chen et al. [26] obtained the following two results:

Theorem 1.2 *Let $r > 1$ and $1 \le p < 2$. Let $\{X, X_n, n \ge 1\}$ be a sequence of identically distributed NOD random variables, $\{a_n, n \ge 1\}$ be a sequence of real numbers with $0 < a_n/n^{1/p} \uparrow$. Then the following statements are equivalent:*

$$\sum_{n=1}^{\infty} n^{r-1} P(|X| > a_n) < \infty; \tag{1.3}$$

$$\sum_{n=1}^{\infty} n^{r-2} P\Big(\max_{1\le j\le n} \Big|\sum_{i=1}^{j} (X_i - EX_i I(|X_i| \le a_n)\Big| > \varepsilon a_n\Big) < \infty, \quad \forall\, \varepsilon > 0. \tag{1.4}$$

Theorem 1.3 *Let $\{X, X_n, n \ge 1\}$ be a sequence of identically distributed pairwise negatively quadrant dependent (PNQD) random variables, $\{a_n, n \ge 1\}$ be a sequence of real numbers with $0 < a_n/n \uparrow$. Then the following statements are equivalent:*

$$\sum_{n=1}^{\infty} P(|X| > a_n) < \infty; \tag{1.5}$$

$$\sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\le j\le n} \Big|\sum_{i=1}^{j} (X_i - EX_i I(|X_i| \le a_n)\Big| > \varepsilon a_n\Big) < \infty, \quad \forall\, \varepsilon > 0; \tag{1.6}$$

$$a_n^{-1} \sum_{i=1}^{n} (X_i - EX_i I(|X_i| \le a_i)) \to 0 \quad a.s. \tag{1.7}$$

The aim of this paper is to obtain the complete convergence for Sung's type weighted sums of END random variables with general moment conditions (1.3) and the complete convergence for Sung's type weighted sums of PNQD random variables with general moment conditions (1.5). Our main results include Theorems 1.1, 1.2 and 1.3.

Now we state the main results. Some lemmas and the proofs of the main results will be detailed in the next section.

Theorem 1.4 *Let $r > 1$ and $1 \le p < 2$. Let $\{X, X_n, n \ge 1\}$ be a sequence of identically distributed END random variables, $\{a_n, n \ge 1\}$ be a sequence of real numbers such that $0 < a_n/n^{1/p} \uparrow$. Then (1.3) is equivalent to*

$$\sum_{n=1}^{\infty} n^{r-2} P\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^{k} a_{ni}(X_i - EX_i I(|a_{ni}X_i| \le a_n))\Big| > \epsilon a_n\Big) < \infty \tag{1.8}$$

for all $\varepsilon > 0$ and all arrays of real numbers $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ satisfying (1.1) for some $q > rp$.

Theorem 1.5 *If (1.8) in Theorem 1.4 is replaced by*

$$\sum_{n=1}^{\infty} n^{r-2} P\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^{k} a_{ni}(X_i - EX_i I(|X_i| \le a_n))\Big| > \epsilon a_n\Big) < \infty,$$

then Theorem 1.4 still holds.

Corollary 1.6 *Let $r > 1$ and $1 \le p < 2$. Let $\{X, X_n, n \ge 1\}$ be a sequence of identically distributed END random variables. Then the following statements are equivalent:*

$$EX = 0 \text{ and } E|X|^{rp} < \infty; \tag{1.9}$$

$$\sum_{n=1}^{\infty} n^{r-2} P\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^{k} a_{ni}X_i\Big| > \epsilon n^{1/p}\Big) < \infty \tag{1.10}$$

for all $\varepsilon > 0$ and all arrays of real numbers $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ satisfying (1.1) for some $q > rp$.

Theorem 1.7 *Let $\{X, X_n, n \ge 1\}$ be a sequence of identically distributed PNQD random variables, $\{a_n, n \ge 1\}$ be a sequence of real numbers with $0 < a_n/n \uparrow$. Then (1.5) and (1.7) and the following statement are equivalent*

$$\sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j} a_{ni}(X_i - EX_i I(|X_i| \le a_n))\Big| > \varepsilon a_n\Big) < \infty \tag{1.11}$$

for all $\varepsilon > 0$ and all arrays of real numbers $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ satisfying (1.1) for some $q > 1$.

Corollary 1.8 *Let $\{X, X_n, n \ge 1\}$ be a sequence of identically distributed PNQD random variables. Then the following statements are equivalent*

$$EX = 0; \tag{1.12}$$

$$\sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| > \varepsilon n\Big) < \infty \tag{1.13}$$

for all $\varepsilon > 0$ and all arrays of real numbers $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ satisfying (1.1) for some $q > 1$.

Remark 1.9 Taking $a_{ni} = 1$ for $1 \le i \le n$ and $n \ge 1$, we can immediately get Theorem 1.2 from Theorem 1.4 and get Theorem 1.3 from Theorem 1.7. But the contents of Theorems 1.4 and 1.7 are riches. For example, (i) Taking $a_{ni} = a + \sin i$ or $a + \cos i$ or $a + \sin(ni)$ or $a + \mathrm{con}(ni)$ for $1 \le i \le n$ and $n \ge 1$, where a is a constant; (ii) Taking $a_{ni} = 1$ for $1 \le i \le n-1$ and $a_{nn} = n^{1/q}$ for all $n \ge 1$; (iii) Taking $a_{ni} = i^\tau / n^\tau$ for $1 \le i \le n$ and $n \ge 1$, where $\tau > -1/q$, and so on.

Next, C denotes a positive constant not depending on n whose value may differ from one place to another, $I(A)$ denotes the indicator function of the event A, $[x]$ denotes the integer part of x.

2. Lemmas and proofs

To prove the main results, we need the following lemmas.

Lemma 2.1 ([11]) *Let $X_1, X_2, \ldots, X_n$ be END random variables. Assume that $f_1, f_2, \ldots, f_n$ are Borel functions all of which are monotone increasing (or all monotone decreasing), then $f_1(X_1), f_2(X_2), \ldots, f_n(X_n)$ are END random variables and the dominating constant of the definition of END random variables remains unchanged.*

Lemma 2.2 ([3]) *If $\{X_n, n \ge 1\}$ is a sequence of END random variables with $EX_n = 0$ for every $n \ge 1$, then for any $v \ge 2$, there is a positive constant C depending on v and the dominating constant of the definition of END random variables such that for all $n \ge 2$,*

$$E\Big|\sum_{i=1}^n X_i\Big|^v \le C\Big\{\sum_{i=1}^n E|X_i|^v + \Big(\sum_{i=1}^n E|X_i|^2\Big)^{v/2}\Big\};$$

$$E\max_{1\le k\le n}\Big|\sum_{i=1}^k X_i\Big|^v \le C(\log n)^v\Big\{\sum_{i=1}^n E|X_i|^v + \Big(\sum_{i=1}^n E|X_i|^2\Big)^{v/2}\Big\}.$$

Lemma 2.3 ([27]) *For any $1 < v \le 2$, there is a positive constant C_v depending only on v such that if $\{X_n, n \ge 1\}$ is a sequence of PNQD random variables with $EX_n = 0$ for every $n \ge 1$, then for all $n \ge 2$, the following statements hold:*

$$E\Big|\sum_{i=1}^n X_i\Big|^v \le C_v\sum_{i=1}^n E|X_i|^v;$$

$$E\max_{1\le k\le n}\Big|\sum_{i=1}^k X_i\Big|^v \le C_v(\log n)^v\sum_{i=1}^n E|X_i|^v.$$

Lemma 2.4 *Let $r > 0$ and $p > 0$. Let $\{X, X_n, n \ge 1\}$ be a sequence of identically distributed random variables and $\{a_n, n \ge 1\}$ a sequence of constants with $0 < a_n/n^{1/p} \uparrow$. Assume that $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers satisfying (1.1) for some $q > rp$. Set $\alpha = \min\{rp, 2\}$. If (1.3) holds, then the following statements hold:*

$$\sum_{n=1}^\infty n^{r-2}\sum_{i=1}^n P(|a_{ni}X_i| > \varepsilon a_n) < \infty \text{ for all } \varepsilon > 0; \tag{2.1}$$

$$\sum_{n=1}^{\infty} n^{r-2} a_n^{-v} \sum_{i=1}^{n} E|a_{ni}X_i|^v I(|a_{ni}X_i| \le a_n) < \infty \text{ for all } v \ge q; \tag{2.2}$$

$$\sum_{i=1}^{n} P(|a_{ni}X_i| > a_{[n^\theta]}) \le Cn^{1-\theta\alpha/p} \text{ for all } \theta \in (0,1]; \tag{2.3}$$

$$\frac{1}{a_n^\alpha} \sum_{i=1}^{n} E|a_{ni}X_i|^\alpha I(|a_{ni}X_i| \le a_{[n^\theta]}) \le Cn^{1-\alpha/p} \text{ for all } \theta \in (0,1]. \tag{2.4}$$

Proof By $0 < a_n/n^{1/p} \uparrow$ we have $0 < a_n \uparrow$ and $a_k/a_n \le (k/n)^{1/p}$ for any $n \ge k$. Hence, we have by $q > rp$, mean valued theorem and (1.3) that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-1} a_n^{-q} E|X|^q I(|X| \le a_n) \\
&= \sum_{n=1}^{\infty} n^{r-1} a_n^{-q} \sum_{k=1}^{n} E|X|^q I(a_{k-1} < |X| \le a_k) \text{ (where and the following set } a_0 = 0) \\
&\le \sum_{n=1}^{\infty} n^{r-1} \sum_{k=1}^{n} (a_k/a_n)^q P(a_{k-1} < |X| \le a_k) \\
&\le \sum_{k=1}^{\infty} k^{q/p} P(a_{k-1} < |X| \le a_k) \sum_{n=k}^{\infty} n^{r-1-q/p} \\
&\le C \sum_{k=1}^{\infty} k^r P(a_{k-1} < |X| \le a_k) \\
&= C \sum_{k=1}^{\infty} k^r \{P(|X| > a_{k-1}) - P(|X| > a_k)\} \\
&\le C + C \sum_{k=1}^{\infty} ((k+1)^r - k^r) P(|X| > a_k) \\
&\le C + C \sum_{k=1}^{\infty} k^{r-1} P(|X| > a_k) < \infty.
\end{aligned} \tag{2.5}$$

Therefore, by (1.1), (1.3) and (2.5),

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} \sum_{i=1}^{n} P(|a_{ni}X_i| > \varepsilon a_n) \\
&= \sum_{n=1}^{\infty} n^{r-2} \sum_{i=1}^{n} \{P(|a_{ni}X_i| > \varepsilon a_n, |X_i| > a_n) + P(|a_{ni}X_i| > \varepsilon a_n, |X_i| \le a_n)\} \\
&\le \sum_{n=1}^{\infty} n^{r-2} \sum_{i=1}^{n} P(|X_i| > a_n) + \varepsilon^{-q} \sum_{n=1}^{\infty} n^{r-2} a_n^{-q} \sum_{i=1}^{n} E|a_{ni}X_i|^q I(|X_i| \le a_n) \\
&\le \sum_{n=1}^{\infty} n^{r-1} P(|X| > a_n) + C \sum_{n=1}^{\infty} n^{r-1} a_n^{-q} E|X|^q I(|X| \le a_n) < \infty
\end{aligned}$$

and

$$\sum_{n=1}^{\infty} n^{r-2} a_n^{-v} \sum_{i=1}^{n} E|a_{ni}X_i|^v I(|a_{ni}X_i| \le a_n)$$

$$
\begin{aligned}
&=\sum_{n=1}^{\infty} n^{r-2}a_n^{-v}\sum_{i=1}^{n}E|a_{ni}X_i|^v\Big\{I(|a_{ni}X_i|\le a_n,|X_i|>a_n)+I(|a_{ni}X_i|\le a_n,|X_i|\le a_n)\Big\}\\
&\le C\sum_{n=1}^{\infty} n^{r-2}\Big\{\sum_{i=1}^{n}P(|X_i|>a_n)+a_n^{-q}\sum_{i=1}^{n}E|a_{ni}X_i|^qI(|X_i|\le a_n)\Big\}\quad(\text{ since } v\ge q)\\
&\le C\sum_{n=1}^{\infty} n^{r-1}P(|X|>a_n)+C\sum_{n=1}^{\infty} n^{r-1}a_n^{-q}E|X|^qI(|X|\le a_n)<\infty.
\end{aligned}
$$

Thus, (2.1) and (2.2) hold. By (1.3) and $0<a_n\uparrow$, we have

$$
\begin{aligned}
&n(a_{[n^\theta]})^{-\alpha}E|X|^\alpha I(|X|\le a_{[n^\theta]})\le n(a_{[n^\theta]})^{-\alpha}\sum_{k=1}^{[n^\theta]}Ea_k^\alpha I(a_{k-1}<|X|\le a_k)\\
&\le n\sum_{k=1}^{[n^\theta]}(k/[n^\theta])^{\alpha/p}P(a_{k-1}<|X|\le a_k)\\
&\le Cn^{1-\theta\alpha/p}\sum_{k=1}^{\infty}k^{\alpha/p}P(a_{k-1}<|X|\le a_k)\\
&\le Cn^{1-\theta\alpha/p}\sum_{k=1}^{\infty}k^{r}P(a_{k-1}<|X|\le a_k)\le Cn^{1-\theta\alpha/p},
\end{aligned}\tag{2.6}
$$

and

$$
n^rP(|X|>a_n)\to 0\ \text{ as }\ n\to\infty. \tag{2.7}
$$

By (1.1) and the Hölder inequality, we have for any $\tau\in(0,q]$ that

$$
\sum_{k=1}^{n}|a_{nk}|^\tau\le\Big(\sum_{k=1}^{n}|a_{nk}|^q\Big)^{\tau/q}\Big(\sum_{k=1}^{n}1\Big)^{1-\tau/q}\le Cn. \tag{2.8}
$$

By (2.6)$\sim$ (2.8), we have

$$
\begin{aligned}
&\sum_{i=1}^{n}P(|a_{ni}X_i|>a_{[n^\theta]})\\
&=\sum_{i=1}^{n}\{P(|a_{ni}X_i|>a_{[n^\theta]},|X_i|\le a_{[n^\theta]})+P(|a_{ni}X_i|>a_{[n^\theta]},|X_i|>a_{[n^\theta]})\}\\
&\le(a_{[n^\theta]})^{-\alpha}\sum_{i=1}^{n}E|a_{ni}X_i|^\alpha I(|X_i|\le a_{[n^\theta]})+\sum_{i=1}^{n}P(|X_i|>a_{[n^\theta]})\\
&\le Cn(a_{[n^\theta]})^{-\alpha}E|X|^\alpha I(|X|\le a_{[n^\theta]})+nP(|X|>a_{[n^\theta]})\\
&\le Cn^{1-\theta\alpha/p}
\end{aligned}
$$

and

$$
\begin{aligned}
&a_n^{-\alpha}\sum_{i=1}^{n}E|a_{ni}X_i|^\alpha I(|a_{ni}X_i|\le a_{[n^\theta]})\\
&=a_n^{-\alpha}\sum_{i=1}^{n}E|a_{ni}X_i|^\alpha\{I(|a_{ni}X_i|\le a_{[n^\theta]},|X_i|\le a_n)+I(|a_{ni}X_i|\le a_{[n^\theta]},|X_i|>a_n)\}
\end{aligned}
$$

$$\le a_n^{-\alpha}\sum_{i=1}^n E|a_{ni}X_i|^\alpha I(|X_i|\le a_n)+\sum_{i=1}^n P(|X_i|>a_n)$$
$$\le Cna_n^{-\alpha}E|X|^\alpha I(|X|\le a_n)+nP(|X|>a_n)\le Cn^{1-\alpha/p}.$$

Therefore, (2.3) and (2.4) hold. □

Lemma 2.5 *Let X be a random variable and $\{a_n, n\ge 1\}$ be a sequence of constants with $0<a_n/n\uparrow$. If (1.5) holds, then*

$$\lim_{n\to\infty} na_n^{-1}EXI(a_{[n^\theta]}<X\le a_n)=0 \text{ for all } \theta\in(0,1). \tag{2.9}$$

Proof Note that (2.7) holds for $r=1$ by $0<a_n/n\uparrow$ and (1.5). Hence,

$$\begin{aligned} na_n^{-1}EXI(a_{[n^\theta]}<X\le a_n) &\le na_n^{-1}\sum_{k=[n^\theta]+1}^n a_kEI(a_{k-1}<X\le a_k)\\ &\le \sum_{k=[n^\theta]+1}^n kEI(a_{k-1}<X\le a_k)\\ &\le Cn^\theta P(|X|>a_{[n^\theta]})+\sum_{k=[n^\theta]}^n P(|X|>a_k)\to 0\end{aligned}$$

as $n\to\infty$. Thus, (2.9) holds. □

Proof of Theorem 1.4 Sufficiency. Since $a_{ni}=a_{ni}^+-a_{ni}^-$, where $a_{ni}^+=\max\{0,a_{ni}\}, a_{ni}^-=\max\{0,-a_{ni}\}$, without loss of generality, we assume that $a_{ni}\ge 0, 1\le i\le n, n\ge 1$. Set

$$X_{ni}=-a_nI(a_{ni}X_i<-a_n)+a_{ni}X_iI(|a_{ni}X_i|\le a_n)+a_nI(a_{ni}X_i>a_n)$$

for all $1\le i\le n$ and $n\ge 1$. By Lemma 2.4 and $r>1$ and $1\le p<2$,

$$\begin{aligned} &a_n^{-1}\max_{1\le k\le n}|\sum_{i=1}^k(-a_nP(a_{ni}X_i<-a_n)+a_nP(a_{ni}X_i>a_n))|\\ &\le\sum_{i=1}^n P(|a_{ni}X_i|>a_n)\to 0\end{aligned}$$

as $n\to\infty$. Therefore, to prove (1.8), it suffices to prove that

$$\sum_{n=1}^\infty n^{r-2}P\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^k(a_{ni}X_i-EX_{ni})\Big|>\varepsilon a_n\Big)<\infty. \tag{2.10}$$

Note that

$$\begin{aligned} &\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^k(a_{ni}X_i-EX_{ni})\Big|>\varepsilon a_n\Big)\\ &\subset\cup_{i=1}^n(|a_{ni}X_i|>a_n)\cup\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^k(X_{ni}-EX_{ni})\Big|>\varepsilon a_n\Big).\end{aligned}$$

Hence, by Lemma 2.4, to prove (2.10), it is enough to prove

$$\sum_{n=1}^{\infty} n^{r-2} P\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^{k}(X_{ni}-EX_{ni})\Big| > \varepsilon a_n\Big) < \infty. \tag{2.11}$$

For any fixed $\theta \in (\max\{1/r, p/2\}, 1)$, set

$$X_{ni}^{(1)} = -a_{[n^\theta]} I(a_{ni}X_i < -a_{[n^\theta]}) + a_{ni}X_i I(|a_{ni}X_i| \le a_{[n^\theta]}) + a_{[n^\theta]} I(a_{ni}X_i > a_{[n^\theta]}),$$
$$X_{ni}^{(2)} = (a_{ni}X_i - a_{[n^\theta]}) I(a_{[n^\theta]} < a_{ni}X_i \le a_n) + (a_n - a_{[n^\theta]}) I(a_{ni}X_i > a_n),$$
$$X_{ni}^{(3)} = (a_{ni}X_i + a_{[n^\theta]}) I(-a_n \le a_{ni}X_i < -a_{[n^\theta]}) - (a_n - a_{[n^\theta]}) I(a_{ni}X_i < -a_n).$$

Then $X_{ni} = \sum_{l=1}^{3} X_{ni}^{(l)}$, and $\{X_{ni}^{(1)}, 1 \le k \le n\}, \{X_{ni}^{(2)}, 1 \le k \le n\}, \{X_{ni}^{(3)}, 1 \le k \le n\}$ are all END by Lemma 2.1. Hence to prove (2.11), it is enough to prove that for $l = 1, 2, 3$,

$$I_l = \sum_{n=1}^{\infty} n^{r-2} P\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^{k}(X_{ni}^{(l)} - EX_{ni}^{(l)})\Big| > \varepsilon a_n/3\Big) < \infty.$$

Firstly, we prove that $I_1 < \infty$. By the Markov inequality, Lemma 2.2, the C_r inequality and the Jensen inequality, for any $v \ge 2$,

$$\begin{aligned} I_1 &\le C\sum_{n=1}^{\infty} n^{r-2} a_n^{-v} E\max_{1\le k\le n} |\sum_{i=1}^{k}(X_{ni}^{(1)} - EX_{ni}^{(1)})|^v \\ &\le C\sum_{n=1}^{\infty} n^{r-2}(\log n)^v a_n^{-v}\sum_{i=1}^{n} E|X_{ni}^{(1)}|^v + C\sum_{n=1}^{\infty} n^{r-2}(\log n)^v [a_n^{-2}\sum_{i=1}^{n} E(X_{ni}^{(1)})^2]^{v/2} \\ &:= I_{11} + I_{12}. \end{aligned}$$

Taking v large enough such that $r - v(1-\theta)/p < -1$ and $r - (\theta\cdot\min\{r, 2/p\} - 1)v/2 < 0$, then by the definition of $X_{ni}^{(1)}$ and $0 < a_n/n^{1/p} \uparrow$, we get that

$$I_{11} \le C\sum_{n=1}^{\infty} n^{r-2}(\log n)^v \sum_{i=1}^{n}(a_{[n^\theta]}/a_n)^v \le C\sum_{n=1}^{\infty} n^{r-1-v(1-\theta)/p}(\log n)^v < \infty.$$

By the definition of $X_{ni}^{(1)}$ again, Lemma 2.4 and $0 < a_n \uparrow$,

$$\begin{aligned} a_n^{-2}\sum_{i=1}^{n} E(X_{ni}^{(1)})^2 &\le a_n^{-2}\sum_{i=1}^{n} E\{|a_{ni}X_i|^2 I(|a_{ni}X_i| \le a_{[n^\theta]}) + a_{[n^\theta]}^2 I(|a_{ni}X_i| > a_{[n^\theta]})\} \\ &\le a_n^{-\min\{rp,2\}}\sum_{i=1}^{n} E|a_{ni}X_i|^{\min\{rp,2\}} I(|a_{ni}X_i| \le a_{[n^\theta]}) + \sum_{i=1}^{n} P(|a_{ni}X_i| > a_{[n^\theta]}) \\ &\le Cn^{1-\theta\cdot\min\{r,2/p\}}. \end{aligned}$$

Therefore, we have by the choice of v that

$$I_{12} \le C\sum_{n=1}^{\infty} n^{r-2-(\theta\cdot\min\{r,2/p\}-1)v/2}(\log n)^v < \infty.$$

Thus, $I_1 < \infty$. Secondly, we prove that $I_2 < \infty$. By the definition of $X_{ni}^{(2)}$ and Lemma 2.4,

$$a_n^{-1}\max_{1\le k\le n}\Big|\sum_{i=1}^{k} EX_{ni}^{(2)}\Big| \le a_n^{-1}\sum_{i=1}^{n} E\{a_{ni}X_i I(a_{[n^\theta]} < a_{ni}X_i \le a_n) + a_n I(a_{ni}X_i > a_n)\}$$

$$\leq 2\sum_{i=1}^{n} P(|a_{ni}X_i| > a_{[n^\theta]}) \leq Cn^{1-\theta\cdot\min\{r,2/p\}} \to 0$$

as $n \to \infty$. Therefore, by the definition of $X_{ni}^{(2)}$ again, to prove $I_2 < \infty$, it is to prove that

$$I_2^* = \sum_{n=1}^{\infty} n^{r-2} P\Big(\sum_{i=1}^{n}(X_{ni}^{(2)} - EX_{ni}^{(2)}) > \varepsilon a_n/12\Big) < \infty.$$

By the Markov inequality, Lemma 2.2, the C_r inequality and the Jensen inequality, for any v such that $v > \max\{2, q, 2r/(\theta \cdot \min\{r, 2/p\} - 1)\}$,

$$\begin{aligned} I_2^* &\leq C\sum_{n=1}^{\infty} n^{r-2} a_n^{-v} E\Big|\sum_{i=1}^{n}(X_{ni}^{(2)} - EX_{ni}^{(2)})\Big|^v \\ &\leq C\sum_{n=1}^{\infty} n^{r-2} a_n^{-v} \sum_{i=1}^{n} E|X_{ni}^{(2)}|^v + C\sum_{n=1}^{\infty} n^{r-2} a_n^{-v} \Big(\sum_{i=1}^{n} E|X_{ni}^{(2)}|^2\Big)^{v/2} \\ &:= I_{21}^* + I_{22}^*. \end{aligned}$$

By Lemma 2.4, we have

$$\begin{aligned} I_{21}^* \leq & C\sum_{n=1}^{\infty} n^{r-2} a_n^{-v} \sum_{i=1}^{n} E|a_{ni}X_i|^v I(|a_{ni}X_i| \leq a_n) + \\ & C\sum_{n=1}^{\infty} n^{r-2} \sum_{i=1}^{n} P(|a_{ni}X_i| > a_n) < \infty \end{aligned}$$

and

$$\begin{aligned} I_{22}^* &\leq C\sum_{n=1}^{\infty} n^{r-2} \Big\{\sum_{i=1}^{n}(E(a_{ni}X_i/a_n)^2 I(a_{[n^\theta]} < a_{ni}X_i \leq a_n) + P(a_{ni}X_i > a_n))\Big\}^{v/2} \\ &\leq C\sum_{n=1}^{\infty} n^{r-2} \Big(2\sum_{i=1}^{n} P(|a_{ni}X_i| > a_{[n^\theta]})\Big)^{v/2} \\ &\leq C\sum_{n=1}^{\infty} n^{r-2-(\theta\cdot\min\{r,2/p\}-1)v/2} < \infty. \end{aligned}$$

Hence, $I_2 < \infty$. By the same argument as $I_2 < \infty$, we have $I_3 < \infty$. Thus, (1.8) holds.

Necessity. Set $a_{ni} = 1$ for all $1 \leq i \leq n$ and $n \geq 1$. Then (1.8) can be rewritten to

$$\sum_{n=1}^{\infty} n^{r-2} P\Big(\max_{1\leq k\leq n} \Big|\sum_{i=1}^{k}(X_i - EX_i I(|X_i| \leq a_n))\Big| > \varepsilon a_n\Big) < \infty \text{ for all } \varepsilon > 0.$$

Hence, the proof of (1.3) can be accomplished completely in a similar way as Theorem 1.1 of [26]. Thus, we omit the details of the proof. □

Proof of Theorem 1.5 In order to prove Theorem 1.5, we need only to prove sufficiency. By (1.8) in Theorem 1.4, we only need to prove that

$$a_n^{-1}\sum_{i=1}^{n} E|a_{ni}X_i||I(|a_{ni}X_i| \leq a_n) - I(|X_i| \leq a_n)| = 0.$$

Set $A_n = \{i : |a_{ni}| > 1\}$, $B_n = \{i : 0 < |a_{ni} \leq 1\}$. By (2.7) and $r > 1$, utilizing a similar method to the proof of (2.6), we have

$$\begin{aligned}
& a_n^{-1}\sum_{i=1}^{n} E|a_{ni}X_i||I(|a_{ni}X_i|\le a_n)-I(|X_i|\le a_n)| \\
&= a_n^{-1}\sum_{i\in A_n} E|a_{ni}X_i|I(a_n<|a_{ni}X_i|\le |a_{ni}|a_n)+ \\
&\quad a_n^{-1}\sum_{i\in B_n} E|a_{ni}X_i|I(|a_{ni}|a_n<|a_{ni}X_i|\le a_n) \\
&\le a_n^{-1}\sum_{i\in A_n} E|a_{ni}X_i|(\frac{|a_{ni}X_i|}{a_n})^{rp-1}I(a_n<|a_{ni}X_i|\le |a_{ni}|a_n)+ \\
&\quad a_n^{-1}\sum_{i\in B_n} E|a_{ni}X_i|I(|a_{ni}|a_n<|a_{ni}X_i|\le a_n) \\
&\le a_n^{-rp}\sum_{i\in A_n}|a_{ni}|^{rp}E|X|^{rp}I(|X|\le a_n)+a_n^{-1}\sum_{i\in B_n}a_nP(|X|>a_n) \\
&\le na_n^{-rp}E|X|^{rp}I(|X|\le a_n)+nP(|X|>a_n) \\
&\le Cn^{1-r}+nP(|X|>a_n)\to 0.
\end{aligned}$$

Thus, Theorem 1.5 holds. □

Proof of Corollary 1.6 Firstly, we prove (1.9)⇒ (1.10). By (1.9) and using Theorem 1.4 for $a_n=n^{1/p}$, to prove (1.10), it is enough to prove

$$\lim_{n\to\infty} n^{-1/p}\max_{1\le k\le n}\Big|\sum_{i=1}^{k} a_{ni}EX_iI(|a_{ni}X_i|\le n^{1/p})\Big|=0.$$

By (1.9) and (2.8),

$$\begin{aligned}
& n^{-1/p}\max_{1\le k\le n}|\sum_{i=1}^{k} a_{ni}EX_iI(|a_{ni}X_i|\le n^{1/p})|\le n^{-1/p}\sum_{i=1}^{n}E|a_{ni}X_i|I(|a_{ni}X_i|>n^{1/p}) \\
&\quad \le n^{-r}\sum_{i=1}^{n}E|a_{ni}X|^{rp}I(|a_{ni}X|>n^{1/p})\le Cn^{1-r}\to 0
\end{aligned}$$

as $n\to\infty$. Therefore, (1.10) holds. The proof of (1.10)⇒ (1.9) can be accomplished in a similar way as Theorem 1.1 of [3], we omit the details of the proof. □

Proof of Theorem 1.7 The proof of (1.5)⇔(1.7) can be done by the same method as in [23, Theorem 2.3]. Next, we prove that (1.5)⇔(1.11). Firstly, we prove that (1.5)⇒(1.11). Without loss of generality, we assume that $a_{ni}\ge 0$ and (1.1) holds for $1<q\le 2$ by the Hölder inequality. Set

$$X_{ni}=-a_nI(X_i<-a_n)+X_iI(|X_i|\le a_n)+a_nI(X_i>a_n)\ \text{ for }\ 1\le i\le n,\ n\ge 1.$$

Since (2.7) holds for $r=1$ by (1.5) and $0<a_n/n\uparrow$, we have by (2.8) that

$$a_n^{-1}\max_{1\le k\le n}\Big|\sum_{i=1}^{k} a_{ni}(-a_nP(X_i<-a_n)+a_nP(X_i>a_n))\Big|\le CnP(|X|>a_n)\to 0$$

as $n \to \infty$. Therefore, to prove that (1.11), it suffices to prove for all $\varepsilon > 0$ and all arrays of real numbers $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ satisfying (1.1) for some $1 < q \le 2$ that

$$\sum_{n=1}^{n} n^{-1} P\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^{k} a_{ni}(X_i - EX_{ni})\Big| > \varepsilon a_n\Big) < \infty. \tag{2.12}$$

Note that

$$\begin{aligned}&\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^{k} a_{ni}(X_i - EX_{ni})\Big| > \varepsilon a_n\Big)\\ &\quad\subset \cup_{i=1}^{n}(|X_i| > a_n) \cup \Big(\max_{1\le k\le n}\Big|\sum_{i=1}^{k} a_{ni}(X_{ni} - EX_{ni})\Big| > \varepsilon a_n\Big).\end{aligned}$$

Hence, by (1.5), to prove (2.12), it is enough to prove

$$\sum_{n=1}^{n} n^{-1} P\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^{k} a_{ni}(X_{ni} - EX_{ni})\Big| > \varepsilon a_n\Big) < \infty. \tag{2.13}$$

For any fixed $\theta \in (0, 1 - 1/q)$ set

$$\begin{aligned}X_{ni}^{(1)} &= -a_{[n^\theta]} I(X_i < -a_{[n^\theta]}) + X_i I(|X_i| \le a_{[n^\theta]}) + a_{[n^\theta]} I(X_i > a_{[n^\theta]}),\\ X_{ni}^{(2)} &= (X_i - a_{[n^\theta]}) I(a_{[n^\theta]} < X_i \le a_n) + (a_n - a_{[n^\theta]}) I(X_i > a_n),\\ X_{ni}^{(3)} &= (X_i + a_{[n^\theta]}) I(-a_n \le X_i < -a_{[n^\theta]}) - (a_n - a_{[n^\theta]}) I(X_i < -a_n).\end{aligned}$$

Then $X_{ni} = \sum_{l=1}^{3} X_{ni}^{(l)}$, and $\{X_{ni}^{(1)}, 1 \le i \le n\}$, $\{X_{ni}^{(2)}, 1 \le i \le n\}$, $\{X_{ni}^{(3)}, 1 \le i \le n\}$ are all END by Lemma 2.1. Hence, to prove (2.13), it is enough to prove that for $l = 1, 2, 3$,

$$I_l = \sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^{k} a_{ni}(X_{ni}^{(l)} - EX_{ni}^{(l)})\Big| > \varepsilon a_n/3\Big) < \infty.$$

Firstly, we prove that $I_1 < \infty$. By the Markov inequality, Lemma 2.3, the C_r inequality, the Jensen inequality, the definition of $X_{ni}^{(1)}$ and (1.1),

$$\begin{aligned}I_1 &\le C\sum_{n=1}^{\infty} n^{-1}(\log n)^q a_n^{-q} \sum_{i=1}^{n} E|a_{ni}X_{ni}^{(1)}|^q\\ &\le C\sum_{n=1}^{\infty} n^{-1}(\log n)^q a_n^{-q} \sum_{i=1}^{n} (a_{ni})^q n^{q\theta} \le C\sum_{n=1}^{\infty} n^{-(1-\theta)q}(\log n)^q < \infty.\end{aligned}$$

Secondly, we prove that $I_2 < \infty$. By (2.7) and (2.8) and Lemma 2.5,

$$\begin{aligned}a_n^{-1}\max_{1\le k\le n}\Big|\sum_{i=1}^{k} Ea_{ni}X_{ni}^{(2)}\Big| &\le a_n^{-1}\sum_{i=1}^{n} a_{ni} E\{X_i I(a_{[n^\theta]} < X_i \le a_n) + a_n I(X_i > a_n)\}\\ &\le n a_n^{-1} EXI(a_{[n^\theta]} < X \le a_n) + nP(|X| > a_n) \to 0\end{aligned}$$

as $n \to \infty$. Therefore, by the definition of $X_{ni}^{(2)}$, to prove $I_2 < \infty$, it is to prove that

$$I_2^* = \sum_{n=1}^{\infty} n^{-1} P\Big(\sum_{i=1}^{n} a_{ni}(X_{ni}^{(2)} - EX_{ni}^{(2)}) > \varepsilon a_n/12\Big) < \infty.$$

By the Markov inequality, Lemma 2.3, the C_r inequality, the Jensen inequality, (1.5) and (2.5),

$$\begin{aligned}I_2^* &\le C\sum_{n=1}^{\infty} n^{-1}a_n^{-q}\sum_{i=1}^{n}(a_{ni})^q E|X_{ni}^{(2)}|^q\\ &\le C\sum_{n=1}^{\infty} n^{-1}a_n^{-q}\sum_{i=1}^{n}E|a_{ni}X_i|^q I(|X_i|\le a_n)+C\sum_{n=1}^{\infty}P(|X|>a_n)\\ &\le C\sum_{n=1}^{\infty}a_n^{-q}E|X|^q I(|X|\le a_n)+C\sum_{n=1}^{\infty}P(|X|>a_n)<\infty.\end{aligned}$$

Hence, $I_2<\infty$. By the same argument as $I_2<\infty$, we have $I_3<\infty$. Thus, (1.11) holds.

Next, we prove that (1.11)⇒(1.5). Set $a_{ni}=1$ for all $1\le i\le n$ and $n\ge 1$. Then (1.9) can be rewritten to

$$\sum_{n=1}^{\infty} n^{-1}P\Big(\max_{1\le k\le n}\Big|\sum_{i=1}^{k}(X_i-EX_iI(|X_i|\le a_n))\Big|>\varepsilon a_n\Big)<\infty,\quad \forall\varepsilon>0.$$

Hence, it is similar to the proof of Theorem 1.2 in [26]. So we complete the proof. □

Proof of Corollary 1.8 Firstly, we prove (1.12)⇒ (1.13). By (1.12) and using Theorem 1.7 for $a_n=n$, to prove (1.13), it is enough to prove

$$\lim_{n\to\infty}\frac{1}{n}\max_{1\le k\le n}|\sum_{i=1}^{k}a_{ni}EX_iI(|X_i|\le n)|=0.$$

By (2.8) and $EX=0$,

$$\begin{aligned}\frac{1}{n}\max_{1\le k\le n}|\sum_{i=1}^{k}a_{ni}EX_iI(|X_i|\le n)| &\le \frac{1}{n}\sum_{i=1}^{n}E|a_{ni}X|I(|X|>n)\\ &\le CE|X|I(|X|>n)\to 0\end{aligned}$$

as $n\to\infty$. Therefore, (1.13) holds. The proof of (1.13)⇒ (1.12) can be completed in a similar way as Theorem 1.1 of [3], we omit the details of the proof. □

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