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# On the Spectra of Strong Power Graphs of Finite Groups

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**Abstract** Let $G$ be a finite group of order $n$. The strong power graph of $G$ is the undirected graph whose vertex set is $G$ and two distinct vertices $x$ and $y$ are adjacent if $x^{n_1} = y^{n_2}$ for some positive integers $n_1, n_2 < n$. In this paper, we give the characteristic polynomials of the distance and adjacency matrix of the strong power graph of $G$, and compute its distance and adjacency spectrum.

**Keywords** strong power graph; cyclic group; characteristic polynomial; spectrum

**MR(2010) Subject Classification** 05C25; 05C50

## 1. Introduction

Given a connected graph $\Gamma$, denote by $V(\Gamma)$ and $E(\Gamma)$ the vertex set and edge set of $\Gamma$, respectively. Let $V(\Gamma) = \{v_1, v_2, \ldots, v_n\}$. The distance between the vertices $v_i$ and $v_j$, denoted $d_\Gamma(v_i, v_j)$, is the length of the shortest path between them. The diameter of $\Gamma$, denoted $\mathrm{diam}(\Gamma)$, is the maximum distance between any pair of vertices of $\Gamma$. The set of neighbours of a vertex $v_i$ in $\Gamma$ is denoted by $N_\Gamma(v_i)$, that is, $N_\Gamma(v_i) = \{v_j \in V(\Gamma) : \{v_i, v_j\} \in E(\Gamma)\}$.

The distance matrix $D(\Gamma)$ of $\Gamma$ is the $n \times n$ matrix, indexed by $V(\Gamma)$, such that $D(\Gamma)_{v_i,v_j} = d_\Gamma(v_i, v_j)$. The distance characteristic polynomial $\Theta(\Gamma, x)$ of $\Gamma$ is $|xI - D(\Gamma)|$, where $I$ is the identity matrix of size $n$. Note that $D(\Gamma)$ is symmetric. The distance characteristic polynomial has real roots $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n$. If $\mu_{i_1} \ge \mu_{i_2} \ge \cdots \ge \mu_{i_t}$ are the distinct roots of $\Theta(\Gamma, x)$, then the distance spectrum (also called $D$-spectrum) of $\Gamma$ can be written as

$$\mathrm{spec}_D(\Gamma) = \begin{pmatrix} \mu_{i_1} & \mu_{i_2} & \ldots & \mu_{i_t} \\ m_1 & m_2 & \ldots & m_t \end{pmatrix}, \tag{1.1}$$

where $m_j$ is the algebraic multiplicity of $\mu_{i_j}$. Clearly, $\sum_{j=1}^{t} m_j = n$. The adjacency matrix $A(\Gamma)$ of $\Gamma$ is an $n \times n$ matrix, indexed by $V(\Gamma)$, and the $ij$-th entry of $A(\Gamma)$ is 1 if the vertices $v_i$ and $v_j$ are adjacent, otherwise it is 0. Denote by $\Phi(\Gamma, x)$ the characteristic polynomial of $A(\Gamma)$. Similarly to (1.1), we can define the adjacency spectrum $\mathrm{spec}(\Gamma)$ of $\Gamma$. The largest root of $\Theta(\Gamma, x)$ (resp., $\Phi(\Gamma, x)$) is called the distance spectral radius (resp., adjacency spectral radius) of $\Gamma$ (see [1]).

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Graphs associated with groups and other algebraic structures have been actively investigated, since they have valuable applications [2] and are related to automata theory [3, 4]. Let $G$ be a finite group of order $n$. The undirected power graph of $G$ has the vertex set $G$ and two distinct elements are adjacent if one is a power of the other. In 2000, Kelarev and Quinn [5] introduced the concept of a directed power graph. In 2009, Chakrabarty, Ghosh, and Sen [6] introduced the undirected power graph of a group. Motivated by this, Singh and Manilal [7] defined the strong power graphs as a generalization of the power graphs. The strong power graph $\mathcal{P}_s(G)$ of $G$ is a graph whose vertex set consists of the elements of $G$ and two distinct vertices $x$ and $y$ are adjacent if $x^{n_1} = y^{n_2}$ for some positive integers $n_1, n_2 < n$. Recently, Bhuniya and Berathe [8] gave the Laplacian spectrum of the strong power graphs of finite groups.

In this paper, we give the characteristic polynomials of the distance and adjacency matrix of the strong power graph of a finite group, and compute its distance and adjacency spectrum.

## 2. The results

Throughout this section $G$ denotes a finite group, and $\mathbb{Z}_n$ stands for the cyclic group of order $n$. We always assume $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$. For strong power graphs, the proof of the following result is straightforward.

**Proposition 1.1** *(1) If $G$ is not cyclic, then $\mathcal{P}_s(G)$ is complete.*

*(2) $\mathcal{P}_s(\mathbb{Z}_n)$ is not connected if and only if $n$ is a prime number.*

*(3) $N_{\mathcal{P}_s(\mathbb{Z}_n)}(0) = \{k \in \mathbb{Z}_n : m \neq 0, (m, n) \neq 1\}$, and the subgraph of $\mathcal{P}_s(\mathbb{Z}_n)$ induced by $\mathbb{Z}_n \setminus \{0\}$ is complete. In particular, $\mathrm{diam}(\mathcal{P}_s(\mathbb{Z}_n)) = 2$ if $n$ is not a prime number.*

Now we determine the characteristic polynomial of the distance matrix associated with the strong power graph $\mathcal{P}_s(\mathbb{Z}_n)$ for any composite number $n$.

**Theorem 2.2** *For any composite number $n$,*

$$\Theta(\mathcal{P}_s(\mathbb{Z}_n), x) = (x+1)^{n-3}\big(x^3 + (3-n)x^2 + (3-2n-3\phi(n))x - \phi(n)^2 - \phi(n)(4-n) - n + 1\big),$$

*where $\phi(n)$ is Euler's totient function.*

**Proof** Write $\phi(n) = t$ and $k = n - \phi(n) - 1$. By Proposition 1.1, the distance matrix $D(\mathcal{P}_s(\mathbb{Z}_n))$ is the $n \times n$ matrix given below, where the rows and columns are indexed in order by the vertices in $N_{\mathcal{P}_s(\mathbb{Z}_n)}(0)$, then all generate elements of $\mathbb{Z}_n$, and 0 is in the last position.

$$D(\mathcal{P}_s(\mathbb{Z}_n)) = \begin{pmatrix} 0 & 1 & \ldots & 1 & 1 & \ldots & 1 & 1 \\ 1 & 0 & \ldots & 1 & 1 & \ldots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & 0 & 1 & \ldots & 1 & 1 \\ 1 & 1 & \ldots & 1 & 0 & \ldots & 1 & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & 1 & 1 & \ldots & 0 & 2 \\ 1 & 1 & \ldots & 1 & 2 & \ldots & 2 & 0 \end{pmatrix}.$$

The characteristic polynomial of $D(\mathcal{P}_s(\mathbb{Z}_n))$ is

$$\Theta(\mathcal{P}_s(\mathbb{Z}_n),x)=\begin{vmatrix} x & -1 & \dots & -1 & -1 & \dots & -1 & -1\\ -1 & x & \dots & -1 & -1 & \dots & -1 & -1\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ -1 & -1 & \dots & x & -1 & \dots & -1 & -1\\ -1 & -1 & \dots & -1 & x & \dots & -1 & -2\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ -1 & -1 & \dots & -1 & -1 & \dots & x & -2\\ -1 & -1 & \dots & -1 & -2 & \dots & -2 & x \end{vmatrix}. \tag{2.1}$$

Subtract the first column from the columns $2,3,\dots,n$ of (2.1) to obtain the determinant (2.2):

$$(x+1)^{k-1}\begin{vmatrix} x & -1 & \dots & -1 & -1-x & \dots & -1-x & -1-x\\ -1 & 1 & \dots & 0 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ -1 & 0 & \dots & 1 & 0 & \dots & 0 & 0\\ -1 & 0 & \dots & 0 & x+1 & \dots & 0 & -1\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ -1 & 0 & \dots & 0 & 0 & \dots & x+1 & -1\\ -1 & 0 & \dots & 0 & -1 & \dots & -1 & x+1 \end{vmatrix}. \tag{2.2}$$

Adding the rows $2,3,\dots,n-1$ to the first row of (2.2), and then adding columns $2,3,\dots,k$ to the first column, we arrive at the determinant (2.3):

$$(x+1)^{k-1}\begin{vmatrix} x-n+2 & 0 & \dots & 0 & 0 & \dots & 0 & -1-x-t\\ 0 & 1 & \dots & 0 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & 0 & \dots & 0 & 0\\ -1 & 0 & \dots & 0 & x+1 & \dots & 0 & -1\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ -1 & 0 & \dots & 0 & 0 & \dots & x+1 & -1\\ -1 & 0 & \dots & 0 & -1 & \dots & -1 & x+1 \end{vmatrix}. \tag{2.3}$$

Subtract the first column from the last column of (2.3). Then subtract the row $k+1$ from the rows $k+2,k+3,\dots,n$ to obtain the determinant (2.4):

$$(x+1)^{n-3}\begin{vmatrix} x-n+2 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -2x+k-2\\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0\\ -1 & 0 & \dots & 0 & x+1 & 0 & \dots & 0 & 0\\ 0 & 0 & \dots & 0 & -1 & 1 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 0 & -1 & 0 & \dots & 1 & 0\\ 0 & 0 & \dots & 0 & -2-x & -1 & \dots & -1 & x+2 \end{vmatrix}. \tag{2.4}$$

Adding the rows $k+2, k+3, \dots, n-1$ to the last row of (2.4), we get the determinant (2.5):

$$(x+1)^{n-3}\begin{vmatrix} x-n+2 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -2x+k-2 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & \dots & 0 & x+1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -1 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & -x-t-1 & 0 & \dots & 0 & x+2 \end{vmatrix}. \tag{2.5}$$

Expand it along the first row to obtain the determinant (2.6):

$$(x+1)^{n-3}\big((x-n+2)(x+1)(x+2) + (-1)^{1+n}(-2x+k-2)|A|\big), \tag{2.6}$$

where

$$|A| = \begin{vmatrix} 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ -1 & 0 & \dots & 0 & x+1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & -1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & -1 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 & -x-t-1 & 0 & \dots & 0 \end{vmatrix}. \tag{2.7}$$

In (2.7) the determinant has order $n-1$. Interchange the row $k-1$ and the last row of (2.7). Then expand it along the first column to obtain $|A|$ as follows:

$$|A| = (-1)^n(-x-t-1).$$

It follows that

$$\Theta(\mathcal{P}_s(\mathbb{Z}_n), x) = (x+1)^{n-3}\big(x^3 + (3-n)x^2 + (3-2n-3t)x - n + nt - t^2 - 4t + 1\big).$$

This completes our proof. □

By Proposition 2.1, one has that $\mathcal{P}_s(G)$ is connected if and only if $G$ is not cyclic, or $G \cong \mathbb{Z}_n$ for some composite number $n$. Now we compute the $D$-spectrum of any connected strong power graph.

**Theorem 2.3** *If $G$ is not cyclic, then*

$$\operatorname{spec}_D(\mathcal{P}_s(G)) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}.$$

*If $G \cong \mathbb{Z}_n$ for some composite number $n$, then $\operatorname{spec}_D(\mathcal{P}_s(\mathbb{Z}_n))$ is*

$$\begin{pmatrix} -1 & \frac{n-3+2\cos\frac{\theta}{3}\sqrt{n^2+9\phi(n)}}{3} & \frac{n-3+2\cos\frac{\theta+2\pi}{3}\sqrt{n^2+9\phi(n)}}{3} & \frac{n-3+2\cos\frac{\theta-2\pi}{3}\sqrt{n^2+9\phi(n)}}{3} \\ n-3 & 1 & 1 & 1 \end{pmatrix},$$

*where* $0<\theta<\frac{\pi}{2}$ *and* $\theta=\arccos\frac{2n^3+27\phi(n)^2+27\phi(n)}{2\sqrt{(n^2+9\phi(n))^3}}$.

**Proof** Note that if $G$ is not cyclic, then $\mathcal{P}_s(G)$ is complete. Thus, it suffices to compute the distance spectrum of $\mathcal{P}_s(\mathbb{Z}_n)$ for some composite number $n$. Let

$$f(x)=x^3+(3-n)x^2+(3-2n-3\phi(n))x-\phi(n)^2-\phi(n)(4-n)-n+1.$$

Suppose that $f(-1)=0$. Then $\phi(n)(n-\phi(n)-1)=0$. It follows that $\phi(n)=n-1$. Namely, $n$ is a prime number, a contradiction. This implies that $D$-spectrum of $\mathcal{P}_s(\mathbb{Z}_n)$ has $-1$ with multiplicity $n-3$ by Theorem 2.2. Noticing the canonical solutions of any quadratic and cubic equation, we conclude that $f(x)$ has three pairwise distinct roots, as presented above. $\square$

**Corollary 2.4** *For any composite number $n$, the distance spectral radius of $\mathcal{P}_s(\mathbb{Z}_n)$ is*

$$\frac{n-3+2\cos\frac{\theta}{3}\sqrt{n^2+9\phi(n)}}{3},$$

*where* $0<\theta<\frac{\pi}{2}$ *and* $\theta=\arccos\frac{2n^3+27\phi(n)^2+27\phi(n)}{2\sqrt{(n^2+9\phi(n))^3}}$.

For any positive integer $n$, the adjacency matrix $A(\mathcal{P}_s(\mathbb{Z}_n))$ is given below:

$$A(\mathcal{P}_s(\mathbb{Z}_n))=\begin{pmatrix} 0 & 1 & \dots & 1 & 1 & \dots & 1 & 1\\ 1 & 0 & \dots & 1 & 1 & \dots & 1 & 1\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & 1 & \dots & 0 & 1 & \dots & 1 & 1\\ 1 & 1 & \dots & 1 & 0 & \dots & 1 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & 1 & \dots & 1 & 1 & \dots & 0 & 0\\ 1 & 1 & \dots & 1 & 0 & \dots & 0 & 0 \end{pmatrix},$$

where the rows and columns are indexed in order by the vertices in $N_{\mathcal{P}_s(\mathbb{Z}_n)}(0)$, then all generate elements of $\mathbb{Z}_n$, and 0 is in the last position. Note that $N_{\mathcal{P}_s(\mathbb{Z}_n)}(0)$ may be the empty set.

By an argument similar to the one used in the computing of $\Theta(\mathcal{P}_s(\mathbb{Z}_n),x)$, we can get $\Phi(\mathcal{P}_s(\mathbb{Z}_n),x)$.

**Theorem 2.5** *For any positive integer $n$,*

$$\Phi(\mathcal{P}_s(\mathbb{Z}_n),x)=(x+1)^{n-3}\big(x^3+(3-n)x^2+(3-2n+\phi(n))x+(n-\phi(n)-1)(\phi(n)-1)\big).$$

**Corollary 2.6** *For a prime number $p$,* $\Phi(\mathcal{P}_s(\mathbb{Z}_p),x)=x(x+1)^{p-2}(x+2-p)$.

As an application of Theorem 2.5, we may obtain the adjacency spectrum of the strong power graph of a finite group.

**Theorem 2.7** *If $G$ is not cyclic, then*

$$\mathrm{spec}(\mathcal{P}_s(G))=\begin{pmatrix} n-1 & -1\\ 1 & n-1 \end{pmatrix}.$$

*If $G \cong \mathbb{Z}_p$ for some prime number $p$, then*

$$\mathrm{spec}(\mathcal{P}_s(\mathbb{Z}_n)) = \begin{pmatrix} 0 & -1 & p-2 \\ 1 & p-2 & 1 \end{pmatrix}.$$

*If $G \cong \mathbb{Z}_n$ for some composite number $n$, then* $\mathrm{spec}(\mathcal{P}_s(\mathbb{Z}_n))$ *is*

$$\begin{pmatrix} -1 & \frac{n-3+2\cos\frac{\theta}{3}\sqrt{n^2-3\phi(n)}}{3} & \frac{n-3+2\cos\frac{\theta+2\pi}{3}\sqrt{n^2-3\phi(n)}}{3} & \frac{n-3+2\cos\frac{\theta-2\pi}{3}\sqrt{n^2-3\phi(n)}}{3} \\ n-3 & 1 & 1 & 1 \end{pmatrix},$$

*where $0<\theta<\frac{\pi}{2}$ and $\theta = \arccos\frac{2n^3+27\phi(n)^2+27\phi(n)-36n\phi(n)}{2\sqrt{(n^2-3\phi(n))^3}}$.*

**Corollary 2.8** *For any composite number $n$, the adjacency spectral radius of $\mathcal{P}_s(\mathbb{Z}_n)$ is*

$$\frac{n-3+2\cos\frac{\theta}{3}\sqrt{n^2-3\phi(n)}}{3},$$

*where $0<\theta<\frac{\pi}{2}$ and $\theta = \arccos\frac{2n^3+27\phi(n)^2+27\phi(n)-36n\phi(n)}{2\sqrt{(n^2-3\phi(n))^3}}$.*

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