Journal of Mathematical Research with Applications
Mar., 2020, Vol. 40, No. 2, pp. 165–168
DOI:10.3770/j.issn:2095-2651.2020.02.005
Http://jmre.dlut.edu.cn

On the Regularity Criteria for 3-D Liquid Crystal Flows in Terms of the Horizontal Derivative Components of the Pressure

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Abstract This paper is devoted to investigating regularity criteria for the 3-D nematic liquid crystal flows in terms of horizontal derivative components of the pressure and gradient of the orientation field. More precisely, we mainly proved that the strong solution (u,d) can be extended beyond T, provided that the horizontal derivative components of the pressure $\nabla_h P=(\partial_{x_1}P,\partial_{x_2}P)$ and gradient of the orientation field satisfy

$$\nabla_h P\in L^s(0,T;L^q(\mathbb{R}^3)),\ \frac{2}{s}+\frac{3}{q}\le\frac{5}{2},\ \frac{18}{13}\le q\le 6$$

and

$$\nabla d\in L^\beta(0,T;L^\gamma(\mathbb{R}^3)),\ \frac{2}{\gamma}+\frac{3}{\beta}\le\frac{3}{4},\ \frac{36}{7}\le\beta\le 12.$$

Keywords regularity criteria; nematic liquid crystal

MR(2010) Subject Classification 35B65; 35Q35; 76A15

1. Introduction

We will consider the following problems:

$$\begin{cases} u_t+(u\cdot\nabla)u+\nabla P=\nu\Delta u-\lambda\nabla\cdot(\nabla d\otimes\nabla d),\\ d_t+(u\cdot\nabla)d=\gamma(\Delta d-f(d)),\\ \operatorname{div}u=0,\end{cases}\tag{1.1}$$

with the initial condition

$$u(x,0)=u_0(x),\operatorname{div}u_0=0,d(x,0)=d_0(x),x\in\mathbb{R}^3,\tag{1.2}$$

where u is the velocity field, P is the scalar pressure and d represents the macroscopic molecular orientation field of the liquid crystal materials. $\nabla\cdot$ denotes the divergence operator, and the (i,j)-th entry of $\nabla d\otimes\nabla d$ is given by $\nabla_{x_i}d\cdot\nabla_{x_j}d$ for $1\le i,j\le 3$. In addition, $f(d)=\frac{1}{\eta^2}(|d|^2-1)d$. Since ν, λ, γ and η are positive constants, for simplicity, we assume that they are all one.

In the 1960s, the hydrodynamic theory of liquid crystals was established by Ericksen and Leslie [1,2]. The above system is a simplified approximate version of the Ericksen-Leslie equations

Received July 8, 2019; Accepted October 29, 2019
Supported by the National Natural Science Foundation of China (Grant No. 11571057).
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for liquid crystal flows, and it was first introduced by Lin [3]. Lin and Liu [4] have established a global existence theorem for weak solutions and local well-posed results for classical solutions, which is one of the most significant developments in this field.

When the orientation field d equals a constant, the above equations become the incompressible Navier-Stokes equations. Some regularity results on the solutions to the 3-D Navier-Stokes equations have been well studied [5–8]. For example, it was proved in [5, 6] that the strong solution can not blow up provided that the regularity criteria of a component of the velocity are satisfied. Many regularity extension of the strong solution can be obtained in terms of one directional derivative $\partial_3 u$ of the velocity and some conditions for ∇d, see [9–12] and so on. More interesting results on the regularity criteria for the liquid crystal equations have been established such as [12–14] and the references therein.

In [6], Zhou and Pokorný give a corollary that the solution to Navier-Stokes equations can be regular in terms of one derivative component of the pressure provided

$$\partial_{x_3} P \in L^p(0,T;L^q(\mathbb{R}^3)),\ \frac{2}{p}+\frac{3}{q}<\frac{29}{10},\ \frac{30}{23}<q\le\frac{10}{3}.$$

Motivated by their ideas, we are interested in the regularity criteria for the system (1.1). For the horizontal derivative components of the pressure, we obtain the following result.

Theorem 1.1 *Let $u_0\in H^1(\mathbb{R}^3), d_0\in H^2(\mathbb{R}^3)$, (u,d) be a strong solution of (1.1)-(1.2) on $[0,T)$ for some $0<T<\infty$. Then (u,d) can be extended beyond T, provided that*

$$\nabla_h P\in L^s(0,T;L^q(\mathbb{R}^3)),\ \frac{2}{s}+\frac{3}{q}\le\frac{5}{2},\ \frac{18}{13}\le q\le 6, \tag{1.3}$$

and

$$\nabla d\in L^\beta(0,T;L^\gamma(\mathbb{R}^3)),\ \frac{2}{\gamma}+\frac{3}{\beta}\le\frac{3}{4},\ \frac{36}{7}\le\beta\le 12. \tag{1.4}$$

2. Main result

Let $u_h=(u_1,u_2)$ denote the horizontal velocity components, and we know the strong solutions to 3D liquid crystal equations (1.1) and (1.2) are regular in terms of two velocity components by [12].

Lemma 2.1 ([12]) *Let $u_0\in H^1(\mathbb{R}^3), d_0\in H^2(\mathbb{R}^3)$, (u,d) be a strong solution of (1.1) and (1.2) on $[0,T)$ for some $0<T<\infty$. Then (u,d) can be extended beyond T, provided that*

$$u_h\in L^s(0,T;L^q(\mathbb{R}^3)),\ \frac{2}{s}+\frac{3}{q}\le\frac{1}{2},\ 6\le q\le\infty. \tag{2.1}$$

In the following we will give the proof of Theorem 1.1.

Proof of Theorem 1.1 Firstly, for convenience, we assume the values of ν,λ take one, considering the equation that u_h satisfies

$$\frac{\partial u_h}{\partial t}+(u\cdot\nabla)u_h+\nabla_h P=\Delta u_h-\nabla\cdot(\nabla_h d\otimes\nabla d). \tag{2.2}$$

Multiplying (2.2) by $|u_h|^{p-2}u_h$, we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{R^3}|u_h|^p\mathrm{d}x+C(p)\int_{R^3}|\nabla|u_h|^{\frac{p}{2}}|^2\mathrm{d}x\\
&=-\int_{R^3}\nabla_h P|u_h|^{p-2}u_h\mathrm{d}x-\int_{R^3}\nabla\cdot(\nabla_h d\otimes\nabla d)|u_h|^{p-2}u_h\mathrm{d}x\\
&=I+II.
\end{aligned}$$

For the term I, we have

$$\begin{aligned}
I&=-\int_{\mathbb{R}^3}\nabla_h P|u_h|^{p-2}u_h\mathrm{d}x\le\int_{\mathbb{R}^3}|\nabla_h P||u_h|^{p-1}\mathrm{d}x\\
&\le\|\nabla_h P\|_q\|u_h\|^{p-1}_{\frac{(p-1)q}{q-1}}\le C\|\nabla_h P\|_q\|u_h\|_p^{\frac{2pq-3p+q}{2q}}\|u_h\|_{3p}^{\frac{3(p-q)}{2q}}\\
&\le\epsilon\|u_h\|_{3p}^p+C(\epsilon)\|\nabla_h P\|_q^{\frac{2pq}{2pq+3q-3p}}\|u_h\|_p^{p\frac{2pq+q-3p}{2pq+3q-3p}},
\end{aligned}$$

where $\frac{3p}{2p+1}\le q\le p$.

For the last term, we get

$$\begin{aligned}
II&=-\int_{\mathbb{R}^3}\nabla\cdot(\nabla_h d\otimes\nabla d)|u_h|^{p-2}u_h\mathrm{d}x=\int_{\mathbb{R}^3}(\nabla_h d\otimes\nabla d)\nabla(|u_h|^{p-2}u_h)\mathrm{d}x\\
&\le C\int_{\mathbb{R}^3}|\nabla d|^2|\nabla|u_h|^{\frac{p}{2}}||u_h|^{\frac{p}{2}-1}\mathrm{d}x\\
&\le C\||\nabla d|^2\|_\alpha\|\nabla|u_h|^{\frac{p}{2}}\|_2\||u_h|^{\frac{p}{2}-1}\|_{\frac{2\alpha}{\alpha-2}}\\
&\le C\||\nabla d|^2\|_\alpha\|\nabla|u_h|^{\frac{p}{2}}\|_2\|u_h\|_p^{\frac{p\alpha-3p+\alpha}{2\alpha}}\|u_h\|_{3p}^{\frac{3p-3\alpha}{2\alpha}}\\
&\le\epsilon\|\nabla|u_h|^{\frac{p}{2}}\|_2^2+\epsilon\|u_h\|_{3p}^p+C(\epsilon)\|\nabla d\|_{2\alpha}^{\frac{4\alpha p}{\alpha p-3p+3\alpha}}\|u_h\|_p^{p\frac{p\alpha-3p+\alpha}{p\alpha-3p+3\alpha}},
\end{aligned}$$

where $\frac{3p}{p+1}\le\alpha\le p$.

Consequently, we obtain

$$\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|u_h\|_p^p&\le C\|\nabla_h P\|_q^{\frac{2pq}{2pq+3q-3p}}\|u_h\|_p^{p\frac{2pq+q-3p}{2pq+3q-3p}}+C\|\nabla d\|_{2\alpha}^{\frac{4\alpha p}{\alpha p-3p+3\alpha}}\|u_h\|_p^{p\frac{p\alpha-3p+\alpha}{p\alpha-3p+3\alpha}}\\
&\le C(\|\nabla_h P\|_q^{\frac{2pq}{2pq+3q-3p}}+\|\nabla d\|_{2\alpha}^{\frac{4\alpha p}{\alpha p-3p+3\alpha}})(1+\|u_h\|_p^p).
\end{aligned}$$

Consider

$$\nabla_h P\in L^s(0,T;L^q(\mathbb{R}^3)),\quad\nabla d\in L^\gamma(0,T;L^\beta(\mathbb{R}^3))$$

then it follows from the Gronwall's inequality that $\sup_t\|u_h\|_p<\infty$ if

$$\frac{3}{q}+\frac{2}{s}=2+\frac{3}{p},$$

and

$$\frac{3}{\beta}+\frac{2}{\gamma}=\frac{1}{2}+\frac{3}{2p}.$$

Combining the result of Lemma 2.1 for $p=6$, we get the strong solution on $(0,T)$ can be extended if

$$\frac{18}{13}\le q\le 6,\quad\frac{36}{7}\le\beta\le 12.$$

Then, the proof is completed. □

Acknowledgements We thank the referees for their time and comments.

References

[1] J. L. ERICKSEN. *Hydrostatic theory of liquid crystals*. Arch. Rational Mech. Anal., 1962, **9**: 371–378.

[2] F. M. LESLIE. *Some constitutive equations for liquid crystals*. Arch. Rational Mech. Anal., 1968, **28**(4): 265–283.

[3] Fanghua LIN. *Nonlinear theory of defects in nematic liquid crystals; phase transition and flow phenomena*. Comm. Pure Appl. Math., 1989, **42**: 789–814.

[4] Fanghua LIN, Chun LIU. *Partial regularity of the dynamic system modeling the flow of liquid crystals*. Discrete Contin. Dynam. Systems, 1996, **2**(1): 1–22.

[5] Chongsheng CAO, E. S. TITI. *Regularity criteria for the three-dimensional Navier-Stokes equations*. Indiana Univ. Math. J., 2008, **57**(6): 2643–2661.

[6] Yong ZHOU, M. POKORNÝ. *On the regularity of the solutions of the Navier-Stokes equations via one velocity component*. Nonlinearity, 2010, **23**(5): 1097–1107.

[7] Yong ZHOU. *On a regularity criterion in terms of the gradient of pressure for the Navier-Stokes equations in R^N*. Z. Angew. Math. Phys., 2006, **57**(3): 384–392.

[8] Yong ZHOU. *On regularity criteria in terms of pressure for the Navier-Stokes equations in R^3*. Proc. Amer. Math. Soc., 2006,**134**(1): 149–156.

[9] Qiao LIU, Jihong ZHAO, Shangbin CUI. *A regularity criterion for the three-dimensional nematic liquid crystal flow in terms of one directional derivative of the velocity*. J. Math. Phys., 2011, **52**(3): 1–8.

[10] Ruiying WEI, Yin LI, Zheng-an YAO. *Two new regularity criteria for nematic liquid crystal flows*. J. Math. Anal. Appl., 2015, **424**(1): 636–650.

[11] Lingling ZHAO, Fengquan LI. *On the regularity criteria for liquid crystal flows*. Z. Angew. Math. Phys., 2018, **69**(5): Art. 125, 13 pp.

[12] Lingling ZHAO, Wendong WANG, Suyu WANG. *Blow-up criteria for the 3D liquid crystal flows involving two velocity components*. Appl. Math. Lett., 2019, **96**: 75–80.

[13] Xiangao LIU, Jianzhong MIN, Kui WANG, et al. *Serrin's regularity results for the incompressible liquid crystals system*. Discrete Contin. Dyn. Syst., 2016, **36**(10): 5579–5594.

[14] Chenyin QIAN. *Remarks on the regularity criterion for the nematic liquid crystalflows in R^3*. Appl. Math. Comput., 2016, **274**: 679–689.