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Annihilator Condition on Power Values of Commutators with Derivations

Shuliang HUANG
School of Mathematics and Finance, Chuzhou University, Anhui 239000*, P. R. China*

Abstract Let R be a prime ring with center $Z(R)$, I a nonzero ideal of R, d a nonzero derivation of R and $0 \neq a \in R$. In the present paper, our object is to study the situation $a[d(x^k), x^k]^n \in Z(R)$ for all $x \in I$ under certain conditions, where n (≥ 1), k (≥ 1) are fixed integers.

Keywords prime ring; derivation; extended centroid

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1. Introduction

Let R be a prime ring with center $Z(R)$. For $x, y \in R$, we set $[x, y]_1 = [x, y] = xy - yx$ and $[x, y]_n = [[x, y]_{n-1}, y]$ where $n \geq 2$ is a positive integer. By d we mean a derivation of R. s_4 denotes the standard identity in four variables. In [1], a well-known result proved by Posner states that if $[d(x), x] \in Z(R)$ for all $x \in R$, then either $d = 0$ or R is commutative. In [2], Lanski generalizes the Posner's result to a Lie ideal. Lanski proved that if L is a noncommutative Lie ideal of R and $d \neq 0$ such that $[d(x), x] \in Z(R)$ for all $x \in L$, then either R is commutative, or char $R = 2$ and R satisfies s_4. In [3], Carini and Filippis studied more generalized situation of this result by considering power values. They proved that if $[d(u), u]^n \in Z(R)$ for all u in a noncentral Lie ideal of R, $n \geq 1$ a fixed integer and char $R \neq 2$, then either $d = 0$ or R satisfies s_4. In [4], Wang and You removed the restriction on characteristic and they proved that the same conclusion holds when char $R = 2$.

On the other hand, some results concerning annihilators of power values in prime and semiprime rings have been obtained in literature. In [5], Bresar proved that if R is a semiprime ring, d a nonzero derivation of R and $a \in R$ such that $ad(x)^n = 0$, then $ad(R) = 0$ when R is $(n-1)!$-torsion free. In [6], Lee and Lin proved Bresar's result on Lie ideals of prime rings without the $(n-1)!$-torsion free assumption on R. In [7], Filippis established a similar version of Bresar's result for multilinear polynomials in prime rings. Furthermore, Filippis studied the left annihilator of power values of commutators with derivations. In [8], he proved if char $R \neq 2$, $0 \neq d$ and $0 \neq a \in R$ such that $a[d(x), x]^n \in Z(R)$ for all $x \in L$, where L is a noncentral Lie

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E-mail address: shulianghuang@sina.com

ideal of R and $n \geq 1$ a fixed integer, then R satisfies s_4. In [9], Wang removed the assumption of char $R \neq 2$. In [10], Du and Wang proved a result on both sided ideal in prime ring. They proved that if char $R \neq 2$, $0 \neq I$ a both sided ideal of R and $0 \neq d$ such that $[d(x^k), x^k]^n \in Z(R)$ for all $x \in I$, where k, n are fixed positive integer, then R satisfies s_4. For more related results concerning annihilators we refer to [11–13].

The purpose of the present paper is to study the same situation of Du and Wang with left annihilator condition.

First we recall some basic notations. We denote by Q the two sided Martindale quotient ring of a prime ring R and by C the center of Q. We call C the extended centroid of R. This C is a field. It is well known that every derivation d of R can be uniquely extended to a derivation of Q, which will be also denoted by d. The derivation d of R is called a Q-inner induced by some $q \in Q$ if $d(x) = [q, x]$ holds for all $x \in R$. If d is not Q-inner, then d is called Q-outer derivation of R.

By Kharchenko's theorem [14], we have the following result:

Let R be a prime ring, d a derivation on R and I a nonzero ideal of R. If I satisfies the differential identity $f(r_1, r_2, \ldots, r_n, d(r_1), d(r_2), \ldots, d(r_n)) = 0$ for any $r_1, r_2, \ldots, r_n \in I$, then either

(i) I satisfies the generalized polynomial identity $f(r_1, r_2, \ldots, r_n, x_1, x_2, \ldots, x_n) = 0$

or (ii) d is Q-inner.

2. Main results

We begin with lemmas.

Lemma 2.1 *Let $R = M_m(F)$ be the ring of all $m \times m$ matrices over a field F of characteristic different from 2 and $m \geq 3$. Let a be an invertible element in R. If for some $b \in R$, $([b, x^k]_2)^n \in F \cdot a^{-1}$ for all $x \in R$, where k (≥ 1), n (≥ 1) are fixed integers, then $b \in F \cdot I_m$.*

Proof Let $a = (a_{ij})_{m\times m}, b = (b_{ij})_{m\times m}$. By assumption, for every $x \in R$, $([b, x^k]_2)^n$ is zero or invertible. Write $b = \left(\begin{smallmatrix} b_{11} & A \\ B & C \end{smallmatrix}\right)$, where $A = (b_{12}, \ldots, b_{1m})$, $B = (b_{21}, \ldots, b_{m1})^T$ and $C = (b_{ij})$ where $2 \leq i, j \leq m$. We choose $x = e_{11}$. Then $[b, e_{11}]_2 = \left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right) = (be_{11} - 2b_{11}e_{11} + e_{11}b)$. Since rank of $[b, e_{11}]_2$ is ≤ 2, $([b, e_{11}]_2)^n$ cannot be invertible, since $m \geq 3$, and so it must be zero. Therefore, $([b, e_{11}]_2)^n = 0$ and so $([b, e_{11}]_2)^{2n} = 0$. By simple manipulation, we have

$$0 = ([b, e_{11}]_2)^{2n} = \begin{pmatrix} (AB)^n & 0 \\ 0 & (BA)^n \end{pmatrix}. \tag{2.1}$$

Therefore, $(AB)^n = 0$. Since $(AB) \in F$, $AB = 0$. Let ϕ be an inner automorphism of R defined by $\phi(x) = (1 + e_{21})x(1 - e_{21})$ for all $x \in R$. Then $\phi(b)$ satisfies the same property as b does, that is, either $([\phi(b), x^k]_2)^n$ is zero or invertible for every $x \in R$. Now, we have

$$\phi(b) = \begin{pmatrix} b_{11} - b_{12} & A \\ b_{11}E - b_{12}E + B - CE & EA + C \end{pmatrix}, \tag{2.2}$$

where $E = ((1, 0, \dots, 0)_{1\times m-1})^T$. As above, we have

$$A(b_{11}E - b_{12}E + B + CE) = 0. \tag{2.3}$$

Recalling $AB = 0$, above relation implies $b_{11}b_{12} - b_{12}^2 - ACE = 0$. Now we choose $x = e_{11} + e_{21}$.

$$[b, x^k]_2 = [b, e_{11} + e_{21}]_2 = \begin{pmatrix} -b_{12} & A \\ D & EA \end{pmatrix}, \tag{2.4}$$

where $D = B + CE - (b_{11} + 2b_{12})E$. We see in the matrix $[b, e_{11} + e_{21}]_2$ that number of distinct column vectors are 2. Hence, rank of $[b, e_{11} + e_{21}]_2$ is ≤ 2 and so rank of $([b, e_{11} + e_{21}]_2)^n$ is also ≤ 2. Therefore, $([b, e_{11} + e_{21}]_2)^n$ can not be invertible in R for $m \ge 3$, and hence it must be zero. Therefore, we can write $([b, e_{11} + e_{21}]_2)^{2n} = 0$. Now we calculate

$$([b, x^k]_2)^2 = ([b, e_{11} + e_{21}]_2)^2 = \begin{pmatrix} b_{12}^2 + AD & 0 \\ -b_{12}D + EAD & DA + b_{12}EA \end{pmatrix}. \tag{2.5}$$

Now the facts $AB = 0$ and $b_{11}b_{12} - b_{12}^2 - ACE = 0$ together imply $AD = -3b_{12}^2$. Thus, we have

$$([b, x^k]_2)^2 = ([b, e_{11} + e_{21}]_2)^2 = \begin{pmatrix} -2b_{12}^2 & 0 \\ -b_{12}D - 3b_{12}^2E & DA + b_{12}EA \end{pmatrix}, \tag{2.6}$$

and hence

$$0 = ([b, x^k]_2)^{2n} = ([b, e_{11} + e_{21}]_2)^{2n} = \begin{pmatrix} (-2b_{12}^2)^n & 0 \\ U & (DA + b_{12}EA)^n \end{pmatrix}, \tag{2.7}$$

where U is an $(m-1)\times 1$ matrix. This gives $(-2b_{12}^2)^n = 0$, implying $b_{12} = 0$. Since for any F-automorphism φ, b and b^φ satisfies the same properties, we can write $(b^\varphi)_{12} = 0$. Therefore, $0 = ((1 - e_{i2})b(1 + e_{i2}))_{12}$ for any $i \ne 1, 2$. This implies $b_{1i} = 0$ for all $i \ne 1, 2$. Since $b_{12} = 0$, all the entries in 1st row of the matrix b are zeros, except b_{11}. Hence, we can write, $0 = ((1 - e_{1j})b(1 + e_{1j}))_{1t}$ for any $j \ne 1$ and $t \ne 1$. This implies $b_{jt} = 0$ for all $j \ne t$. Thus, the matrix b is diagonal. Let $b = \sum\limits_{i=1}^{m} b_{ii}e_{ii}$. Then for $s \ne t$, we have $(1 + e_{ts})b(1 - e_{ts}) = \sum_{i=0}^{m} b_{ii}e_{ii} + (b_{ss} - b_{tt})e_{ts}$ is diagonal. Hence, $b_{ss} = b_{tt}$ and so b is a scalar matrix, that is, $b \in F \cdot I_m$. □

Lemma 2.2 ([15]) *Let R be a noncommutative simple algebra, finite-dimensional over its center Z. If $g(x_1, \dots, x_t) \in R*_Z Z\{x_j\}$, the free product over Z, is an identity for R that is homogeneous in $\{x_1, \dots, x_t\}$ of degree d, then for some field F and $n > 1$, $R \subseteq M_n(F)$ and $g(x_1, \dots, x_t)$ is an identity for $M_n(F)$.*

Theorem 2.3 *Let R be a prime ring of characteristic different from 2 with center $Z(R)$, I a nonzero ideal of R, d a nonzero derivation of R and $0 \ne a \in R$. Suppose that there exists $x \in I$ such that $a[d(x^k), x^k]^n \ne 0$. If $a[d(x^k), x^k]^n \in Z(R)$ for all $x \in I$, where n (≥ 1), k (≥ 1) are fixed integers, then R satisfies s_4, the standard identity in four variables.*

Proof Suppose that R does not satisfy s_4. By our assumption, we have

$$a[d(x^k), x^k]^n \in Z(R), \tag{2.8}$$

for all $x \in I$. Since there exists $r \in I$ such that $a[d(r^k), r^k]^n \neq 0$, $a[d(x^k), x^k]^n$ is a central differential identity for I. It follows from [16, Theorem 1] that R is a prime PI-ring and so $RC(= Q)$ is a finite-dimensional central simple C-algebra by Posner's theorem for prime PI-ring. Now we divide the proof in the following two cases:

Case 1 Let d be inner derivation of R induced by $p \in Q$. Then

$$[a([p, x^k]_2)^n, x_3] = 0, \tag{2.9}$$

for all $x \in I$ and so for all $x \in Q$, sine I and Q satisfy same GPI [17]. Since $a[d(r^k), r^k]^n \neq 0$ for some $r \in I$, (2.9) is a nontrivial GPI for Q. Also, since Q is a finite-dimensional central simple C-algebra, Lemma 2.2 is applicable. By Lemma 2.2, there exists a suitable field F such that $Q \subseteq M_k(F)$, the ring of all $k \times k$ matrices over F, and moreover $M_k(F)$ satisfies (2.9). Since by assumption, R does not satisfy s_4, $k \geq 3$. Therefore, we have

$$a([p, x^k]_2)^n \in Z(M_k(F))$$

for all $x \in M_k(F)$. Since $I \subseteq Q \subseteq M_k(F)$, there exists $r \in M_k(F)$, such that $a([p, r^k]_2)^n \neq 0$. Then a is invertible and so $([p, x^k]_2)^n \in F \cdot a^{-1}$ for all $x \in M_k(F)$. By Lemma 2.1, $p \in Z(R)$ implying $d = 0$, a contradiction.

Case 2 Let d be outer derivation of R. We rewrite the relation (2.8) as

$$a[\sum_{i=0}^{k-1} x^i d(x) x^{k-i-1}, x^k]^n \in Z(R). \tag{2.10}$$

By Kharchenko's theorem [14], we have that I satisfies

$$a[\sum_{i=0}^{k-1} x^i y x^{k-i-1}, x^k]^n \in Z(R). \tag{2.11}$$

Since we assumed that R does not satisfy s_4, R cannot be commutative. Therefore, we may choose $b \in R$ such that $b \notin Z(R)$. Replacing y with $[b, x]$ in (2.11), we obtain that for all $x \in I$

$$[a([[b, x^k], x^k])^n, x_3] = 0. \tag{2.12}$$

Then by the same argument as given in case-I, $b \in Z(R)$, a contradiction. □

The following example demonstrates that in the hypothesis the condition $a[d(r^k), r^k]^n \neq 0$ for some $r \in I$ cannot be omitted.

Example 2.4 Let R_1 be any ring not satisfying s_4 and $R_2 = \left\{\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right) \mid a, b \in F\right\}$, where F is a field. Set $R = R_1 \bigoplus R_2$, we define a map $d : R \to R$ by $d(r, s) = (0, t)$ with $t = \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right)$ for all $r \in R_1$ and $s = \left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right) \in R_2$. It is easy to check d is a nonzero derivation of R. Now let $I = \{0\} \times \left\{\left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right) \mid a \in F\right\}$ be a nonzero ideal of R. It is straightforward to check that d satisfies the property $a[d(x^k), x^k]^n = 0$ for all $x \in I$, however R does not satisfy s_4.

Now to prove our next theorem we need the following lemma.

Lemma 2.5 *Let n be a fixed positive integer, R be an $n!$-torsion free ring with center $Z(R)$.*

Suppose $y_1, y_2, \ldots, y_n \in R$ satisfy $\lambda y_1 + \lambda^2 y_2 + \cdots + \lambda^n y_n \in Z(R)$ for $\lambda = 1, 2, \ldots, n$. Then $y_i \in Z(R)$ for all i.

Proof The proof of this lemma is analogous to the proof of Lemma 1 in [18]. □

Now we prove our next theorem.

Theorem 2.6 *Let n (≥ 1), k (≥ 1) be fixed integers, R a noncommutative $2n(k-1)!$-torsion free prime ring with center $Z(R)$, $0 \neq I$ an ideal of R, $0 \neq a \in R$ and d a derivation of R. If $a[d(x^k), x^k]^n \in Z(R)$ for all $x \in I$, then either $d = 0$ or R satisfies s_4.*

Proof By [19], since I, R and Q satisfies the same differential identities, we have

$$a[d(x^k), x^k]^n \in C, \tag{2.13}$$

for all $x \in Q$. Since $1 \in Q$, we may replace x with $x + 1$. By this replacement, we obtain

$$a[d((x+1)^k), (x+1)^k]^n \in C, \tag{2.14}$$

for all $x \in Q$. We have the facts $(x+1)^k = x^k + \binom{k}{1}x^{k-1} + \binom{k}{2}x^{k-2} + \cdots + 1$ and $d(1) = 0$. Using these facts, (2.14) implies that

$$a\left[d(x^k) + \binom{k}{1}d(x^{k-1}) + \cdots + \binom{k}{k-1}d(x), x^k + \binom{k}{1}x^{k-1} + \cdots + \binom{k}{k-1}x\right]^n \in C, \tag{2.15}$$

that is

$$a\left\{[d(x^k), x^k] + \binom{k}{1}\left([d(x^k), x^{k-1}] + [d(x^{k-1}), x^k]\right) + \cdots + \binom{k}{k-1}\binom{k}{k-1}[d(x), x]\right\}^n \in C, \tag{2.16}$$

for all $x \in Q$. Now expanding the expression completely and then using (2.13), the above expression can be rewritten as

$$af_{2kn-1}(x) + af_{2kn-2}(x) + \cdots + af_{2n}(x) \in C, \tag{2.17}$$

where $f_n(x)$ denotes a suitable homogeneous function of degree n in x. Putting $x = \lambda x$, where $\lambda \in C$, in (2.17), we get

$$\lambda^{2n-1}\{\lambda^{2kn-2n}af_{2kn-1}(x) + \lambda^{2kn-2n-1}af_{2kn-2}(x) + \cdots + \lambda af_{2n}(x)\} \in C. \tag{2.18}$$

Since $\lambda \in C$ is invertible in C, above relation yields that

$$\lambda^{2kn-2n}af_{2kn-1}(x) + \lambda^{2kn-2n-1}af_{2kn-2}(x) + \cdots + \lambda af_{2n}(x) \in C. \tag{2.19}$$

Putting $\lambda = 1, 2, \ldots, 2kn - 2n$ and then using Lemma 2.5, we have $af_{2n}(x) \in C$ for all $x \in Q$, since R is $(2kn - 2n)!$-torsion free. Now, $af_{2n}(x) \in C$ is $a\{\binom{k}{k-1}\binom{k}{k-1}[d(x), x]\}^n \in C$ for all $x \in Q$ i.e., $ak^{2n}[d(x), x]^n \in C$ for all $x \in Q$. Since R is $2n(k-1)!$-torsion free, $a[d(x), x]^n \in C$ for all $x \in Q$. This implies that either $d = 0$ or R satisfies s_4 (see [8,9]). □

We conclude with an example in a prime ring R satisfying the differential identity in above theorem.

Example 2.7 Let $R = M_2(F)$ be a 2×2 matrix ring over a field F. Then for any $0 \neq a \in Z(R)$

and any derivation d of R, we have $a[d(x^k), x^k]^{2n} \in Z(R)$ for all $x \in R$, where k and n are any positive integers.

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