On the Regularity Criteria for 3-D Liquid Crystal Flows in Terms of the Horizontal Derivative Components of the Pressure

Lingling ZHAO*, Fengquan LI
School of Mathematical Sciences, Dalian University of Technology, Liaoning 116024, P. R. China

Abstract This paper is devoted to investigating regularity criteria for the 3-D nematic liquid crystal flows in terms of horizontal derivative components of the pressure and gradient of the orientation field. More precisely, we mainly proved that the strong solution $(u, d)$ can be extended beyond $T$, provided that the horizontal derivative components of the pressure $\nabla_h P = (\partial_{x_1} P, \partial_{x_2} P)$ and gradient of the orientation field satisfy

$$\nabla_h P \in L^s(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{s} + \frac{3}{q} \leq \frac{5}{2} \cdot \frac{18}{13} \leq q \leq 6$$

and

$$\nabla d \in L^\beta(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{\gamma} + \frac{3}{\beta} \leq \frac{3}{4}, \quad \frac{36}{7} \leq \beta \leq 12.$$ 

Keywords regularity criteria; nematic liquid crystal

MR(2010) Subject Classification 35B65; 35Q35; 76A15

1. Introduction

We will consider the following problems:

$$\begin{cases}
  u_t + (u \cdot \nabla) u + \nabla P = \nu \Delta u - \lambda \nabla \cdot (\nabla d \otimes \nabla d), \\
  d_t + (u \cdot \nabla) d = \gamma (\Delta d - f(d)), \\
  \text{div} u = 0,
\end{cases}\tag{1.1}$$

with the initial condition

$$u(x, 0) = u_0(x), \text{div} u_0 = 0, d(x, 0) = d_0(x), x \in \mathbb{R}^3,\tag{1.2}$$

where $u$ is the velocity field, $P$ is the scalar pressure and $d$ represents the macroscopic molecular orientation field of the liquid crystal materials. $\nabla \cdot$ denotes the divergence operator, and the $(i, j)$-th entry of $\nabla d \otimes \nabla d$ is given by $\nabla_{x_i} d \cdot \nabla_{x_j} d$ for $1 \leq i, j \leq 3$. In addition, $f(d) = \frac{1}{\eta^2} (|d|^2 - 1)d$. Since $\nu, \lambda, \gamma$ and $\eta$ are positive constants, for simplicity, we assume that they are all one.

In the 1960s, the hydrodynamic theory of liquid crystals was established by Ericksen and Leslie [1,2]. The above system is a simplified approximate version of the Ericksen-Leslie equations...
for liquid crystal flows, and it was first introduced by Lin [3]. Lin and Liu [4] have established a global existence theorem for weak solutions and local well-posed results for classical solutions, which is one of the most significant developments in this field.

When the orientation field \( d \) equals a constant, the above equations become the incompressible Navier-Stokes equations. Some regularity results on the solutions to the 3-D Navier-Stokes equations have been well studied [5–8]. For example, it was proved in [5, 6] that the strong solution can not blow up provided that the regularity criteria of a component of the velocity are satisfied. Many regularity extension of the strong solution can be obtained in terms of one directional derivative \( \partial_3 u \) of the velocity and some conditions for \( \nabla d \), see [9–12] and so on. More interesting results on the regularity criteria for the liquid crystal equations have been established such as [12–14] and the references therein.

In [6], Zhou and Pokorný give a corollary that the solution to Navier-Stokes equations can be regular in terms of one derivative component of the pressure provided
\[
\partial x_3 P \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} < \frac{29}{10}, \quad \frac{30}{23} < q \leq \frac{10}{3}.
\]

Motivated by their ideas, we are interested in the regularity criteria for the system (1.1). For the horizontal derivative components of the pressure, we obtain the following result.

**Theorem 1.1** Let \( u_0 \in H^1(\mathbb{R}^3), d_0 \in H^2(\mathbb{R}^3), (u, d) \) be a strong solution of (1.1)-(1.2) on \( [0, T) \) for some \( 0 < T < \infty \). Then \( (u, d) \) can be extended beyond \( T \), provided that
\[
\nabla P \in L^s(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{s} + \frac{3}{q} \leq \frac{5}{2}, \quad \frac{18}{13} \leq q \leq 6,
\]
and
\[
\nabla d \in L^\beta(0, T; L^\gamma(\mathbb{R}^3)), \quad \frac{2}{\gamma} + \frac{3}{\beta} \leq \frac{3}{4}, \quad \frac{36}{7} \leq \beta \leq 12.
\]

2. Main result

Let \( u_h = (u_1, u_2) \) denote the horizontal velocity components, and we know the strong solutions to 3D liquid crystal equations (1.1) and (1.2) are regular in terms of two velocity components by [12].

**Lemma 2.1** ([12]) Let \( u_0 \in H^1(\mathbb{R}^3), d_0 \in H^2(\mathbb{R}^3), (u, d) \) be a strong solution of (1.1) and (1.2) on \( [0, T) \) for some \( 0 < T < \infty \). Then \( (u, d) \) can be extended beyond \( T \), provided that
\[
u_h \in L^s(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{s} + \frac{3}{q} \leq \frac{1}{2}, \quad 6 \leq q \leq \infty.
\]

In the following we will give the proof of Theorem 1.1.

**Proof of Theorem 1.1** Firstly, for convenience, we assume the values of \( \nu, \lambda \) take one, considering the equation that \( u_h \) satisfies
\[
\frac{\partial u_h}{\partial t} + (u \cdot \nabla) u_h + \nabla P = \Delta u_h - \nabla \cdot (\nabla_h d \otimes \nabla d).
\]
On the regularity criteria for 3-D liquid crystal flows

Multiplying (2.2) by $|u_h|^{p-2}u_h$, we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^3} |u_h|^p \, dx + C(p) \int_{\mathbb{R}^3} |\nabla |u_h||^2 \, dx$$

$$= - \int_{\mathbb{R}^3} \nabla_h P |u_h|^{p-2} u_h \, dx - \int_{\mathbb{R}^3} \nabla \cdot (\nabla_h d \otimes \nabla d) |u_h|^{p-2} u_h \, dx$$

$$= I + II.$$  

For the term $I$, we have

$$I = - \int_{\mathbb{R}^3} \nabla_h P |u_h|^{p-2} u_h \, dx \leq \int_{\mathbb{R}^3} |\nabla_h P| |u_h|^{p-1} \, dx$$

$$\leq \|\nabla_h P\|_q \|u_h\|_{\frac{p-1}{q}} \leq C \|\nabla_h P\|_q \|u_h\|_p^{\frac{2p-3q+q}{2q}} \|u_h\|_3^{\frac{3(p-q)}{2q}}$$

$$\leq \epsilon \|u_h\|_p^{3p} + C(\epsilon) \|\nabla_h P\|_q^{\frac{2p}{2p-3q+q}} \|u_h\|_p^{\frac{2p}{2p-3q+q}},$$

where $\frac{3p}{2p+1} \leq q \leq p$.

For the last term, we get

$$II = - \int_{\mathbb{R}^3} \nabla \cdot (\nabla_h d \otimes \nabla d) |u_h|^{p-2} u_h \, dx = \int_{\mathbb{R}^3} (\nabla_h d \otimes \nabla d) \nabla (|u_h|^{p-2} u_h) \, dx$$

$$\leq C \int_{\mathbb{R}^3} |\nabla d|^2 |\nabla |u_h||^2 \|u_h\|^{p-1} \, dx$$

$$\leq C \|\nabla d\|_2^2 \|\nabla |u_h||^2 \|u_h\|^{p-1} \|u_h\|_{\frac{3p-3\alpha}{3p}}$$

$$\leq C \|\nabla d\|_2^2 \|\nabla |u_h||^2 \|u_h\|_{\frac{3p-3\alpha}{3p}} \|u_h\|_{\frac{3p-3\alpha}{3p}} + C(\epsilon) \|\nabla d\|_2^{\frac{4\alpha p}{4\alpha p + 3} - \frac{2}{3}} \|u_h\|_{\frac{p}{p-3p+\alpha}}^{\frac{p\alpha - 3p + \alpha}{p-3p + \alpha}},$$

where $\frac{3p}{2p+1} \leq \alpha \leq p$.

Consequently, we obtain

$$\frac{d}{dt} \|u_h\|_p^p \leq C \|\nabla_h P\|_q^{\frac{2pq}{2pq - 3q + 3p}} \|u_h\|_p^{\frac{2p + q - 3p}{2p + q - 3p}} \|u_h\|_{\frac{3p - 3\alpha}{3p}}^{\frac{3p - 3\alpha}{3p}}$$

$$\leq C \|\nabla_h P\|_q^{\frac{2pq}{2pq - 3q + 3p}} \|\nabla d\|_2^{\frac{4\alpha p}{4\alpha p + 3} - \frac{2}{3}} (1 + \|u_h\|_{\frac{p}{p-3p+\alpha}}^p).$$

Consider

$$\nabla_h P \in L^s(0, T; L^4(\mathbb{R}^3)), \quad \nabla d \in L^\gamma(0, T; L^\beta(\mathbb{R}^3))$$

then it follows from the Gronwall’s inequality that $\sup_t \|u_h\|_p < \infty$ if

$$\frac{3}{q} + \frac{2}{s} = 2 + \frac{3}{p},$$

and

$$\frac{3}{\beta} + \frac{2}{\gamma} = 1 + \frac{3}{2p}.$$
Then, the proof is completed. □

Acknowledgements  We thank the referees for their time and comments.

References