Journal of Mathematical Research with Applications
Sept., 2019, Vol. 39, No. 5, pp. 495–505
DOI:10.3770/j.issn:2095-2651.2019.05.006
Http://jmre.dlut.edu.cn

On Split Regular Hom-Poisson Color Algebras

Shuangjian GUO
School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guizhou 550025*, P. R. China*

Abstract We introduce the class of split regular Hom-Poisson color algebras as the natural generalization of split regular Hom-Poisson algebras and the one of split regular Hom-Lie color algebras. By developing techniques of connections of roots for this kind of algebras, we show that such a split regular Hom-Poisson color algebras A is of the form $A = U + \sum_\alpha I_\alpha$ with U a subspace of a maximal abelian subalgebra H and any I_α, a well described ideal of A, satisfying $[I_\alpha, I_\beta] + I_\alpha I_\beta = 0$ if $[\alpha] \neq [\beta]$. Under certain conditions, in the case of A being of maximal length, the simplicity of the algebra is characterized.

Keywords Hom-Lie color algebra; Hom-Poisson color algebra; root; structure theory

MR(2010) Subject Classification 17A30; 17B22

1. Introduction

As generalizations of Lie algebras, Hom-Lie algebras were introduced motivated by applications to physics and to deformations of Lie algebras, especially Lie algebras of vector fields. The notion of Hom-Lie algebras was firstly introduced by Hartwig, Larsson and Silvestrov to describe the structure of certain q-deformations of the Witt and the Virasoro algebras [1]. More precisely, a Hom-Lie algebras are different from Lie algebras as the Jacobi identity is replaced by a twisted form using a morphism.

The twisting of parts of the defining identities was transferred to other algebraic structures. Makhlouf and Silvestrov [2, 3] introduced the notions of Hom-associative algebras, Hom-coassociative coalgebras, Hom-bialgebras and Hom-Hopf algebras. The original definition of a Hom-bialgebra involved two linear maps, one twisting the associativity condition and the other one twisting the coassociativity condition. In the case of Hom-Lie algebras, the relevant structure for a tensor theory is a Hom-Poisson algebra structure. A Hom-Poisson algebra has simultaneously a Hom-Lie algebra structure and a Hom-associative algebra structure, satisfying the Hom-Leibniz identity in [4]. Yuan [5] introduced the notations of Hom-Lie color algebras and presented the methods to construct these color algebras, which can be viewed as an extension of Hom-Lie algebras to Γ-graded algebras, where Γ is any abelian group.

The class of the split algebras is specially related to addition quantum numbers, graded contractions and deformations. For instance, for a physical system which displays a symmetry, it

Received December 20, 2018; Accepted March 3, 2019
Supported by the National Natural Science Foundation of China (Grant No. 11761017) and the Youth Project for Natural Science Foundation of Guizhou Provincial Department of Education (Grant No. KY[2018]155).
E-mail address: shuangjianguo@126.com

is interesting to know in detail the structure of the split decomposition because its roots can be seen as certain eigenvalues which are the additive quantum numbers characterizing the state of such system. Determining the structure of split algebras will become more and more meaningful in the area of research in mathematical physics. Recently, the structure of different classes of split algebras have been determined by the techniques of connections of rootsin [6–15]. The purpose of this paper is to consider the class of split regular Hom-Poisson color algebras as the natural extension of one of split regular Hom-Lie color algebras.

In Section 2, we show that such an arbitrary split regular Hom-Poisson color algebras A is of the form $A = U + \sum_\alpha I_\alpha$ with U a subspace of a maximal abelian subalgebra H and any I_α, a well described ideal of A, satisfying $[I_\alpha, I_\beta] + I_\alpha I_\beta = 0$ if $[\alpha] \neq [\beta]$.

In Section 3, we show that under certain conditions, in the case of A being of maximal length, the simplicity of the algebra is characterized.

Throughout this paper, we will denote by $\mathbb{N}$ the set of all nonnegative integers and by $\mathbb{Z}$ the set of all integers. Split regular Hom-Poisson color algebras are considered of arbitrary dimension and over an arbitrary base field $\mathbb{K}$.

2. Decomposition

Definition 2.1 ([5]) *Let Γ be an abelian group. A bi-character on Γ is a map $\varepsilon : \Gamma \times \Gamma \to \mathbb{K}\backslash\{0\}$ satisfying*

(1) $\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = 1$*;*

(2) $\varepsilon(\alpha, \beta + \gamma) = \varepsilon(\alpha, \beta)\varepsilon(\alpha, \gamma)$*;*

(3) $\varepsilon(\alpha + \beta, \gamma) = \varepsilon(\alpha, \gamma)\varepsilon(\beta, \gamma)$*, for all any* $\alpha, \beta \in \Gamma$.

Definition 2.2 ([5]) *A Hom-Lie color algebra is a quadruple $(L, [\cdot, \cdot], \phi, \varepsilon)$ consisting of a Γ-graded space L, an even bilinear mapping $[\cdot, \cdot] : L \times L \to L$, a homomorphism $\phi : L \to L$ and a bi-character ε on Γ satisfying the following conditions,*

$$[x, y] = -\varepsilon(\overline{x}, \overline{y})[y, x],$$
$$\varepsilon(\overline{z}, \overline{x})[\phi(x), [y, z]] + \varepsilon(\overline{x}, \overline{y})[\phi(y), [z, x]] + \varepsilon(\overline{y}, \overline{z})[\phi(z), [x, y]] = 0,$$

for all homogeneous elements $x, y, z \in L$, $\overline{x}, \overline{y}, \overline{z}$ denote the homogeneous degree of x, y, z. When ϕ is an algebra automorphism it is said that L is a regular Hom-Lie color algebra.

Definition 2.3 *A Hom-Poisson color algebra is a Hom-Lie color algebra $(A, [\cdot, \cdot], \phi, \varepsilon)$ endowed with a Hom-associative color product, that is, a bilinear product denoted by juxtaposition such that*

$$\phi(x)(yz) = (xy)\phi(y),$$

for all $x, y, z \in A$, and such that the Hom-Leibniz color identity

$$[xy, \phi(z)] = \phi(x)[y, z] + \varepsilon(\overline{y}, \overline{z})[x, z]\phi(y)$$

holds for any $x, y, z \in A$, $\overline{y}, \overline{z}$ denote the homogeneous degree of y, z.

If ϕ is furthermore a Poisson automorphism, that is, a linear bijective on such that $\phi([x,y]) = [\phi(x), \phi(y)]$ and $\phi(xy) = \phi(x)\phi(y)$ for any $x, y \in A$, then A is called a regular Hom-Poisson color algebra.

Example 2.4 Let $(A, \mu, [\cdot,\cdot], \varepsilon)$ be a Poisson color algebra and $\phi : A \to A$ be a Possion color algebra automorphism, If we endow the underlying linear space A with new products $[\cdot,\cdot]', \mu'$ defined by $[x,y]' = \phi[x,y], \mu'(x \otimes y) = \phi \circ \mu(x \otimes y)$ for any $x, y \in A$, we have that $(A, \mu', [\cdot,\cdot]', \phi, \varepsilon)$ becomes a regular Hom-Poisson color algebra.

A subalgebra H of A is a graded subspace $H = \oplus_{g\in\Gamma} H_g$ such that $[H,H] + HH \subset A$ and $\phi(H) = H$. A graded subspace $I = \oplus_{g\in\Gamma} I_g$ of A is called an ideal if $[I,A] + IA + AI \subset I$ and $\phi(I) = I$. A Hom-Poisson color algebra A will be called simple if $[A,A] + AA \neq 0$ and its only ideals are $\{0\}$ and A.

We recall from [14] that a Hom-Lie color algebra $(A, [\cdot,\cdot], \phi, \varepsilon)$ and a maximal abelian subalgebra H of A, for a linear functional

$$\alpha : H_0 \to \mathbb{K},$$

we define the root space of A associated to α as the subspace

$$A_\alpha := \{v_\alpha \in A : [h_0, \phi(v_\alpha)] = \alpha(h)\phi(v_\alpha), \text{for any } h_0 \in H_0\}.$$

The elements $\alpha : H_0 \to \mathbb{K}$ satisfying $A_\alpha \neq 0$ are called roots of A with respect to H and we denote $\Lambda := \{\alpha \in (H_0)^*/\{0\} : A_\alpha \neq 0\}$. We say that A is a split regular Hom-Lie color algebra with respect to H if

$$A = H \oplus \bigoplus_{\alpha\in\Lambda} A_\alpha.$$

We also say that Λ is the root system of A.

To ease notation, the mappings $\phi|_H, \phi^{-1}|_H : H \to H$ will be denoted by ϕ and ϕ^{-1}.

We recall some properties of split regular Hom-Lie color algebras that can be found in [14].

Lemma 2.5 *Let $(A, [\cdot,\cdot], \phi, \varepsilon)$ be a split regular Hom-Lie color algebra. Then for any $\alpha, \beta \in \Lambda \cup \{0\}$,*

(1) $\phi(A_\alpha) = A_{\alpha\phi^{-1}}$, $\phi^{-1}(A_\alpha) = A_{\alpha\phi}$;

(2) $[A_\alpha, A_\beta] \subset A_{\alpha\phi^{-1}+\beta\phi^{-1}}$;

(3) If $\alpha \in \Lambda$, then $\alpha\phi^{-z} \in \Lambda$ for any $z \in \mathbb{Z}$;

(4) $A_0 = H$.

Lemma 2.6 *Let A be a split regular Hom-Poisson color algebra. Then for any $\alpha, \beta \in \Lambda \cup \{0\}$, we have $A_\alpha A_\beta \subset A_{\alpha\phi^{-1}+\beta\phi^{-1}}$.*

Proof Let $h_0 \in H_0, v_\alpha \in A_\alpha$ and $v_\beta \in A_\beta$. We can write

$$[h_0, v_\alpha v_\beta] = [\phi\phi^{-1}(h_0), v_\alpha v_\beta],$$

and denote $h_0' = \phi^{-1}(h_0)$. By applying the Hom-Leibniz color identity, we get

$$[\phi\phi^{-1}(h_{\overline{0}}), v_\alpha v_\beta] = [\phi(h'_{\overline{0}}), v_\alpha v_\beta]$$

$$\begin{aligned}&= -\varepsilon(\overline{h_0}, \overline{v_\beta})[v_\alpha, h']\phi(v_\beta) - \phi(v_\alpha)[v_\beta, h']\\&= [h', v_\alpha]\phi(v_\beta) + \phi(v_\alpha)[h', v_\beta]\\&= [\phi^{-1}(h), v_\alpha]\phi(v_\beta) + \phi(v_\alpha)[\phi^{-1}(h), v_\beta]\\&= \alpha\phi^{-1}(h)\phi(v_\alpha)\phi(v_\beta) + \beta\phi^{-1}(h)\phi(v_\alpha)\phi(v_\beta)\\&= (\alpha\phi^{-1} + \beta\phi^{-1})(h)\phi(v_\alpha)\phi(v_\beta).\end{aligned}$$

That is $A_\alpha A_\beta \subset A_{\alpha\phi^{-1}+\beta\phi^{-1}}$. $\square$

In the following, A denotes a split regular Hom-Poisson color algebra and

$$A = H \oplus (\bigoplus_{\alpha\in\Lambda} A_\alpha)$$

the corresponding root spaces decomposition. Given a linear functional $\alpha : H_0 \to \mathbb{K}$, we denote by $-\alpha : H_0 \to \mathbb{K}$ the element in H_0^* defined by $(-\alpha)(h_0) := -\alpha(h_0)$. We also denote by

$$-\Lambda := \{-\alpha : \alpha \in \Lambda\} \quad \text{and} \quad \pm\Lambda : \Lambda \cup (-\Lambda).$$

Example 2.7 Let $A = H \oplus (\bigoplus_{\alpha\in\Gamma} A_\alpha)$ be a split Possion color algebra, $\phi : A \to A$ an automorphism such that $\phi(H) = H$. By Example 2.4, we know that $(A, \mu', [\cdot,\cdot]', \phi, \varepsilon)$ is a regular Hom-Possion color algebra. Then we have

$$A = H \oplus (\bigoplus_{\alpha\in\Gamma} A_{\alpha\phi^{-1}})$$

makes of the regular Hom-Possion color algebra $(A, \mu', [\cdot,\cdot]', \phi, \varepsilon)$ being the roots system $\Lambda = \{\alpha\phi^{-1} : \alpha \in \Gamma\}$.

Definition 2.8 *Let $\alpha, \beta \in \Lambda$. We will say that α is connected to β if either*

$$\beta = \epsilon\alpha\phi^z \quad \text{for some } z \in \mathbb{Z} \text{ and } \epsilon \in \{-1, 1\}$$

or there exists $\{\alpha_1, \cdots, \alpha_k\} \subset \pm\Lambda$ with $k \geq 2$, such that

(1) $\alpha_1 \in \{\alpha\phi^{-n} : n, r \in \mathbb{N}\}$.

(2) $\alpha_1\phi^{-1} + \alpha_2\phi^{-1} \in \pm\Lambda$, $\alpha_1\phi^{-2} + \alpha_2\phi^{-2} + \alpha_3\phi^{-1} \in \pm\Lambda$, $\alpha_1\phi^{-3} + \alpha_2\phi^{-3} + \alpha_3\phi^{-2} + \alpha_4\phi^{-1} \in \pm\Lambda$, $\ldots$, $\alpha_1\phi^{-i} + \alpha_2\phi^{-i} + \alpha_3\phi^{-i+1} + \cdots + \alpha_i\phi^{-2} + \alpha_i \in \pm\Lambda$, $\alpha_1\phi^{-k+2} + \alpha_2\phi^{-k+2} + \alpha_3\phi^{-k+3} + \cdots + \alpha_{k-2}\phi^{-2} + \alpha_{k-1}\phi^{-1} \in \pm\Lambda$.

(3) $\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \alpha_3\phi^{-k+2} + \cdots + \alpha_i\phi^{-k+i-1} + \cdots + \alpha_{k-1}\phi^{-2} + \alpha_k\phi^{-1} \in \{\pm\beta\phi^{-m} : m \in \mathbb{N}\}$.

We will also say that $\{\alpha_1, \ldots, \alpha_k\}$ is a connection from α to β.

The proof of the next result is analogous to the one of [14] . For the sake of completeness, we give a sketch of the proof. $\square$

Proposition 2.9 *The relation $\sim$ in Λ, defined by $\alpha \sim \beta$ if and only if α is connected to β, is an equivalence relation.*

Proof If $\alpha \sim \beta$, then either $\beta = \epsilon\alpha\phi^z$ for some $z \in \mathbb{Z}$ and $\epsilon \in \{-1, 1\}$, and so β is connected to

α; or there exists $\{\alpha_1, \dots, \alpha_k\} \subset \pm\Lambda$ with $k \geq 2$, from α to β with

$$\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \alpha_3\phi^{-k+2} + \cdots + \alpha_i\phi^{-k+i-1} + \cdots + \alpha_k\phi^{-1} = \epsilon\beta\phi^{-m},$$

for some $m, s \in \mathbb{N}$, $\epsilon \in \{-1, 1\}$. Then we can verify that

$$\{\beta\phi^{-m}, -\epsilon\alpha_k\phi^{-1}, -\epsilon\alpha_{k-1}\phi^{-3}, -\epsilon\alpha_{k-2}\phi^{-5}, \dots, -\epsilon\alpha_2\phi^{-2k+3}\}$$

is a connection from β to α and the relation $\sim$ is symmetric.

Finally, suppose $\alpha \sim \beta$ and $\beta \sim \gamma$. If $\beta = \epsilon\alpha\phi^z$ for some $z \in \mathbb{Z}$, $\epsilon \in \{-1, 1\}$ and $\gamma = \epsilon'\alpha\phi^{z'}$ for some $z' \in \mathbb{Z}$, $\epsilon' \in \{-1, 1\}$, it is clear that $\alpha \sim \gamma$. Hence suppose $\{\alpha_1, \dots, \alpha_k\}$ with $k \geq 2$ is a connection from α to β which satisfies

$$\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \cdots + \alpha_k\phi^{-1} = \epsilon\beta\phi^{-m}$$

for some $m \in \mathbb{N}$, $\epsilon \in \{-1, 1\}$, and $\{h_1, \dots, h_p\}$ is a connection from β to γ. Then $\{\alpha_1, \dots, \alpha_k, \epsilon h_2, \dots, \epsilon h_p\}$ is connection from α to γ, so the connection relation is also transitive. □

By Proposition 2.9 we can consider the quotient set

$$\Lambda/\sim = \{[\alpha] : \alpha \in \Lambda\},$$

with $[\alpha]$ being the set of nonzero roots which are connected to α. Our next goal is to associate an ideal $I_{[\alpha]}$ to $[\alpha]$. Fix $[\alpha] \in \Lambda/\sim$, we start by defining

$$I_{H,[\alpha]} = \operatorname{span}_{\mathbb{K}}\{[A_{\beta\phi^{-1}}, A_{-\beta\phi^{-1}}] + A_{\beta\phi^{-1}}A_{-\beta\phi^{-1}} : \beta \in [\alpha]\}.$$

Now we define

$$V_{[\alpha]} := \bigoplus_{\beta \in [\alpha]} A_\beta.$$

Finally, we denote by $I_{[\alpha]}$ the direct sum of the two subspaces above:

$$I_{[\alpha]} := I_{H,[\alpha]} \oplus V_{[\alpha]}.$$

Proposition 2.10 *For any $[\alpha] \in \Lambda/\sim$, the following assertions hold.*

(1) $[I_{[\alpha]}, I_{[\alpha]}] + I_{[\alpha]}I_{[\alpha]} \subset I_{[\alpha]}$;

(2) $\phi(I_{[\alpha]}) = I_{[\alpha]}$;

(3) For any $[\beta] \neq [\alpha]$, we have $[I_{[\alpha]}, I_{[\beta]}] + I_{[\alpha]}I_{[\beta]} = 0$.

Proof (1) First we check that $[I_{[\alpha]}, I_{[\alpha]}] \subset I_{[\alpha]}$, we can write

$$\begin{aligned}[I_{[\alpha]}, I_{[\alpha]}] &= [I_{H,[\alpha]} \oplus V_{[\alpha]}, I_{H,[\alpha]} \oplus V_{[\alpha]}] \\ &\subset [I_{H,[\alpha]}, V_{[\alpha]}] + [V_{[\alpha]}, I_{H,[\alpha]}] + [V_{[\alpha]}, V_{[\alpha]}]. \end{aligned} \tag{2.1}$$

Given $\beta \in [\alpha]$, we have $[I_{H,[\alpha]}] \subset A_{\beta\phi^{-1}}$. Since $\beta\phi^{-1} \in [\alpha]$, we have $[I_{H,[\alpha]}, A_\beta] \subset V_{[\alpha]}$. In a similar way we get $[A_\beta, I_{H,\alpha}] \subset V_{[\alpha]}$. Next we consider $[V_{[\alpha]}, V_{[\alpha]}]$. If we take $\beta, \gamma \in [\alpha]$ such that $[A_\beta, A_\gamma] \neq 0$, then $[A_\beta, A_\gamma] \subset A_{\beta\phi^{-1}+\gamma\phi^{-1}}$. If $\beta\phi^{-1} + \gamma\phi^{-1} = 0$ we have $[A_\beta, A_{-\gamma}] \subset H$ and so $[A_\beta, A_{-\gamma}] \subset I_{H,[\alpha]}$. Suppose that $\beta\phi^{-1} + \gamma\phi^{-1} \in \Lambda$. We have that $\{\beta, \gamma\}$ is connection from β to $\beta\phi^{-1} + \gamma\phi^{-1}$. The transitivity of $\sim$ gives now that $\beta\phi^{-1} + \gamma\phi^{-1} \in [\alpha]$ and so $[A_\beta, A_\gamma] \subset V_{[\alpha]}$.

Hence

$$[V_{[\alpha]}, V_{[\alpha]}] \in I_{[\alpha]}. \tag{2.2}$$

From (2.1) and (2.2), we get $[I_{[\alpha]}, I_{[\alpha]}] \subset I_{[\alpha]}$.

Second, we will check that $I_{[\alpha]} I_{[\alpha]} \subset I_{[\alpha]}$. We have

$$\begin{aligned} I_{[\alpha]} I_{[\alpha]} &= (I_{H,[\alpha]} \oplus V_{[\alpha]})(I_{H,[\alpha]} \oplus V_{[\alpha]}) \\ &\subset I_{H,[\alpha]} I_{H,[\alpha]} + I_{H,[\alpha]} V_{[\alpha]} + V_{[\alpha]} I_{H,[\alpha]} + V_{[\alpha]} V_{[\alpha]}. \end{aligned} \tag{2.3}$$

By arguing as above, we have

$$I_{H,[\alpha]} V_{[\alpha]} + V_{[\alpha]} I_{H,[\alpha]} + V_{[\alpha]} V_{[\alpha]} \subset I_{H,[\alpha]}.$$

Hence, it just remains to check that $I_{H,[\alpha]} I_{H,[\alpha]}$, observe that

$$\begin{aligned} I_{H,[\alpha]} I_{H,[\alpha]} &\subset \Big(\sum_{\beta \in [\alpha]} [A_{\beta\phi^{-1}}, A_{-\beta\phi^{-1}}] + A_{\beta\phi^{-1}} A_{-\beta\phi^{-1}} \Big) H \\ &\subset \Big(\sum_{\beta \in [\alpha]} [A_{\beta\phi^{-1}}, A_{-\beta\phi^{-1}}] \Big) H + \Big(\sum_{\beta \in [\alpha]} A_{\beta\phi^{-1}} A_{-\beta\phi^{-1}} \Big) H. \end{aligned} \tag{2.4}$$

Consider the first summand on the right hand side of (2.4). By Hom-Leibniz color identity, we have

$$\begin{aligned} &[A_{\beta\phi^{-1}}, A_{-\beta\phi^{-1}}]\phi\phi^{-1}(H) \\ &\quad \subset [A_{-\beta\phi^{-1}}\phi^{-1}(H), \phi(A_{-\beta\phi^{-1}})] + \phi(A_{\beta\phi^{-1}})[\phi^{-1}(H), \phi\phi^{-1}(A_{-\beta\phi^{-1}})] \\ &\quad \subset [A_{-\beta\phi^{-2}}, A_{-\beta\phi^{-2}}] + A_{\beta\phi^{-2}} A_{-\beta\phi^{-2}} \subset I_{H,[\alpha]}. \end{aligned}$$

Next we consider the last summand on the right hand side of (2.4). By Hom-associativity, we have

$$\begin{aligned} (A_{\beta\phi^{-1}} A_{-\beta\phi^{-1}})\phi(\phi^{-1}(H)) &= \phi(A_{\beta\phi^{-1}})(A_{-\beta\phi^{-1}}\phi^{-1}(H)) \\ &\subset A_{\beta\phi^{-1}\phi^{-1}} A_{-\beta\phi^{-1}\phi^{-1}} \subset I_{H,[\alpha]}. \end{aligned}$$

(2) It is easy to check that $\phi(I_{[\alpha]}) = I_{[\alpha]}$.

(3) We will study the expression $[I_{[\alpha]}, I_{[\beta]}] + I_{[\alpha]} I_{[\beta]}$. Observe that

$$\begin{aligned} [I_{[\alpha]}, I_{[\beta]}] &= [I_{H,[\alpha]} \oplus V_{[\alpha]}, I_{H,[\beta]} \oplus V_{[\beta]}] \\ &\subset [I_{H,[\alpha]}, V_{[\beta]}] + [V_{[\alpha]}, I_{H,[\beta]}] + [V_{[\alpha]}, V_{[\beta]}], \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} I_{[\alpha]} I_{[\beta]} &= (I_{H,[\alpha]} \oplus V_{[\alpha]})(I_{H,[\beta]} \oplus V_{[\beta]}) \\ &\subset I_{H,[\alpha]} I_{H,[\beta]} + I_{H,[\alpha]} V_{[\beta]} + V_{[\alpha]} I_{H,[\beta]} + V_{[\alpha]} V_{[\beta]}. \end{aligned} \tag{2.6}$$

First we consider $[V_{[\alpha]}, V_{[\beta]}] + V_{[\alpha]} V_{[\beta]}$ and suppose there exist $\alpha_1 \in [\alpha]$ and $\beta_1 \in [\beta]$ such that $[A_{\alpha_1}, A_{\beta_1}] + A_{\alpha_1} A_{\beta_1} \neq 0$. As necessarily $\alpha_1\phi^{-1} \neq -\beta_1\phi^{-1}$, then $\alpha_1\phi^{-1} + \beta_1\phi^{-1} \in \Lambda$. So $\{\alpha_1, \beta_1, -\alpha_1\phi^{-1}\}$ is a connection between α_1 and β_1. By the transitivity of the connection relation we have $\alpha \in [\beta]$, a contradicition. Hence $[A_{\alpha_1}, A_{\beta_1}] + A_{\alpha_1} A_{\beta_1} = 0$ and so

$$[V_{[\alpha]}, V_{[\beta]}] + V_{[\alpha]} V_{[\beta]} = 0.$$

Next we consider the first summand $[I_{H,[\alpha]}, V_{[\beta]}]$ on the right hand side of (2.5) and the second one $I_{H,[\alpha]}V_{[\beta]}$ of (2.6), and suppose there exist $\alpha_1 \in [\alpha]$ and $\beta_1 \in [\beta]$ and such that

$$[[A_{\alpha_1}, A_{-\alpha_1}], A_{\beta_1}] + [A_{\alpha_1}A_{-\alpha_1}, A_{\beta_1}] + [A_{\alpha_1}, A_{-\alpha_1}]A_{\beta_1} + (A_{\alpha_1}A_{-\alpha_1})A_{\beta_1} \neq 0.$$

Then some of the four summands are different from zero.

If $[[A_{\alpha_1}, A_{-\alpha_1}], A_{\beta_1}] \neq 0$, then Hom-Leibniz identity gives

$$\begin{aligned} 0 &\neq [[A_{\alpha_1}, A_{-\alpha_1}], \phi\phi^{-1}(A_{\beta_1})] \\ &\subset [[A_{\alpha_1}, \phi^{-1}(A_{\beta_1})], \phi(A_{-\alpha_1})] + [\phi(A_{\alpha_1}), [A_{-\alpha_1}, \phi^{-1}(A_{\beta_1})]] \\ &\subset [[A_{\alpha_1}, \phi^{-1}(A_{\beta_1})], \phi(A_{-\alpha_1})] + [[A_{-\alpha_1}, \phi^{-1}(A_{\beta_1})], \phi(A_{\alpha_1})]. \end{aligned}$$

Hence

$$[A_{\alpha_1}, \phi^{-1}(A_{\beta_1})] + [A_{-\alpha_1}, \phi^{-1}(A_{\beta_1})] \neq 0$$

which contradicts (2.6). Hence, $[[A_{\alpha_1}, A_{-\alpha_1}], A_{\beta_1}] = 0$.

If the second, third or fourth summand were nonzero, we can argue as above but using the Hom-Leibniz or Hom-associativity color identities to show that these products are zero. Consequently,

$$[I_{H,[\alpha]}, V_{[\beta]}] + I_{H,[\alpha]}V_{[\beta]} = 0.$$

In a similar way we prove that the remaining summands in (2.5) and (2.6) are zero, and the proof is completed. □

Proposition 2.11 *For any $[\alpha] \in \Lambda/\sim$, we have $I_{H,[\alpha]}H + HI_{H[\alpha]} \in I_{H[\alpha]}$.*

Proof Fix any $\beta \in [\alpha]$. On the one hand, by the Hom-Leibniz color identity, we get

$$[A_\beta, A_{-\beta}]H + H[A_\beta, A_{-\beta}] \in I_{H[\alpha]}.$$

On the other hand, by Hom-associativity

$$(A_\beta A_{-\beta})H + H(A_\beta A_{-\beta}) \in I_{H[\alpha]}. \quad \square$$

Theorem 2.12 *(1) For any $[\alpha] \in \Lambda/\sim$, the linear space $I_{[\alpha]} = I_{H,[\alpha]} + V_{[\alpha]}$ of A associated to $[\alpha]$ is an ideal of A.*

(2) If A is simple, then there exists a connection from α to β for any $\alpha, \beta \in \Lambda$ and $H = \sum_{\alpha\in\Lambda}([A_{\alpha\phi^{-1}}, A_{-\alpha\phi^{-1}}] + A_{\alpha\phi^{-1}}A_{-\alpha\phi^{-1}})$.

Proof (1) Since $[I_{[\alpha]}, H] \subset I_{[\alpha]}$, by Proposition 2.10 we have

$$[I_{[\alpha]}, A] = \Big[I_{[\alpha]}, H \oplus \Big(\bigoplus_{\beta\in[\alpha]} A_\beta\Big) \oplus \Big(\bigoplus_{\gamma\in[\alpha]} A_\gamma\Big)\Big] \subset I_{[\alpha]}.$$

By Propositions 2.10 and 2.11, we have

$$\begin{aligned} I_{[\alpha]}A + AI_{[\alpha]} =& I_{[\alpha]}\Big(H \oplus \Big(\bigoplus_{\beta\in[\alpha]} A_\beta\Big) \oplus \Big(\bigoplus_{\gamma\notin[\alpha]} A_\gamma\Big)\Big) + \\ & \Big(H \oplus \Big(\bigoplus_{\beta\in[\alpha]} A_\beta) \oplus \Big(\bigoplus_{\gamma\notin[\alpha]} A_\gamma\Big)\Big)I_{[\alpha]} \subset I_{[\alpha]}. \end{aligned}$$

As we also have $\phi(I_{[\alpha]}) = I_{[\alpha]}$. So we conclude $I_{[\alpha]}$ is an ideal of A.

(2) The simplicity of A implies $I_{[\alpha]} = A$. From here, it is clear that $[\alpha] = \Lambda$ and $H = \sum_{\alpha\in\Lambda}([A_{\alpha\phi^{-1}}, A_{-\alpha\phi^{-1}}] + A_{\alpha\phi^{-1}}A_{-\alpha\phi^{-1}})$. □

Theorem 2.13 *We have*

$$A = U + \sum_{[\alpha]\in\Lambda/\sim} I_{[\alpha]},$$

where U is a linear complement in H of $\mathrm{span}_{\mathbb{K}}\{[A_{\alpha\phi^{-1}}, A_{-\alpha\phi^{-1}}] + A_{\alpha\phi^{-1}}A_{-\alpha\phi^{-1}} : \alpha \in \Lambda\}$ *and any $I_{[\alpha]}$ is one of the ideals of A described in Theorem 2.12, satisfying $[I_{[\alpha]}, I_{[\beta]}] + I_{[\alpha]}I_{[\beta]} = 0$ if $[\alpha] \neq [\beta]$.*

Proof $I_{[\alpha]}$ is well defined and an ideal of A, being clear that

$$A = H \oplus \sum_{[\alpha]\in\Lambda} A_{[\alpha]} = U + \sum_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}.$$

Finally, Proposition gives us $[I_{[\alpha]}, I_{[\beta]}] + I_{[\alpha]}I_{[\beta]} = 0$ if $[\alpha] \neq [\beta]$. □

Let us denote by $Z(A) := \{v \in A : [v, A] + vA + Av = 0\}$ the center of A.

Corollary 2.14 *If $Z(A) = 0$ and $H = \sum_{\alpha\in\Lambda}([A_{\alpha\phi^{-1}}, A_{-\alpha\phi^{-1}}] + A_{\alpha\phi^{-1}}A_{-\alpha\phi^{-1}})$. Then A is the direct sum of the ideals given in Theorem 2.12,*

$$A = \bigoplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]},$$

Furthermore $[I_{[\alpha]}, I_{[\beta]}] + I_{[\alpha]}I_{[\beta]} = 0$ if $[\alpha] \neq [\beta]$.

Proof Since $H = \sum_{\alpha\in\Lambda}([A_{\alpha\phi^{-1}}, A_{-\alpha\phi^{-1}}] + A_{\alpha\phi^{-1}}A_{-\alpha\phi^{-1}})$, we get $A = \sum_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$. Finally, to verify the direct character of the sum, take some $v \in I_{[\alpha]} \cap (\sum_{[\beta]\in\Lambda/\sim,[\beta]\neq[\alpha]} I_{[\beta]})$. Since $v \in I_{[\alpha]}$, the fact $[I_{[\alpha]}, I_{[\beta]}] + I_{[\alpha]}I_{[\beta]} = 0$ when $[\alpha] \neq [\beta]$ gives us

$$[v, \sum_{[\beta]\in\Lambda/\sim,[\beta]\neq[\alpha]} I_{[\beta]}] + v\Big(\sum_{[\beta]\in\Lambda/\sim,[\beta]\neq[\alpha]} I_{[\beta]}\Big) + \Big(\sum_{[\beta]\in\Lambda/\sim,[\beta]\neq[\alpha]} I_{[\beta]}\Big)v = 0.$$

In a similar way, since $v \in \sum_{[\beta]\in\Lambda/\sim,[\beta]\neq[\alpha]} I_{[\beta]}$, we have $[v, I_{[\alpha]}] + vI_{[\alpha]} + I_{[\alpha]}v = 0$. That is $v \in Z(A)$ and so $v = 0$. □

3. The simple components

In this section we focus on the simplicity of split regular Hom-Poisson color algebras by centering our attention in those of maximal length, we recall that a roots system Λ of a split regular Hom-Poisson color algebra A is called symmetric if it satisfies that $\alpha \in \Lambda$ implies $-\alpha \in \Lambda$. From now on we will suppose Λ is symmetric.

Observe the grading of I, we have

$$I = \oplus_{g\in\Gamma} I_g = \bigoplus_{g\in\Gamma}(I_g \cap H_g) \oplus \Big(\bigoplus_{\alpha\in\Lambda}(I_g \cap A_{\alpha,g})\Big).$$

Lemma 3.1 *Suppose $H = \sum_{\alpha\in\Lambda}([A_{\alpha\phi^{-1}}, A_{-\alpha\phi^{-1}}] + A_{\alpha\phi^{-1}}A_{-\alpha\phi^{-1}})$. If I is an ideal of A such* *that $I \subset H$, then $I \subset Z(A)$.*

Proof Observe that $[I,H] \subset [H,H] = 0$ and

$$\Big[I, \bigoplus_{\alpha\in\Lambda} A_\alpha\Big] + I\Big(\bigoplus_{\alpha\in\Lambda} A_\alpha\Big) + \Big(\bigoplus_{\alpha\in\Lambda} A_\alpha\Big)I \subset I \cap \Big(\bigoplus_{\alpha\in\Lambda} A_\alpha\Big) \subset H \cap \Big(\bigoplus_{\alpha\in\Lambda} A_\alpha\Big) = 0.$$

Since $H = \sum_{\alpha\in\Lambda}([A_{\alpha\phi^{-1}}, A_{-\alpha\phi^{-1}}] + A_{\alpha\phi^{-1}}A_{-\alpha\phi^{-1}})$, by the Hom-Leibniz color identity and the above observation, that $HI + IH = 0$. So $I \subset Z(A)$.

In a similar way, we use the notions of [14], for each $g \in \Gamma$, we denote $\Lambda_g := \{\alpha \in \Lambda : L_{\alpha,g} \neq 0\}$.

Definition 3.2 *A split regular Hom-Poisson color algebra A is root multiplicative if $\alpha \in \Lambda_{g_i}, \beta \in \Lambda_{g_j}$ with $g_i, g_j \in \Gamma$ such that $\alpha + \beta \in \Lambda$, then $[A_{\alpha,g_i}, A_{\beta,g_j}] + A_{\alpha,g_i}A_{\beta,g_j} \neq 0$.*

Definition 3.3 *A split regular Hom-Poisson color algebra A is of maximal length if $\dim A_{\kappa\alpha,\kappa g} = 1$ for any $\alpha \in \Lambda_g, \kappa \in \{-1,1\}$ and $g \in \Gamma$.*

Observe that if A is of maximal length, then we have

$$I = \oplus_{g\in\Gamma}((I_g \cap H_g) \oplus (\oplus_{\alpha\in\Lambda'}(I_g \cap A_{\alpha,g})), \tag{3.1}$$

where $\Lambda'_g = \{\alpha \in \Lambda : I_g \cap A_{\alpha,g} \neq 0\}$.

Theorem 3.4 *Let A be a split regular Hom-Poisson color algebra of maximal length and root multiplicative. Then A is simple if and only if $Z(A) = 0$, $H = \sum_{\alpha\in\Lambda}([A_{\alpha\phi^{-1}}, A_{-\alpha\phi^{-1}}] + A_{\alpha\phi^{-1}}A_{-\alpha\phi^{-1}})$ and Λ has all of its elements connected.*

Proof Suppose A is simple. Since $Z(A)$ is an ideal of A, we have $Z(A) = 0$. Now Theorem 2.12(2) completes the proof of the direct implication. To prove the converse, consider a nozero ideal of A. By (3.1), we can write $I = \oplus_{g\in\Gamma}((I_g \cap H_g) \oplus (\oplus_{\alpha\in\Lambda'}(I_g \cap A_{\alpha,g}))$, where $\Lambda'_g \subset \Lambda_g$, and some $\Lambda'_g \neq \emptyset$ as consequence of Lemma 3.1. Let us fix some $\alpha_0 \in \Lambda'_{\bar{i}}$ with $0 \neq A_{\alpha_0,g} \subset I$. Since $\phi(I) = I$, we can obtain that

$$\text{if } \alpha \in \Lambda', \text{ then } \ \{\alpha\phi^z : z \in \mathbb{Z}\} \subset \Lambda'. \tag{3.2}$$

In particular

$$\{A_{\alpha_0\phi^z,g} : z \in \mathbb{Z}\} \subset I. \tag{3.3}$$

Now, let us take any $\beta \in \Lambda$ satisfying $\beta \notin \{\alpha_0\phi^z : z \in \mathbb{Z}\}$. Since α_0 and β are connected, we have a connection $\{\alpha_1, \dots, \alpha_k\}, k \geq 2$, from α_0 to β satisfying:

$$\begin{aligned}
&\alpha_1 = \alpha_0\phi^{-n} : \quad n \in \mathbb{N},\\
&\alpha_1\phi^{-1} + \alpha_2\phi^{-1} \in \Lambda,\\
&\alpha_1\phi^{-2} + \alpha_2\phi^{-2} + \alpha_3\phi^{-1} \in \Lambda,\\
&\alpha_1\phi^{-3} + \alpha_2\phi^{-3} + \alpha_3\phi^{-2} + \alpha_4\phi^{-1} \in \Lambda,\\
&\cdots\\
&\alpha_1\phi^{-i} + \alpha_2\phi^{-i} + \alpha_3\phi^{-i+1} + \cdots + \alpha_i\phi^{-2} + \alpha_{i+1}\phi^{-1} \in \Lambda,
\end{aligned}$$

$$\alpha_1\phi^{-k+2}+\alpha_2\phi^{-k+2}+\alpha_3\phi^{-k+3}+\cdots+\alpha_{k-2}\phi^{-2}+\alpha_{k-1}\phi^{-1}\in\Lambda$$
$$\alpha_1\phi^{-k+1}+\alpha_2\phi^{-k+1}+\alpha_3\phi^{-k+2}+\cdots+\alpha_i\phi^{-k+i-1}+\cdots+$$
$$\alpha_{k-1}\phi^{-2}+\alpha_k\phi^{-1}=\epsilon\beta\phi^{-m}:m\in\mathbb{N}.$$

Taking into account that $\alpha_1,\alpha_2\in\Lambda$, there exists $g_1\in\Gamma$ such that $A_{\alpha_2,g_1}\neq 0$. From here, the root multiplicativity and maximal length of A allow us to assert that either $0\neq[A_{\alpha_1,g},A_{\alpha_2,g_1}]=A_{\alpha_1\phi^{-1}+\alpha_2\phi^{-1},g+g_1}$ or $0\neq A_{\alpha_1,g}A_{\alpha_2,g_1}+A_{\alpha_2,g_1}A_{\alpha_1,g}=A_{\alpha_1\phi^{-1}+\alpha_2\phi^{-1},g+g_1}$.

Since $0\neq A_{\alpha_1,g}\subset I$, as a consequence of (3.3) we get

$$0\neq A_{\alpha_1\phi^{-1}+\alpha_2\phi^{-1},g+g_1}\subset I.$$

A similar argument applied to $\alpha_1\phi^{-1}+\alpha_2\phi^{-1},\alpha_3$, and

$$(\alpha_1\phi^{-1}+\alpha_2\phi^{-1})\phi^{-1}+\alpha_3\phi^{-1}=\alpha_1\phi^{-2}+\alpha_2\phi^{-2}+\alpha_3\phi^{-1}$$

gives us $0\neq A_{\alpha_1\phi^{-2}+\alpha_2\phi^{-2}+\alpha_3\phi^{-1},g_2}\subset I$ with $g_2\in\Gamma$. We can follow this process with the connection $\{\alpha_1,\ldots,\alpha_k\}$ to get

$$0\neq A_{\alpha_1\phi^{-k+1}+\alpha_2\phi^{-k+1}+\cdots+\alpha_k\phi^{-1},g_3}\subset I$$

and then

$$\text{either}\ \ A_{\beta\phi^{-m},g_3}\subset I\ \ \text{or}\ \ A_{-\beta\phi^{-m},g_3}\subset I.$$

From (3.2) and (3.3), we have

$$\text{either}\ \ A_{\alpha\phi^{-z},g_3}\subset I\ \ \text{or}\ \ A_{-\alpha\phi^{-z},g_3}\subset I.$$

This can be reformulated by saying that for any $\alpha\in\Lambda$, either $\{\alpha\phi^{-z}\}$ or $\{-\alpha\phi^{-z}\}$ is contained in Λ_I. Taking now into account $H=\sum_{\alpha\in\Lambda}([A_{\alpha\phi^{-1}},A_{-\alpha\phi^{-1}}]+A_{\alpha\phi^{-1}}A_{-\alpha\phi^{-1}})$, we have

$$H\subset I. \tag{3.4}$$

Now for any $\alpha\in\Lambda$, since $A_\alpha=[H,A_{\alpha\phi}]$ by the maximal length of A, (3.4) gives us $A_\alpha\subset I$ and so $A=I$. That is, A is simple. □

Theorem 3.5 *Let A be a split regular Hom-Poisson color algebra of maximal length and root multiplicative with $Z(A)=0$ satisfying $H=\sum_{\alpha\in\Lambda}([A_{\alpha\phi^{-1}},A_{-\alpha\phi^{-1}}]+A_{\alpha\phi^{-1}}A_{-\alpha\phi^{-1}})$. Then*

$$A=\bigoplus_{[\alpha]\in\Lambda/\sim}I_{[\alpha]},$$

where any $I_{[\alpha]}$ is a simple split ideal having its roots system $\Lambda_{I_{[\alpha]}}$, with all of its elements $\Lambda_{I_{[\alpha]}}$-connected.

Proof By Corollary 2.14, we can write A as the direct sum $\bigoplus_{[\alpha]\in\Lambda/\sim}I_{[\alpha]}$ of the family of ideals

$$I_{[\alpha]}=I_{H,[\alpha]}\oplus V_{[\alpha]}=\mathrm{span}_{\mathbb{K}}\{[A_{\beta\phi^{-1}},A_{-\beta\phi^{-1}}]+A_{\beta\phi^{-1}}A_{-\beta\phi^{-1}}:\beta\in[\alpha]\}\oplus_{\beta\in[\alpha]}A_\beta,$$

where each $I_{[\alpha]}$ is a split regular Hom-Poisson color algebra with root system $A_{I_{[\alpha]}}=[\alpha]$. To make use of Theorem 3.4 in each $I_{[\alpha]}$, we observe that the root multiplicativity of A and Proposition 2.11 show that $A_{I_{[\alpha]}}$ has all of its elements $A_{I_{[\alpha]}}$ connected, that is, connected through connections contained in $A_{I_{[\alpha]}}$. Moreover, each $I_{[\alpha]}$ is root multiplicative by the root multiplicativity of A. So we obtain $I_{[\alpha]}$ is of maximal length, and finally its center $Z(I_{[\alpha]}) = 0$ as consequence $[I_\alpha, I_\beta] + I_\alpha I_\beta = 0$ if $[\alpha] \neq [\beta]$. Applying Theorem 3.4, we have that $I_{[\alpha]}$ is simple and $A = \bigoplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$. $\square$

Acknowledgements The author is grateful to the referee for carefully reading the manuscript and for many valuable comments which largely improved the article.

References

[1] J. T. HARTWIG, D. LARSSON, S. D. SILVESTROV. *Deformations of Lie algebras using σ-derivations*. J. Algebra, 2016, **295**(2): 314–361.

[2] A. MAKHLOUF, S. SILVESTROV. *Hom-algebra structures*. J. Gen. Lie Theory Appl., 2008, **2**(2): 51–64.

[3] A. MAKHLOUF, S. SILVESTROV. *Hom-algebras and Hom-coalgebras*. J. Algebra Appl., 2010, **9**(4): 553–589.

[4] A. MAKHLOUF, S. SILVESTROV. *Notes on formal deformations of Hom-associative and Hom-Lie algebras*. Forum Math., 2010, **22**(4): 715–739.

[5] Chunyue WANG, Qingcheng ZHANG, Zhu WEI. *Hom-Leibniz superalgebras and Hom-Leibniz Poisson superalgebras*. Hacet. J. Math. Stat., 2015, **44**(5): 1163–1179.

[6] H. ALBUQUERQUE, E. BARREIRO, A. J. CALDERÓN, et al. *On split regular Hom-Lie superalgebras*. J. Geom. Phys., 2018, **128**: 1–11.

[7] M. J. ARAGÓN, A. J. CALDERÓN. *Split regular Hom-Lie algebras*. J. Lie Theory, 2015, **25**(3): 875–888.

[8] M. J. ARAGÓN, A. J. CALDERÓN. *The structure of split regular Hom-Poisson algebras*. Colloq. Math., 2016, **145**(1): 1–13.

[9] A. J. CALDERÓN. *On split Lie algebras with symmetric root systems*. Proc. Indian Acad. Sci. Math. Sci., 2008, **118**(3): 351–356.

[10] A. J. CALDERÓN. *On split Lie triple systems*. Proc. Indian Acad. Sci., Math. Sci., 2009, **119**(2): 165–177.

[11] A. J. CALDERÓN, J. M. SÁNCHEZ. *On the structure of split Lie color algebras*. Linear Algebra Appl., 2012, **436**(2): 307–315.

[12] Yan CAO, Liangyun CHEN. *On the structure of split Leibniz triple systems*. Acta Math. Sin., (Engl. Ser.) 2015, **31**(10): 1629–1644.

[13] Yan CAO, Liangyun CHEN. *On split Leibniz triple systems*. J. Korean Math. Soc., 2017, **54**(4): 1265–1279.

[14] Yan CAO, Liangyun CHEN. *On split regular Hom-Lie color algebras*. Colloq. Math., 2017, **146**(1): 143–155.

[15] Jian ZHANG, Chiping ZHANG, Yan CAO. *On the structure of split involutive regular Hom-Lie algebras*. Oper. Matrices, 2017, **11**(3): 783–792.