Journal of Mathematical Research with Applications
Jan., 2020, Vol. 40, No. 1, pp. 47–56
DOI:10.3770/j.issn:2095-2651.2020.01.005
Http://jmre.dlut.edu.cn

On $\mathcal{I}$-Covering Mappings and 1-$\mathcal{I}$-Covering Mappings

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Abstract In this paper, we introduce the concepts of $\mathcal{I}$-covering mappings and 1-$\mathcal{I}$-covering mappings, discuss the difference between sequence-covering and $\mathcal{I}$-covering mappings by some examples. With those concepts, we get some interesting properties of $\mathcal{I}$-covering (1-$\mathcal{I}$-covering) mappings and some characterizations of $\mathcal{I}$-covering (1-$\mathcal{I}$-covering) and compact mapping images of metric spaces.

Keywords ideal convergence; $\mathcal{I}$-covering mappings; 1-$\mathcal{I}$-covering mappings; compact mappings

MR(2010) Subject Classification 54A20; 54B15; 54C08; 54D55; 40A05; 26A03

1. Introduction

Statistical convergence, which is a generalization of the usual notion of convergence, was introduced independently by Fast [1] and Steinhaus [2]. There is not doubt that the study of statistical convergence and its various generalizations has become an active research area [3–9]. In particular, Kostyrko, Šalát and Wilczynski [10] introduced two interesting generalizations of statistical convergence by using the notion of ideals of subsets of positive integers, which were named as $\mathcal{I}$ and $\mathcal{I}^*$-convergence, and studied some properties of $\mathcal{I}$ and $\mathcal{I}^*$-convergence in metric spaces. Later, Lahiri and Das [11] discussed $\mathcal{I}$ and $\mathcal{I}^*$-convergence in topological spaces. Some further results connected with $\mathcal{I}$ and $\mathcal{I}^*$-convergence can be found in [12–17].

On the other hand, to find the internal characterizations of certain images of metric spaces is one of the central questions in general topology. In 1971, Siwiec [18] introduced the concept of sequence-covering mappings. Thereafter, the research in this area has been well developed [19–25]. In 2017, Renukadevi and Prakash [26] extended sequence-covering mappings to statistical sequence-covering mappings. Naturally, we wonder if we can combine sequence-covering mappings with $\mathcal{I}$-convergence. For this reason, this paper draws into $\mathcal{I}$-covering mappings and 1-$\mathcal{I}$-covering mappings for an ideal $\mathcal{I}$ on $\mathbb{N}$, and discusses some basic properties of them.

Recently, the researches on $\mathcal{I}$-convergence are mainly focused on aspects of $\mathcal{I}^*$-convergence [11], $\mathcal{I}$-limit points [10], $\mathcal{I}$-cluster points [16], $\mathcal{I}$-Cauchy sequences [13], and selection principles [14] and so on. As we know, continuous mappings, sequence-covering mappings and sequentially quotient mappings are the most important tools to study convergence, sequential and metric

Received December 21, 2018; Accepted April 11, 2019
Supported by the National Natural Science Foundation of China (Grant No. 11801254) and the Ningde Normal University (Grant No. 2017T01).
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spaces. It is expected that $\mathcal{I}$-covering mappings and 1-$\mathcal{I}$-covering mappings will also play an active role.

The paper is organized as follows. In Section 2, we introduce some basic concepts and propositions in topological spaces. In Section 3, we define $\mathcal{I}$-covering mappings and discuss the difference between sequence-covering mappings and $\mathcal{I}$-covering mappings by some examples. In Section 4, we discuss the $\mathcal{I}$-covering (1-$\mathcal{I}$-covering) compact mappings and obtain some characterizations of $\mathcal{I}$-covering (1-$\mathcal{I}$-covering) compact mapping images of metric spaces.

In this paper, the letter X will always denote a topological space. $\mathcal{U}_x$ denotes the family of all neighborhoods of a point x in a topological space X. The cardinality of the set B is denoted by $|B|$. The set of all positive integers, the first infinite ordinal, and the first uncountable ordinal are denoted by $\mathbb{N}$, ω and ω_1, respectively. Let $\mathscr{P}$ be a family of subsets of X, $\mathrm{st}(x, \mathscr{P})$ and $(\mathscr{P})_x$ denote the union $\cup\{P : P \in \mathscr{P}, x \in P\}$ and the subfamily $\{P \in \mathscr{P} : x \in P\}$, respectively. $\langle x_n \rangle$ denotes the subset $\{x_n : n \in \mathbb{N}\} \subset X$. A point $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ of a Tychonoff-product space is abbreviated to (α_n). The readers may refer to [27,28] for notation and terminology not explicitly given here.

2. Preliminaries

Recall some related concepts and notations. For each subset A of $\mathbb{N}$ the asymptotic density of A, denoted $\delta(A)$, is given by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} |\{k \in A : k \le n\}|,$$

if this limit exists. Let X be a topological space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to converge statistically to a point $x \in X$, if $\delta(\{n \in \mathbb{N} : x_n \notin U\}) = 0$, i.e., $\delta(\{n \in \mathbb{N} : x_n \in U\}) = 1$ for each neighborhood U of x in X, which is denoted by $s\text{-}\lim_{n\to\infty} x_n = x$ or $x_n \xrightarrow{s} x$.

The concept of $\mathcal{I}$-convergence in topological spaces is a generalization of statistical convergence, which is based on the ideal of subsets of the set $\mathbb{N}$. Let $\mathcal{A} = 2^{\mathbb{N}}$ be the family of all subsets of $\mathbb{N}$. An ideal $\mathcal{I} \subset \mathcal{A}$ is a hereditary family of subsets of $\mathbb{N}$ which is stable under finite unions[10], i.e., the following are satisfied: if $B \subset A \in \mathcal{I}$, then $B \in \mathcal{I}$; if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. An ideal $\mathcal{I}$ is said to be non-trivial, if $\mathcal{I} \neq \emptyset$ and $\mathbb{N} \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I} \subset \mathcal{A}$ is called admissible if $\mathcal{I} \supset \{\{n\} : n \in \mathbb{N}\}$. Clearly, every non-trivial ideal $\mathcal{I}$ defines a dual filter $\mathcal{F}_{\mathcal{I}} = \{A \subset \mathbb{N} : \mathbb{N} \setminus A \in \mathcal{I}\}$ on $\mathbb{N}$.

Let $\mathcal{I}_\delta$ be the family of subsets $A \subset \mathbb{N}$ with $\delta(A) = 0$. Then $\mathcal{I}_\delta$ is an admissible ideal, and the dual filter $\mathcal{F}_{\mathcal{I}_\delta} = \{A \subset \mathbb{N} : \delta(A) = 1\}$.

Definition 2.1 ([10]) *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a topological space X is said to be $\mathcal{I}$-convergent to a point $x \in X$ provided for any neighborhood U of x, we have $A_U = \{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$, which is denoted by $\mathcal{I}\text{-}\lim_{n\to\infty} x_n = x$ or $x_n \xrightarrow{\mathcal{I}} x$, and the point x is called the $\mathcal{I}$-limit of the sequence $\{x_n\}_{n\in\mathbb{N}}$.*

Definition 2.2 *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a set is trivial if the set $\{x_n : n \in \mathbb{N}\}$ is finite.*

Let X be a topological space, $P \subset X$ and $x \in P$. P is called a sequential neighborhood of x in X if whenever $\{x_n\}_{n\in\mathbb{N}}$ is a sequence converging to the point x, $\{x_n\}_{n\in\mathbb{N}}$ is eventually in P. A subset $F \subset X$ is called sequentially closed if F is closed with respect to the usual convergence of sequences in F, i.e., for each sequence $\{x_n\}_{n\in\mathbb{N}} \subset F$ with $x_n \to x \in X$, $x \in F$. X is called a sequential space [8,29] if each sequentially closed subset of X is closed. A subset $U \subset X$ is called sequentially open if $X \setminus U$ is sequentially closed. Obviously, a subset $U \subset X$ is sequentially open if and only if for each sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to a point $x \in U$, $\{x_n\}_{n\in\mathbb{N}}$ is eventually in U; a space X is a sequential space if and only if each sequentially open subset of X is open. Every first countable space is a sequential space [29].

Definition 2.3 ([30]) *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and X be a topological space.*

(1) A subset $F \subset X$ is said to be $\mathcal{I}$-closed if for each sequence $\{x_n\}_{n\in\mathbb{N}} \subset F$ with $x_n \xrightarrow{\mathcal{I}} x \in X$, $x \in F$.

(2) A subset $U \subset X$ is said to be $\mathcal{I}$-open if $X \setminus U$ is $\mathcal{I}$-closed.

(3) X is called an $\mathcal{I}$-sequential space if each $\mathcal{I}$-closed subset of X is closed.

Obviously, every sequential space is an $\mathcal{I}$-sequential space.

Definition 2.4 *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$, X, Y be topological spaces and $f : X \to Y$ be a mapping.*

(1) f is called preserving $\mathcal{I}$-convergence provided for each sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with $x_n \xrightarrow{\mathcal{I}} x$, the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ $\mathcal{I}$-converges to $f(x)$ (see [11]).

(2) f is called $\mathcal{I}$-continuous provided U is $\mathcal{I}$-open in Y, then $f^{-1}(U)$ is $\mathcal{I}$-open in X.

Definition 2.5 *Let X be a topological space and $P \subset X$, P is called an $\mathcal{I}$-sequential neighborhood of x, if for each sequence $\{x_n\}_{n\in\mathbb{N}}$ which $\mathcal{I}$-converges to $x \in P$, $\{n \in \mathbb{N} : x_n \notin P\} \in \mathcal{I}$.*

Definition 2.6 *Let X be a topological space and $\mathscr{P}$ be a cover of X.*

(1) $\mathscr{P}$ is a cs-cover [31] of X if for any convergent sequence S in X, there exists $P \in \mathscr{P}$ such that S is eventually in P;

(2) $\mathscr{P}$ is an sn-cover [32] of X if each element of $\mathscr{P}$ is a sequential neighborhood of some point of X and for each $x \in X$, there exists $P \in \mathscr{P}$ such that P is a sequential neighborhood of x.

Similarly, we can define the following two concepts.

Definition 2.7 *Let X be a topological space and $\mathscr{P}$ be a cover of X.*

(1) $\mathscr{P}$ is an $\mathcal{I}$-cs-cover of X if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X which $\mathcal{I}$-converges to x, there exists $P \in \mathscr{P}$ such that $x \in P$ and $\{n \in \mathbb{N} : x_n \notin P\} \in \mathcal{I}$;

(2) $\mathscr{P}$ is an $\mathcal{I}$-sn-cover of X if each element of $\mathscr{P}$ is an $\mathcal{I}$-sequential neighborhood of some point of X and for each $x \in X$, there exists $P \in \mathscr{P}$ such that P is an $\mathcal{I}$-sequential neighborhood of x.

Definition 2.8 ([33]) *A class of mappings is said to be hereditary if whenever $f : X \to Y$ is in* *the class, then for each subspace H of Y, the restriction of f to $f^{-1}(H)$ is in the class.*

Definition 2.9 *Let $\mathscr{P} = \bigcup_{x\in X} \mathscr{P}_x$ be a cover of a space X, where $\mathscr{P}_x \subset (\mathscr{P})_x$. $\mathscr{P}$ is called a network [34] of X if for each $x \in U$ with U open in X, there exists $P \in \mathscr{P}_x$ such that $P \subset U$, where $\mathscr{P}_x$ is called a network at x in X.*

Definition 2.10 *Let $\{\mathscr{P}_n\}$ be a sequence of covers of a space X. $\{\mathscr{P}_n\}$ is called a point-star network [32] of X if $\langle\, st(x, \mathscr{P}_n)\rangle$ is a network at x in X for each $x \in X$.*

Obviously, $\{\mathscr{P}_n\}$ is a point-star network of X if and only if for each $x \in X$ and for given $P_n \in (\mathscr{P}_n)_x$, $\langle P_n\rangle$ is a network at x in X (see [20]).

Definition 2.11 *Let $\{\mathscr{P}_n\}$ be a point-star network of X. For each $n \in \mathbb{N}$, put $\Lambda_n = \Lambda$ and endow Λ_n with the discrete topology. Put $M = \{\alpha = (\alpha_n) \in \prod_{n\in\mathbb{N}} \Lambda_n : \langle P_{\alpha_n}\rangle$ forms a network at some point x_α in $X\}$. Then M, which is a subspace of the product space $\prod_{n\in\mathbb{N}} \Lambda_n$, is a metric space and x_α is unique for each $\alpha \in M$. Define $f : M \to X$ by $f(\alpha) = x_\alpha$; then f is a mapping, and $(f, M, X, \{\mathscr{P}_n\})$ is called a Ponomarevs system.*

Lemma 2.12 ([30]) *Let X, Y be topological spaces and $f : X \to Y$ be a mapping.*

(1) If f is continuous, then f preserves $\mathcal{I}$-convergence[11].

(2) If f preserves $\mathcal{I}$-convergence, then f is $\mathcal{I}$-continuous.

Lemma 2.13 *Let Γ be an index set and $\{x_{\gamma,n}\}_{n\in\mathbb{N}}$ be a sequence in X_γ for each $\gamma \in \Gamma$. Then each $x_{\gamma,n} \xrightarrow{\mathcal{I}} x_\gamma \in X_\gamma$ $(\gamma \in \Gamma)$ if and only if $(x_{\gamma,n})_{\gamma\in\Gamma} \xrightarrow{\mathcal{I}} (x_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} X_\gamma$.*

Proof Necessity. For any neighborhood U of $(x_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} X_\gamma$, there exists a finite subset $\Gamma' \subset \Gamma$ and open set U_γ in X_γ $(\gamma \in \Gamma')$ such that $U \supset \prod_{\gamma\in\Gamma'} U_\gamma \times \prod_{\gamma\in\Gamma\backslash\Gamma'} X_\gamma$. Since each $x_{\gamma,n} \xrightarrow{\mathcal{I}} x_\gamma$, we have $\{n \in \mathbb{N} : x_{\gamma,n} \notin U_\gamma\} \in \mathcal{I}$ for each $\gamma \in \Gamma'$. Obviously, $\{n \in \mathbb{N} : x_{\gamma,n} \notin X_\gamma\} = \emptyset \in \mathcal{I}$ for each $\gamma \in \Gamma\backslash\Gamma'$. Note that $\{n \in \mathbb{N} : (x_{\gamma,n})_{\gamma\in\Gamma} \notin U\} \subset \bigcup_{\gamma\in\Gamma'}\{n \in \mathbb{N} : x_{\gamma,n} \notin U_\gamma\}$. By the definitions of ideal and ideal convergence, it follows that $\{n \in \mathbb{N} : (x_{\gamma,n})_{\gamma\in\Gamma} \notin U\} \in \mathcal{I}$, thus $(x_{\gamma,n})_{\gamma\in\Gamma} \xrightarrow{\mathcal{I}} (x_\gamma)_{\gamma\in\Gamma}$.

Sufficiency. Let $\mathrm{P}_\gamma : \prod_{\gamma\in\Gamma} X_\gamma \to X_\gamma$ be the projection mapping, then P_γ is continuous. By Lemma 2.12, P_γ preserves $\mathcal{I}$-convergence. Hence, $x_{\gamma,n} \xrightarrow{\mathcal{I}} x_\gamma \in X_\gamma$ for each $\gamma \in \Gamma$. $\square$

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$, and X be a topological space. It is easy to see that the ideal $\mathcal{I} = \emptyset$ if and only if each constant sequence $x, x, \ldots, x, \ldots$ in X does not $\mathcal{I}$-converge to the point $x \in X$. If $\mathbb{N} \in \mathcal{I}$, then $\mathcal{I} = 2^{\mathbb{N}}$, and each sequence in X $\mathcal{I}$-converges to any point in X. It is known that if $\mathcal{I}$ is a non-trivial ideal on $\mathbb{N}$ and X is a T_2 space, then each $\mathcal{I}$-convergent sequence in X has a unique $\mathcal{I}$-limit [11].

If no otherwise specified, we consider $\mathcal{I}$ is always an admissible ideal on $\mathbb{N}$, all mappings are surjection and all spaces are Hausdorff.

3. Properties of $\mathcal{I}$-covering mappings

In this section, we define $\mathcal{I}$-covering mappings, discuss the difference between sequence-covering mappings and $\mathcal{I}$-covering mappings by some examples and prove some properties of $\mathcal{I}$-covering mappings.

Let X, Y be topological spaces, and f be a mapping from X onto Y. f is said to be *sequence-covering* if whenever $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in Y converging to y in Y, there exist a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points $x_n \in f^{-1}(y_n)$ for each $n \in \mathbb{N}$ and $x \in f^{-1}(y)$ such that $x_n \to x$ (see [18]).

Definition 3.1 *Let f be a mapping from a topological space X onto a topological space Y. f is said to be $\mathcal{I}$-covering if whenever $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in Y $\mathcal{I}$-converging to y in Y, there exist a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points $x_n \in f^{-1}(y_n)$ for each $n \in \mathbb{N}$ and $x \in f^{-1}(y)$ such that $x_n \xrightarrow{\mathcal{I}} x$.*

Proposition 3.2 *If $f : X \to Y$ and $g : Y \to Z$ are $\mathcal{I}$-covering mappings, then $g \circ f : X \to Z$ is an $\mathcal{I}$-covering mapping.*

Proof Let $z \in Z$ and $\{z_n\}_{n\in\mathbb{N}}$ be a sequence such that $z_n \xrightarrow{\mathcal{I}} z$. Since g is an $\mathcal{I}$-covering mapping, there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ $\mathcal{I}$-converging to y with each $y_n \in g^{-1}(z_n)$ and $y \in g^{-1}(z)$. And because f is an $\mathcal{I}$-covering mapping, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ $\mathcal{I}$-converging to x with each $x_n \in f^{-1}(y_n)$ and $x \in f^{-1}(y)$. Note that $x_n \in f^{-1}(y_n) \subset f^{-1}(g^{-1}(z_n)) = (g \circ f)^{-1}(z_n)$ and $x \in f^{-1}(y) \subset f^{-1}(g^{-1}(z)) = (g \circ f)^{-1}(z)$. Thus $g \circ f$ is an $\mathcal{I}$-covering mapping. $\square$

Proposition 3.3 *(1) The product mapping of $\mathcal{I}$-covering mappings is an $\mathcal{I}$-covering mapping;*

(2) $\mathcal{I}$-covering mappings are hereditarily $\mathcal{I}$-covering mappings.

Proof (1) Let $\prod_{\alpha\in\Lambda} f_\alpha : \prod_{\alpha\in\Lambda} X_\alpha \to \prod_{\alpha\in\Lambda} Y_\alpha$ be a mapping, where each $f_\alpha : X_\alpha \to Y_\alpha$ is an $\mathcal{I}$-covering mapping. Let $\{(y_{\alpha,n})\}_{n\in\mathbb{N}} \subset \prod_{\alpha\in\Lambda} Y_\alpha$ be a sequence $\mathcal{I}$-converging to (y_α) in $\prod_{\alpha\in\Lambda} Y_\alpha$. By Lemma 2.13, each $\{y_{\alpha,n}\}_{n\in\mathbb{N}}$ is a sequence $\mathcal{I}$-converging to y_α in Y_α. Since each f_α is an $\mathcal{I}$-covering mapping, there exists a sequence $\{x_{\alpha,n}\}_{n\in\mathbb{N}}$ $\mathcal{I}$-converging to x_α such that $f_\alpha(x_{\alpha,n}) = y_{\alpha,n}$, $f_\alpha(x_\alpha) = y_\alpha$ for each $\alpha \in \Lambda$. By Lemma 2.13 again, the sequence $\{(x_{\alpha,n})\}_{n\in\mathbb{N}}$ $\mathcal{I}$-converges to (x_α) with $(y_\alpha) \in \prod_{\alpha\in\Lambda} Y_\alpha$ and $(y_{\alpha,n}) \in \prod_{\alpha\in\Lambda} f_\alpha(x_{\alpha,n})$. Therefore, $\prod_{\alpha\in\Lambda} f_\alpha$ is an $\mathcal{I}$-covering mapping.

(2) Let $f : X \to Y$ be an $\mathcal{I}$-covering mapping and H be a subspace of Y. Define $g : f^{-1}(H) \to H$ by $g = f|_{f^{-1}(H)}$. Then g is a mapping.

Given a sequence $\{y_n\}_{n\in\mathbb{N}}$ $\mathcal{I}$-converging to y in H, since f is an $\mathcal{I}$-covering mapping, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $x_n \xrightarrow{\mathcal{I}} x$, $x_n \in f^{-1}(y_n) \subset f^{-1}(H)$ and $x \in f^{-1}(y) \subset f^{-1}(H)$. And $g = f|_{f^{-1}(H)}$ implies that each $x_n \in g^{-1}(y_n)$ and $x \in g^{-1}(y)$. $\square$

Two examples below show that sequence-covering mappings and $\mathcal{I}$-covering mappings are independent.

Example 3.4 There exists a continuous and sequence-covering mapping which is not an $\mathcal{I}$-covering mapping.

Proof Let $\mathcal{I} = \mathcal{I}_\delta$ and $S = \{x_n : n \in \mathbb{N}\}$ be a sequence with different terms. Take $x \notin S$ and put $X = S \cup \{x\}$. The topology on X is defined as follows [8]:

(1) Each point x_n is isolated;

(2) Each open neighborhood of the point x is a set U of the form $U = \{x\} \cup M$, where $M \subset S$ and $\{n \in \mathbb{N} : x_n \in M\} \in \mathcal{F}_{\mathcal{I}} = \{A \subset \mathbb{N} : \mathbb{N} \setminus A \in \mathcal{I}\}$, i.e., $\delta(\{n \in \mathbb{N} : x_n \in M\}) = 1$.

It was obtained that the space X is a Hausdorff statistically sequential space but no sequence of S converges to the point x (see [8, Example 2.1]).

Now, let Z be the set X endowed with the discrete topology. Define a mapping $f : Z \to X$ to be the identity mapping. Obviously, f is continuous. Since there is no non-trivial convergent sequence in X, f is a sequence-covering mapping. But f is not an $\mathcal{I}$-covering mapping. In fact, $S = \{x_n\}_{n\in\mathbb{N}} \subset X$ $\mathcal{I}$-converges to $x \in X$. But $\{n \in \mathbb{N} : x_n \neq x\} = \mathbb{N} \notin \mathcal{I}$, since $\mathcal{I}$ is an admissible ideal. Consequently, $\{x_n\}_{n\in\mathbb{N}} \subset Z$ does not $\mathcal{I}$-converge to $x \in Z$. $\square$

Example 3.5 There exists a continuous $\mathcal{I}$-covering mapping which is not a sequence-covering mapping.

Proof Let $\mathcal{I} = \mathcal{I}_\delta$ and $Y = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ be a subspace of $\mathbb{R}$ with the usual topology. Denote

$$\{\overline{\{y_{k:k\in\mathbb{N}}\}} : \{y_k\}_{k\in\mathbb{N}} \subset Y \text{ is a convergent sequence }\} = \{Y_\alpha : \alpha \in \varLambda\}.$$

Obviously, $\{Y_\alpha : \alpha \in \varLambda\}$ is a cover of Y. For each $\alpha \in \varLambda$, the set Y_α is endowed with the following topology and denoted it by X_α: if Y_α is a finite set, then X_α is a discrete space; if Y_α is an infinite set, the topology on X_α is defined as Example 3.4 with $x_0 = 0$. Put $X = \bigoplus_{\alpha\in\varLambda} X_\alpha \times \{\alpha\}$. Let $p : X \to Y$ be a natural mapping, that is, $p((y, \alpha)) = y$, for each $(y, \alpha) \in X_\alpha \times \{\alpha\}$ and $\alpha \in \varLambda$.

Assume that U is a neighborhood of 0 in Y, then $Y\backslash U$ is a finite set, and further $(X_\alpha \times \{\alpha\}) \cap p^{-1}(Y\backslash U)$ is a finite set for each $\alpha \in \varLambda$. Thus $p^{-1}(Y\backslash U)$ is closed in X, and hence $p^{-1}(U)$ is open in X. Therefore p is continuous.

It was obtained that there is no non-trivial convergent sequence in X_α for each $\alpha \in \varLambda$ (see [8]). Hence there is no non-trivial convergent sequence in X. Consequently, p is not a sequence-covering mapping.

Let $\{y_k\}_{k\in\mathbb{N}} \subset Y$ be an $\mathcal{I}_\delta$-convergent sequence. Without loss of generality, we can assume that $y_k \xrightarrow{\mathcal{I}_\delta} 0$. Since Y is a first countable space, there exists $A \in \mathcal{F}_{\mathcal{I}}$ such that $\lim_{A\ni k\to\infty} y_k = 0$ (see [5]). Hence, there exists $\alpha \in \varLambda$ such that $\{y_k : k \in A\} \cup \{0\} = Y_\alpha$. Since the sequence $\{y_k\}_{k\in A}$ in X_α $\mathcal{I}_\delta$-converges to 0, the sequence $\{(y_k, \alpha)\}_{k\in A}$ $\mathcal{I}_\delta$-converges to $(0, \alpha)$. For each $k \in \mathbb{N}$, put $x_k \in p^{-1}(y_k)$ satisfying $x_k = (y_k, \alpha) \in Y_\alpha \times \{\alpha\}$ as $k \in A$. And because $A \in \mathcal{F}_{\mathcal{I}}$, the sequence $\{x_k\}_{k\in\mathbb{N}}$ in X $\mathcal{I}_\delta$-converges to 0. Thus p is an $\mathcal{I}_\delta$-covering mapping. $\square$

4. $\mathcal{I}$-covering (1-$\mathcal{I}$-covering) and compact mapping of metric spaces

In this section, we discuss $\mathcal{I}$-covering (1-$\mathcal{I}$-covering) and compact mappings and obtain some characterizations of $\mathcal{I}$-covering (1-$\mathcal{I}$-covering) and compact mapping images of metric spaces. Next, we assume all mappings are continuous.

Theorem 4.1 *X is an $\mathcal{I}$-covering and compact image of a metric space if and only if X has a point-star network consisting of point-finite $\mathcal{I}$-cs-covers.*

Proof Necessity. Suppose that $f : M \to X$ is an $\mathcal{I}$-covering and compact mapping, where M is a metric space. Then there exists a sequence $\{\mathscr{B}_i\}_{i\in\mathbb{N}}$ of locally finite open covers of M such that for each compact subset K of M, $\langle\ \mathrm{st}(K, \mathscr{B}_i)\rangle$ is a neighborhood base of K in M. Put $\mathscr{P}_i = f(\mathscr{B}_i)$. As f being a compact mapping, $\mathscr{P}_i$ is a point-finite cover of X. For each $x \in X$, let V be an open neighborhood at x in X. Since $f^{-1}(x)$ is a compact subset in M and $f^{-1}(x) \subset f^{-1}(V)$, there exists $n \in \mathbb{N}$ such that $\mathrm{st}(f^{-1}(x), \mathscr{B}_n) \subset f^{-1}(V)$. Hence $\mathrm{st}(x, \mathscr{P}_n) \subset V$, thus $\langle\ \mathrm{st}(x, \mathscr{P}_i)\rangle$ is a network at x in X. This implies that $\{\mathscr{P}_n\}$ is a point-star network of X.

Let $\{x_n\}_{n\in\mathbb{N}} \subset X$ be a sequence satisfying $x_n \xrightarrow{\mathcal{I}} x \in X$. Since f is an $\mathcal{I}$-covering mapping, there exist a sequence $\{b_n\}_{n\in\mathbb{N}}$ of points $b_n \in f^{-1}(x_n)$ for each $n \in \mathbb{N}$ and $b \in f^{-1}(x)$ such that $b_n \xrightarrow{\mathcal{I}} b$. And since each $\mathscr{B}_i$ is an open cover of X for each $i \in \mathbb{N}$, there exists $B \in \mathscr{B}_i$ such that $b \in B$. Take $P = f(B)$. Note that $b_n \xrightarrow{\mathcal{I}} b$, hence $\{n \in \mathbb{N} : b_n \notin B\} \in \mathcal{I}$, and further $x \in P$ and $\{n \in \mathbb{N} : x_n \notin f(B) = P\} \in \mathcal{I}$. This implies that $\{\mathscr{P}_i\}$ is an $\mathcal{I}$-*cs*-covers of X. Therefore, X has a point-star network consisting of point-finite $\mathcal{I}$-*cs*-covers.

Sufficiency. Let $\{\mathscr{P}_i\}$ be a point-star network consisting of point-finite $\mathcal{I}$-*cs*-covers of X. For each $i \in \mathbb{N}$, put $\mathscr{P}_i = \{\mathscr{P}_\alpha : \alpha \in \Lambda_i\}$ and each Λ_i is endowed with the discrete topology. Put $M = \{\alpha = (\alpha_i) \in \prod_{i\in\mathbb{N}} \Lambda_i : \langle P_{\alpha_i}\rangle$ forms a network at some point x_α in $X\}$, then M, which is a subspace of the product space $\prod_{i\in\mathbb{N}} \Lambda_i$, is a metric space and the point x_α is unique for each $\alpha \in M$. Define a function $f : M \to X$ by $f(\alpha) = x_\alpha$, then f is a mapping, and $(f, M, X, \{\mathscr{P}_i\})$ is a Ponomarevs system. By [20, Lemma 3.3.2], f is a compact mapping.

Next, we shall show that f is an $\mathcal{I}$-covering mapping.

For each $x_0 \in X$, let $\{x_n\}_{n\in\mathbb{N}} \subset X$ be a sequence satisfying $x_n \xrightarrow{\mathcal{I}} x_0 \in X$. Since $\{\mathscr{P}_i\}$ is a point-finite $\mathcal{I}$-*cs*-covers of X, we can choose $\alpha_j \in \Lambda_j$ such that $x_0 \in P_{\alpha_j} \in \mathscr{P}_j$ and $\{n \in \mathbb{N} : x_n \notin P_{\alpha_j}\} \in \mathcal{I}$ for each $j \in \mathbb{N}$. Since $\{\mathscr{P}_i\}$ is a point-star network of X and $P_{\alpha_j} \in (\mathscr{P}_j)_{x_0}$, $\langle P_{\alpha_j}\rangle$ forms a network at x_0 in X. Let $\alpha = (\alpha_j) \in M$. Then $\alpha \in f^{-1}(x_0)$. Now choose a sequence $\{(\alpha_{j,n})\}_{n\in\mathbb{N}}$ in M as follows: Choose $\alpha_{j,n} = \alpha_j$ if $x_n \in P_{\alpha_j}$, otherwise choose $\beta_j \in \Lambda_j$ such that $x_n \in P_{\beta_j}$ so that $\alpha_{j,n} = \beta_j$, for each $j \in \mathbb{N}$. For each neighborhood V_j of α_j in Λ_j, since $\{n \in \mathbb{N} : \alpha_{j,n} \notin V_j\} \subset \{n \in \mathbb{N} : \alpha_{j,n} \neq \alpha_j\} = \{n \in \mathbb{N} : x_n \notin P_{\alpha_j}\} \in \mathcal{I}$, $\alpha_{j,n} \xrightarrow{\mathcal{I}} \alpha_j$, and hence $(\alpha_{j,n}) \xrightarrow{\mathcal{I}} (\alpha_j)$ in M from Lemma 2.13. By the choosing of $(\alpha_{j,n})$, it is easy to see that $P_{\alpha_{j,n}} \in (\mathscr{P}_j)_{x_n}$, hence $\langle P_{\alpha_{j,n}}\rangle$ forms a network at x_n in X, thus $(\alpha_{j,n}) \in f^{-1}(x_n)$ for each $n \in \mathbb{N}$. Therefore, f is an $\mathcal{I}$-covering mapping. □

Definition 4.2 *Let f be a mapping from a topological space X onto a topological space Y. f is said to be $\mathcal{I}$-quotient provided $f^{-1}(U)$ is $\mathcal{I}$-open in X, then U is $\mathcal{I}$-open in Y.*

The following two theorems can be seen in [30].

Theorem 4.3 *Every $\mathcal{I}$-covering mapping is $\mathcal{I}$-quotient.*

Theorem 4.4 *Let f be a mapping from a topological space X onto a topological space Y.*

(1) If X is an $\mathcal{I}$-sequential space and f is continuous quotient, then Y is an $\mathcal{I}$-sequential space and f is $\mathcal{I}$-quotient.

(2) If Y is an $\mathcal{I}$-sequential space and f is $\mathcal{I}$-quotient, then f is quotient.

Corollary 4.5 *The following conditions are equivalent:*

(1) X is an $\mathcal{I}$-covering, quotient and compact mapping image of a metric space;

(2) X is a sequential space and has a point-star network consisting of point-finite $\mathcal{I}$-cs-covers.

Recall the notion of 1-sequence-covering mappings in topological spaces. Let $f: X \to Y$ be a mapping. f is an 1-sequence-covering mapping if for each $y \in Y$, there is $x \in f^{-1}(y)$ such that whenever $\{y_n\}_{n\in\mathbb{N}}$ is a sequence converging to y in Y there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x in X with each $x_n \in f^{-1}(y_n)$ (see [19]). Next, we assume all mappings are continuous.

Definition 4.6 *Let $f: X \to Y$ be a mapping. f is an 1-$\mathcal{I}$-covering mapping if for each $y \in Y$, there is $x \in f^{-1}(y)$ such that whenever $\{y_n\}_{n\in\mathbb{N}}$ is a sequence $\mathcal{I}$-converging to y in Y there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ $\mathcal{I}$-converging to x in X with each $x_n \in f^{-1}(y_n)$.*

Obviously, f is a 1-$\mathcal{I}$-covering mapping, then f is an $\mathcal{I}$-covering mapping.

Proposition 4.7 *Let $f: X \to Y$ and $g: Y \to Z$ be mappings. If f and g are 1-$\mathcal{I}$-covering mappings, then $g \circ f$ is 1-$\mathcal{I}$-covering mapping.*

Theorem 4.8 *Let $f: X \to Y$ be a 1-$\mathcal{I}$-covering mapping. Then for each $y \in Y$, there exists $x \in f^{-1}(y)$ such that whenever U is an open neighborhood of x in X, $f(U)$ is an $\mathcal{I}$-sequential neighborhood of y in Y.*

Proof Let $f: X \to Y$ be a 1-$\mathcal{I}$-covering. Then for each $y \in Y$, there is $x \in f^{-1}(y)$ such that whenever $\{y_n\}_{n\in\mathbb{N}}$ is a sequence $\mathcal{I}$-converging to y in Y there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ $\mathcal{I}$-converging to x in X with each $x_n \in f^{-1}(y_n)$. Let U be an open neighborhood of x. Note that $x_n \xrightarrow{\mathcal{I}} x$, hence $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$, and further $\{n \in \mathbb{N} : y_n \notin f(U)\} = \{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$. Therefore, $f(U)$ is an $\mathcal{I}$-sequential neighborhood of y in Y. $\Box$

Theorem 4.9 *X is a 1-$\mathcal{I}$-covering and compact mapping image of a metric space if and only if X has a point-star network consisting of point-finite $\mathcal{I}$-sn-covers.*

Proof Necessity. The procedure to prove that X has a point-star network $\{\mathscr{P}_i\}$ consisting of a point-finite covers is similar to Theorem 4.1. Next, we shall show that $\{\mathscr{P}_i\}$ is an $\mathcal{I}$-*sn*-covers of X.

Since f is a surjection, there exists $b \in f^{-1}(x)$ satisfying the condition in Theorem 4.8 for each $x \in X$. And since each $\mathscr{B}_i$ is an open cover of X, there exists $B \in \mathscr{B}_i$ such that $b \in B$. Take $P = f(B)$. By Theorem 4.8, P is an $\mathcal{I}$-sequential neighborhood of x. Let $\mathscr{P}_i' = \{P \in \mathscr{P}_i : P$ is an $\mathcal{I}$-sequential neighborhood of some point in $X\}$. Then $\mathscr{P}_i'$ is a point-finite cover of X and $\{\mathscr{P}_i'\}$ is a point-star network consisting of point-finite $\mathcal{I}$-sn-covers of X.

Sufficiency. Let $\{\mathscr{P}_i\}$ be a point-star network consisting of point-finite $\mathcal{I}$-sn-covers of X. And let $(f, M, X, \{\mathscr{P}_i\})$ be a Ponomarevs system. By [20, Lemma 3.3.2], f is a compact mapping.

Next, we shall show that f is a 1-$\mathcal{I}$-covering mapping.

For each $x \in X$, since $\{\mathscr{P}_i\}$ is a point-finite $\mathcal{I}$-sn-covers of X, there exists $P_{\alpha_j} \in \mathscr{P}_j$ such that P_{α_j} is an $\mathcal{I}$-sequential neighborhood of x for each $j \in \mathbb{N}$. And since $\{\mathscr{P}_i\}$ is a point-star network of X and $P_{\alpha_j} \in (\mathscr{P}_j)_x$, $\langle P_{\alpha_j} \rangle$ forms a network at x in X. Let $\alpha = (\alpha_j) \in M$, then $\alpha \in f^{-1}(x)$. And let $\{x_n\}_{n\in\mathbb{N}} \subset X$ be a sequence satisfying $x_n \xrightarrow{\mathcal{I}} x \in X$. Then $\{n \in \mathbb{N} : x_n \notin P_{\alpha_j}\} \in \mathcal{I}$ for each $j \in \mathbb{N}$. Now choose a sequence $\{(\alpha_{j,n})\}_{n\in\mathbb{N}}$ in M as follows: Choose $\alpha_{j,n} = \alpha_j$ if $x_n \in P_{\alpha_j}$, otherwise choose $\beta_j \in \Lambda_j$ such that $x_n \in P_{\beta_j}$ so that $\alpha_{j,n} = \beta_j$, for each $j \in \mathbb{N}$. For each neighborhood V_j of α_j in Λ_j, since $\{n \in \mathbb{N} : \alpha_{j,n} \notin V_j\} \subset \{n \in \mathbb{N} : \alpha_{j,n} \neq \alpha_j\} = \{n \in \mathbb{N} : x_n \notin P_{\alpha_j}\} \in \mathcal{I}$, $\alpha_{j,n} \xrightarrow{\mathcal{I}} \alpha_j$, and hence $(\alpha_{j,n}) \xrightarrow{\mathcal{I}} (\alpha_j)$ in M from Lemma 2.13. By choosing of $(\alpha_{j,n})$, it is easy to see that $P_{\alpha_{j,n}} \in (\mathscr{P}_j)_{x_n}$, hence $\langle P_{\alpha_{j,n}} \rangle$ forms a network at x_n in X, thus $(\alpha_{j,n}) \in f^{-1}(x_n)$ for each $n \in \mathbb{N}$. Therefore, f is a 1-$\mathcal{I}$-covering mapping. $\square$

Question 4.10 Are $\mathcal{I}$-covering and compact mapping images of metric spaces equivalent with 1-$\mathcal{I}$-covering and compact mapping images of metric spaces?

Example 4.11 There exist an admissible ideal $\mathcal{I}$ on $\mathbb{N}$, a topological space X, and two sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ in X, such that $x_n \xrightarrow{\mathcal{I}} x_0$, $y_n \xrightarrow{\mathcal{I}} x_0$, but the sequence $\{x_1, y_1, x_2, y_2, x_3, y_3, \ldots\}$ does not $\mathcal{I}$-converge to x_0.

Proof Let $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$ generated by all subsets of the set of all even positive integers and all finite subsets of the set of all odd positive integers. Let the topological space $(\mathbb{R}, \tau)$ be the set of all real numbers $\mathbb{R}$ endowed with the usual topology τ. Set $x_n = 0$, if n is odd; $x_n = n$, if n is even, $n = 1, 2, 3, \ldots$. And let $y_n = \frac{1}{n}$, $n = 1, 2, 3, \ldots$. It is easy to verify that $x_n \xrightarrow{\mathcal{I}} 0$ and $y_n \xrightarrow{\mathcal{I}} 0$. Now define a sequence $\{x_1, y_1, x_2, y_2, x_3, y_3, \ldots\}$, and denote it by $\{z_n\}_{n\in\mathbb{N}}$. Then, $\{z_n\}_{n\in\mathbb{N}}$ does not $\mathcal{I}$-converge to 0. In fact, if not, then $\{4n-1 : n \in \mathbb{N}\} \subset \{n \in \mathbb{N} : z_n \notin U\} \in \mathcal{I}$, for each neighborhood U of 0 in $\mathbb{R}$. Hence, $\{4n-1 : n \in \mathbb{N}\} \in \mathcal{I}$. But this contradicts the structure of $\mathcal{I}$. $\square$

Acknowledgements Our gratitude goes to Professor Shou LIN, for his friendly encouragement and inspiring suggestions.

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