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Approximate Quadratic Functional Inequality in β-Homogeneous Normed Spaces

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Abstract Using the direct method, we investigate the generalized Hyers-Ulam stability of the following quadratic functional inequality $\|f(x-y)+f(y-z)+f(x-z)-3f(x)-3f(y)-3f(z)\| \leq \|f(x+y+z)\|$ in β-homogeneous complex Banach spaces.

Keywords β-homogeneous space; generalized Hyers-Ulam stability; quadratic functional inequality

MR(2010) Subject Classification 39B82; 39B52

1. Introduction and preliminaries

The stability problems concerning group homomorphisms were raised by Ulam [1] in 1940 and affirmatively answered for Banach spaces by Hyers [2] in the next year. Hyers' result was generalized by Aoki [3] for approximate additive mappings and by Rassias [4] for approximate linear mappings by allowing the Cauchy difference operator $\mathrm{CD}_f(x,y) = f(x+y)-[f(x)+f(y)]$ to be controlled by $\varepsilon(\|x\|^p+\|y\|^p)$. In 1994, a generalization of the Rassias' theorem was obtained by Găvruţă [5], who replaced $\varepsilon(\|x\|^p+\|y\|^p)$ by a general control function $\varphi(x,y)$ in the spirit of the Rassias approach. Since then, the stability of several functional equations has been extensively investigated by several mathematicians [6–9].

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called quadratic functional equation. In fact, every solution of the quadratic equation (1.1) is said to be a quadratic mapping. The Hyers-Ulam stability problem for the quadratic functional equation (1.1) was investigated by Skof [10], Cholewa [11], Czerwik [12] and Lee et al. [13] in different settings. In 2001, Bae and Kim [14] discussed the Hyers-Ulam stability of the quadratic functional equation

$$f(x+y+z)+f(x-y)+f(y-z)+f(z-x)=3f(x)+3f(y)+3f(z) \tag{1.2}$$

which is equivalent to the original quadratic functional equation (1.1). The hyperstability of a pexiderized σ-quadratic functional equation on semigroups was investigated by El-fassi and

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Brzdęk [15]. In 2013, Kim et al. [16] introduced the following quadratic functional inequality:

$$\|f(x-y)+f(y-z)+f(z-x)-3f(x)-3f(y)-3f(z)\| \le \|f(x+y+z)\|. \tag{1.3}$$

They established the general solution of the quadratic functional inequality (1.3), and then investigated the generalized Hyers-Ulam stability of this inequality in Banach spaces and in non-Archimedean Banach spaces. Recently, the Hyers-Ulam stability problem for the additive functional inequality and the quartic functional equation was discussed by Lee et al. [17], Lu and Park [18] in β-homogeneous F-spaces, respectively. In 2017, Park et al. [19] established the Hyers-Ulam stability of the quadratic ρ-functional inequallites in β-homogeneous normed spaces.

The main purpose of this paper is to establish the generalized Hyers-Ulam stability of the quadratic functional inequality (1.3) in β-homogeneous complex Banach spaces by using the direct method. Our results generalize those results of [16] to β-homogeneous complex Banach spaces.

Definition 1.1 ([17–19]) *Let X be a linear space. A nonnegative valued function $\|\cdot\|$ is an F-norm if it satisfies the following conditions:*

(FN1) $\|x\| = 0$ *if and only if* $x = 0$;

(FN2) $\|\lambda x\| = \|x\|$ *for all* $x \in X$ *and all* λ *with* $|\lambda| = 1$;

(FN3) $\|x+y\| \le \|x\| + \|y\|$ *for all* $x, y \in X$;

(FN4) $\|\lambda_n x\| \to 0$ *provided* $\lambda_n \to 0$;

(FN5) $\|\lambda x_n\| \to 0$ *provided* $x_n \to 0$.

Then $(X, \|\cdot\|)$ is called an F^*-space. An F-space is a complete F^*-space.

An F-norm is called β-homogeneous ($\beta > 0$) if $\|tx\| = |t|^\beta \|x\|$ for all $x \in X$ and all $t \in \mathbb{C}$ (see [20]). A β-homogeneous F-space is called a β-homogeneous complex Banach space [19].

2. Main results

In this section, we prove the stability problem of the quadratic functional inequality (1.3) in β-homogeneous complex Banach space. Let β_1, β_2 be positive real numbers with $\beta_1 \le 1$ and $\beta_2 \le 1$. Assume that X is a β_1-homogeneous real or complex normed space with $\|\cdot\|$ and that Y is a β_2-homogeneous complex Banach space with $\|\cdot\|$. Now before taking up the main subject, we need introduce the following lemma.

Lemma 2.1 ([16]) *Let V and W be real vector spaces. A mapping $f : V \to W$ satisfies the functional inequality (1.3) for all $x, y, z \in V$ if and only if f is quadratic.*

Theorem 2.2 *Let θ_i be a nonnegative real number and r_i be a positive real number such that $0 < r_i < \frac{2\beta_2}{\beta_1}$ or $r_i > \frac{2\beta_2}{\beta_1}$ for all $i = 1, 2, 3$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\begin{aligned}&\|f(x-y)+f(y-z)+f(x-z)-3f(x)-3f(y)-3f(z)\| \\ &\quad \le \|f(x+y+z)\| + \theta_1\|x\|^{r_1} + \theta_2\|y\|^{r_2} + \theta_3\|z\|^{r_3}\end{aligned} \tag{2.1}$$

for all $x, y, z \in X$, then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{(2^{\beta_2} + 3^{\beta_2})\theta_1}{|4^{\beta_2} - 2^{r_1\beta_1}|}\|x\|^{r_1} + \frac{3^{\beta_2}\theta_2}{|4^{\beta_2} - 2^{r_2\beta_1}|}\|x\|^{r_2} + \frac{2^{\beta_2}\theta_3}{|4^{\beta_2} - 2^{r_3\beta_1}|}\|x\|^{r_3} \tag{2.2}$$

for all $x \in X$.

Proof Assume that $0 < r_i < \frac{2\beta_2}{\beta_1}$. Replacing z by $-x - y$ in (2.1), we have

$$\begin{aligned}&\|f(x-y) + f(x+2y) + f(2x+y) - 3f(x) - 3f(y) - 3f(-x-y)\|\\ &\quad \le \theta_1\|x\|^{r_1} + \theta_2\|y\|^{r_2} + \theta_3\|x+y\|^{r_3}\end{aligned} \tag{2.3}$$

for all $x, y \in X$. Letting $y = -x$ and $z = 0$ in (2.1), we obtain

$$\|f(2x) - 2f(x) - 2f(-x)\| \le \theta_1\|x\|^{r_1} + \theta_2\|x\|^{r_2} \tag{2.4}$$

for all $x \in X$. Putting $y = 0$ in (2.3), we have

$$\|f(2x) - f(x) - 3f(-x)\| \le \theta_1\|x\|^{r_1} + \theta_3\|x\|^{r_3} \tag{2.5}$$

for all $x \in X$. It follows from (2.4) and (2.5) that

$$\|f(2x) - 4f(x)\| \le (2^{\beta_2} + 3^{\beta_2})\theta_1\|x\|^{r_1} + 3^{\beta_2}\theta_2\|x\|^{r_2} + 2^{\beta_2}\theta_3\|x\|^{r_3} \tag{2.6}$$

for all $x \in X$. So

$$\|f(x) - \frac{f(2x)}{4}\| \le \frac{(2^{\beta_2} + 3^{\beta_2})}{4^{\beta_2}}\theta_1\|x\|^{r_1} + \frac{3^{\beta_2}}{4^{\beta_2}}\theta_2\|x\|^{r_2} + \frac{2^{\beta_2}}{4^{\beta_2}}\theta_3\|x\|^{r_3} \tag{2.7}$$

for all $x \in X$. It follows from (2.7) that

$$\begin{aligned}\|\frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}\| \le &\frac{(2^{\beta_2} + 3^{\beta_2})}{4^{\beta_2}}\sum_{j=m}^{n-1}\frac{2^{r_1\beta_1 j}}{4^{\beta_2 j}}\theta_1\|x\|^{r_1} + \\ &\frac{3^{\beta_2}}{4^{\beta_2}}\sum_{j=m}^{n-1}\frac{2^{r_2\beta_1 j}}{4^{\beta_2 j}}\theta_2\|x\|^{r_2} + \frac{2^{\beta_2}}{4^{\beta_2}}\sum_{j=m}^{n-1}\frac{2^{r_3\beta_1 j}}{4^{\beta_2 j}}\theta_3\|x\|^{r_3}\end{aligned} \tag{2.8}$$

for all nonnegative integers m and n with $n > m$ and all $x \in X$. By virtue of $r_i < \frac{2\beta_2}{\beta_1}$, it follows from (2.8) that the sequence $\{\frac{f(2^n x)}{4^n}\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{f(2^n x)}{4^n}\}$ converges. So, one can define a mapping $Q : X \to Y$ by $Q(x) := \lim_{n\to\infty}\frac{f(2^n x)}{4^n}$ for all $x \in X$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (2.8), we get

$$\|f(x) - Q(x)\| \le \frac{(2^{\beta_2} + 3^{\beta_2})\theta_1}{(4^{\beta_2} - 2^{r_1\beta_1})}\|x\|^{r_1} + \frac{3^{\beta_2}\theta_2}{(4^{\beta_2} - 2^{r_2\beta_1})}\|x\|^{r_2} + \frac{2^{\beta_2}\theta_3}{(4^{\beta_2} - 2^{r_3\beta_1})}\|x\|^{r_3} \tag{2.9}$$

for all $x \in X$.

Next, we claim that the mapping $Q : X \to Y$ is quadratic. In fact, it follows from (2.1) that

$$\begin{aligned}&\|Q(x-y) + Q(y-z) + Q(x-z) - 3Q(x) - 3Q(y) - 3Q(z)\|\\ &= \lim_{n\to\infty}\frac{1}{4^{\beta_2 n}}\|f(2^n(x-y)) + f(2^n(y-z)) + f(2^n(x-z)) -\\ &\quad 3f(2^n x) - 3f(2^n y) - 3f(2^n z)\|\end{aligned}$$

$$\le \lim_{n\to\infty}\frac{1}{4^{\beta_2 n}}\|f(2^n(x+y+z))\| + \lim_{n\to\infty}\frac{2^{r_1\beta_1 n}}{4^{\beta_2 n}}\theta_1\|x\|^{r_1} + \lim_{n\to\infty}\frac{2^{r_2\beta_1 n}}{4^{\beta_2 n}}\theta_2\|y\|^{r_2} + \lim_{n\to\infty}\frac{2^{r_3\beta_1 n}}{4^{\beta_2 n}}\theta_3\|z\|^{r_3}$$

$$= \|Q(x+y+z)\|. \tag{2.10}$$

Thus, the mapping $Q : X \to Y$ is quadratic by Lemma 2.1.

Now, let $Q' : X \to Y$ be another quadratic mapping satisfying (2.9). Then, we obtain

$$\begin{aligned}\|Q(x) - Q'(x)\| &= \frac{1}{2^{2\beta_2 n}}\|Q(2^n x) - Q'(2^n x)\| \\ &\le \frac{1}{4^{\beta_2 n}}(\|Q(2^n x) - f(2^n x)\| + \|Q'(2^n x) - f(2^n x)\|) \\ &\le \frac{2(2^{\beta_2}+3^{\beta_2})2^{r_1\beta_1 n}}{4^{\beta_2 n}(4^{\beta_2}-2^{r_1\beta_1})}\theta_1\|x\|^{r_1} + \frac{2\cdot 3^{\beta_2}2^{r_2\beta_1 n}}{4^{\beta_2 n}(4^{\beta_2}-2^{r_2\beta_1})}\theta_2\|x\|^{r_2} + \\ &\quad \frac{2\cdot 2^{\beta_2}2^{r_3\beta_1 n}}{4^{\beta_2 n}(4^{\beta_2}-2^{r_3\beta_1})}\theta_3\|x\|^{r_3}\end{aligned} \tag{2.11}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So, we can conclude that $Q(x) = Q'(x)$ for all $x \in X$.

Now, assume that $r_i > \frac{2\beta_2}{\beta_1}$. It follows from (2.6) that

$$\|f(x) - 4f(\frac{x}{2})\| \le \frac{(2^{\beta_2}+3^{\beta_2})\theta_1}{2^{r_1\beta_1}}\|x\|^{r_1} + \frac{3^{\beta_2}\theta_2}{2^{r_2\beta_1}}\|x\|^{r_2} + \frac{2^{\beta_2}\theta_3}{2^{r_3\beta_1}}\|x\|^{r_3} \tag{2.12}$$

for all $x \in X$. Hence

$$\begin{aligned}\|4^m f(\frac{x}{2^m}) - 4^n f(\frac{x}{2^n})\| \le &\frac{(2^{\beta_2}+3^{\beta_2})}{2^{r_1\beta_1}}\sum_{j=m}^{n-1}\frac{4^{\beta_2 j}}{2^{r_1\beta_1 j}}\theta_1\|x\|^{r_1} + \\ &\frac{3^{\beta_2}}{2^{r_2\beta_1}}\sum_{j=m}^{n-1}\frac{4^{\beta_2 j}}{2^{r_2\beta_1 j}}\theta_2\|x\|^{r_2} + \frac{2^{\beta_2}}{2^{r_3\beta_1}}\sum_{j=m}^{n-1}\frac{4^{\beta_2 j}}{2^{r_3\beta_1 j}}\theta_3\|x\|^{r_3}\end{aligned} \tag{2.13}$$

for all $x \in X$. Define $Q : X \to Y$ by $Q(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (2.13), we get

$$\|f(x) - Q(x)\| \le \frac{(2^{\beta_2}+3^{\beta_2})\theta_1}{(2^{r_1\beta_1}-4^{\beta_2})}\|x\|^{r_1} + \frac{3^{\beta_2}\theta_2}{(2^{r_2\beta_1}-4^{\beta_2})}\|x\|^{r_2} + \frac{2^{\beta_2}\theta_3}{(2^{r_3\beta_1}-4^{\beta_2})}\|x\|^{r_3} \tag{2.14}$$

for all $x \in X$. The rest of the proof is similar to the proof for the case $0 < r_i < \frac{2\beta_2}{\beta_1}$. By (2.9) and (2.14), we obtain the approximation (2.2) of f by Q, as desired. This completes the proof of the theorem. □

Corollary 2.3 *Let $\theta \ge 0$ be fixed. Let $f : X \to Y$ be a mapping with $f(0) = 0$ such that*

$$\|f(x-y) + f(y-z) + f(x-z) - 3f(x) - 3f(y) - 3f(z)\| \le \|f(x+y+z)\| + \theta \tag{2.15}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{2^{\beta_2}+3^{\beta_2}}{|4^{\beta_2}-1|}\theta \tag{2.16}$$

for all $x \in X$.

From now on, assume that X is a β-homogeneous real or complex normed space and that Y is a β-homogeneous complex Banach space. We prove the stability problem of the quadratic inequality (1.3) with perturbed control function φ.

Theorem 2.4 *Let $\varphi : X^3 \to [0,\infty)$ be a function such that*

$$\begin{aligned}&\sum_{j=0}^{\infty} \tfrac{1}{4^{j\beta}}\varphi(2^j x, 2^j y, 2^j z) < \infty,\\ &(\sum_{j=1}^{\infty} 4^{j\beta}\varphi(\tfrac{x}{2^j}, \tfrac{y}{2^j}, \tfrac{z}{2^j}) < \infty, \textit{resp.})\end{aligned} \tag{2.17}$$

for all $x, y, z \in X$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\begin{aligned}&\|f(x-y) + f(y-z) + f(x-z) - 3f(x) - 3f(y) - 3f(z)\|\\ &\le \|f(x+y+z)\| + \varphi(x,y,z)\end{aligned} \tag{2.18}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\begin{aligned}&\|f(x) - Q(x)\| \le \tfrac{1}{4^\beta}\sum_{j=0}^{\infty} \tfrac{1}{4^{j\beta}}\{3^\beta\varphi(2^j x, -2^j x, 0) + 2^\beta\varphi(2^j x, 0, -2^j x)\},\\ &(\|f(x) - Q(x)\| \le \tfrac{1}{4^\beta}\sum_{j=1}^{\infty} 4^{j\beta}\{3^\beta\varphi(\tfrac{x}{2^j}, -\tfrac{x}{2^j}, 0) + 2^\beta\varphi(\tfrac{x}{2^j}, 0, -\tfrac{x}{2^j})\}, \textit{resp.})\end{aligned} \tag{2.19}$$

for all $x \in X$.

Proof Replacing z by $-x-y$ in (2.18), we have

$$\|f(x-y) + f(x+2y) + f(2x+y) - 3f(x) - 3f(y) - 3f(-x-y)\| \le \varphi(x,y,-x-y) \tag{2.20}$$

for all $x, y \in X$. Letting $y = -x$ and $z = 0$ in (2.18), we obtain

$$\|f(2x) - 2f(x) - 2f(-x)\| \le \varphi(x,-x,0) \tag{2.21}$$

for all $x \in X$. Putting $y = 0$ in (2.20), we have

$$\|f(2x) - f(x) - 3f(-x)\| \le \varphi(x,0,-x) \tag{2.22}$$

for all $x \in X$. It follows from (2.21) and (2.22) that

$$\|f(2x) - 4f(x)\| \le 3^\beta\varphi(x,-x,0) + 2^\beta\varphi(x,0,-x) \tag{2.23}$$

for all $x \in X$. So

$$\|f(x) - \frac{f(2x)}{4}\| \le \frac{1}{4^\beta}\{3^\beta\varphi(x,-x,0) + 2^\beta\varphi(x,0,-x)\} \tag{2.24}$$

for all $x \in X$. It follows from (2.24) that for all nonnegative integers n and m with $n > m$

$$\begin{aligned}\|\frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}\| &\le \sum_{j=m}^{n-1}\frac{1}{4^{j\beta}}\|f(2^j x) - \frac{f(2^{j+1}x)}{4}\|\\ &\le \frac{1}{4^\beta}\sum_{j=m}^{n-1}\frac{1}{4^{j\beta}}\{3^\beta\varphi(2^j x, -2^j x, 0) + 2^\beta\varphi(2^j x, 0, -2^j x)\}\end{aligned} \tag{2.25}$$

for all $x \in X$. It means that the sequence $\{\frac{f(2^n x)}{4^n}\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{f(2^n x)}{4^n}\}$ converges in Y. Therefore, we can define a mapping $Q : X \to Y$

by $Q(x) := \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$ for all $x \in X$. Moreover, letting $m = 0$ and taking the limit $n \to \infty$ in (2.25), we obtain the inequality (2.19), as desired.

By (2.17) and (2.18), we have

$$\begin{aligned}
&\|Q(x-y)+Q(y-z)+Q(x-z)-3Q(x)-3Q(y)-3Q(z)\| \\
&= \lim_{n\to\infty} \frac{1}{4^{n\beta}} \|f(2^n(x-y))+f(2^n(y-z))+f(2^n(x-z))- \\
&\quad 3f(2^n x)-3f(2^n y)-3f(2^n z)\| \\
&\le \lim_{n\to\infty} \frac{1}{4^{n\beta}} \|f(2^n(x+y+z))\| + \lim_{n\to\infty} \frac{1}{4^{n\beta}} \varphi(2^n x, 2^n y, 2^n z) \\
&= \|Q(x+y+z)\|. \qquad (2.26)
\end{aligned}$$

By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic.

Next, we show that the uniqueness of Q. Let $Q' : X \to Y$ be another quadratic mapping satisfying (2.19). Then, we obtain

$$\begin{aligned}
&\|Q(x)-Q'(x)\| = \frac{1}{4^{n\beta}} \|Q(2^n x)-Q'(2^n x)\| \\
&\le \frac{1}{4^{n\beta}} (\|Q(2^n x)-f(2^n x)\| + \|Q'(2^n x)-f(2^n x)\|) \\
&\le \frac{2}{4^\beta} \sum_{j=0}^{\infty} \frac{1}{4^{(j+n)\beta}} \{3^\beta \varphi(2^{j+n}x, -2^{j+n}x, 0) + 2^\beta \varphi(2^{j+n}x, 0, -2^{j+n}x)\} \\
&= \frac{2}{4^\beta} \sum_{j=n}^{\infty} \frac{1}{4^{j\beta}} \{3^\beta \varphi(2^j x, -2^j x, 0) + 2^\beta \varphi(2^j x, 0, -2^j x)\} \qquad (2.27)
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. Hence $Q(x) = Q'(x)$ for all $x \in X$. This completes the proof of the theorem. □

Corollary 2.5 *Let $\varepsilon_i \ge 0$ be a real number and λ_i be a positive real number with $\lambda_i < 2$ or $\lambda_i > 2$ for all $i = 1, 2, 3$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\begin{aligned}
&\|f(x-y)+f(y-z)+f(x-z)-3f(x)-3f(y)-3f(z)\| \\
&\le \|f(x+y+z)\| + \varepsilon_1\|x\|^{\lambda_1} + \varepsilon_2\|y\|^{\lambda_2} + \varepsilon_3\|z\|^{\lambda_3} \qquad (2.28)
\end{aligned}$$

for all $x, y, z \in X$, then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x)\| \le \frac{(2^\beta+3^\beta)\varepsilon_1}{|4^\beta - 2^{\lambda_1\beta}|}\|x\|^{\lambda_1} + \frac{3^\beta\varepsilon_2}{|4^\beta - 2^{\lambda_2\beta}|}\|x\|^{\lambda_2} + \frac{2^\beta\varepsilon_3}{|4^\beta - 2^{\lambda_3\beta}|}\|x\|^{\lambda_3} \qquad (2.29)$$

for all $x \in X$.

Proof Define $\varphi(x,y,z) := \varepsilon_1\|x\|^{\lambda_1} + \varepsilon_2\|y\|^{\lambda_2} + \varepsilon_3\|z\|^{\lambda_3}$ and apply Theorem 2.4 to get the result. □

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