Minimal Critical Sets of Refined Inertias for Irreducible Sign Patterns of Order 3

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Abstract Let $S$ be a nonempty, proper subset of all possible refined inertias of real matrices of order $n$. The set $S$ is a critical set of refined inertias for irreducible sign patterns of order $n$, if for each $n \times n$ irreducible sign pattern $A$, the condition $S \subseteq ri(A)$ is sufficient for $A$ to be refined inertially arbitrary. If no proper subset of $S$ is a critical set of refined inertias, then $S$ is a minimal critical set of refined inertias for irreducible sign patterns of order $n$.

All minimal critical sets of refined inertias for full sign patterns of order 3 have been identified in [Wei GAO, Zhongshan LI, Lihua ZHANG, The minimal critical sets of refined inertias for $3 \times 3$ full sign patterns, Linear Algebra Appl. 458(2014), 183–196]. In this paper, the minimal critical sets of refined inertias for irreducible sign patterns of order 3 are identified.

Keywords sign pattern; refined inertia; refined inertially arbitrary sign pattern; critical set of refined inertias

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1. Introduction

An $n \times n$ matrix $A$ is called a sign pattern if its entries are from the set $\{+, -, 0\}$. For a real matrix $B$, $\text{sgn}(B)$ is the sign pattern matrix obtained by replacing each positive (resp., negative, zero) entry of $B$ by $+$ (resp., $-$, 0). The set of all real matrices with the same sign pattern as the $n \times n$ sign pattern $A$ is the qualitative class

$$Q(A) = \{B = [b_{ij}] \in M_n(R) | \text{sgn}(B) = A\}.$$ 

A subpattern of an $n \times n$ sign pattern $A$ is a sign pattern $B$ obtained by replacing some (possible empty) subset of the nonzero entries of $A$ with zero. If $B$ is a subpattern of $A$, then $A$ is a superpattern of $B$.

Let $A$ be a real matrix of order $n$. The inertia of $A$ is the ordered triple $i(A) = (n_+, n_-, n_0)$, where $n_+, n_-$ and $n_0$ are the numbers of its eigenvalues (counting multiplicities) with positive, negative and zero real parts, respectively. The refined inertia of $A$ is the ordered quadruple $ri(A) = (n_+, n_-, n_z, 2n_p)$ of nonnegative integers that sum to $n$, in which $(n_+, n_-, n_z + 2n_p)$ is

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the inertia of $A$ while $n_z$ is the number of 0 as an eigenvalue of $A$ and $2n_p$ is the number of nonzero pure imaginary eigenvalues of $A$.

For an $n \times n$ sign pattern $A$, the inertia of $A$ is $i(A) = \{ i(B) | B \in Q(A) \}$, and the refined inertia of $A$ is $ri(A) = \{ ri(B) | B \in Q(A) \}$.

The reversal of an inertia (resp., refined inertia) is obtained by exchanging the first two entries in the ordered triple (resp., quadruple), i.e., the reversal of $(n_+, n_-, n_0)$ (resp., $(n_+, n_-, n_z, 2n_p)$) is $(n_-, n_+, n_0)$ (resp., $(n_-, n_+, n_z, 2n_p)$). The reversal of a set of inertias (resp., refined inertias) is the set of reversals of the inertias (resp., refined inertias) in the set. Clearly, for an $n \times n$ sign pattern $A$, $i(-A)$ is the reversal of $i(A)$ and $ri(-A)$ is the reversal of $ri(A)$.

An $n \times n$ sign pattern $A$ is called a spectrally arbitrary pattern (SAP) if for each real monic polynomial $r(x)$ of degree $n$, there exists some $A \in Q(A)$ with characteristic polynomial $p_A(x) = r(x)$. Thus, $A$ is spectrally arbitrary, if given any self-conjugate spectrum, there exists $A \in Q(A)$ with that spectrum [1].

An $n \times n$ sign pattern $A$ is called an inertially arbitrary pattern (IAP) if given any ordered triple $(n_+, n_-, n_0)$ of nonnegative integers with $n_+ + n_- + n_0 = n$, there exists a real matrix $A \in Q(A)$ such that $i(A) = (n_+, n_-, n_0)$. Similarly, $A$ is a refined inertially arbitrary pattern (rIAP) if given any ordered quadruple $(n_+, n_-, n_z, 2n_p)$ of nonnegative integers that sum to $n$, there exists a real matrix $A \in Q(A)$ such that $ri(A) = (n_+, n_-, n_z, 2n_p)$ (see [2, 3]).

Let $S$ be a nonempty, proper subset of all possible refined inertias of real matrices of order $n$. Then, $S$ is a critical set of refined inertias for irreducible sign patterns of order $n$, if for each $n \times n$ irreducible sign pattern $A$, the condition $S \subseteq ri(A)$ is sufficient for $A$ to be refined inertially arbitrary.

If no proper subset of $S$ is a critical set of refined inertias for irreducible sign patterns of order $n$, then $S$ is a minimal critical set of refined inertias for irreducible sign patterns of order $n$.

A permutation sign pattern is a square sign pattern with entries 0 and +, where the entry + occurs precisely once in each row and in each column. A signature sign pattern is a square diagonal sign pattern each of whose diagonal entries is nonzero. Let $A$ and $B$ be two square sign patterns of the same order. We say that $A$ is permutationally similar to $B$ if there exists a permutation sign pattern $P$ such that $B = P^T A P$, and that $A$ is signature similar to $B$ if there exists a signature sign pattern $D$ such that $B = DAD$.

Two square sign patterns $A$ and $B$ of the same order are equivalent if one can be obtained from the other by any combination of negation, transposition, permutation similarity and signature similarity. Clearly, if $A$ and $B$ are equivalent, then $A$ is an rIAP (resp., IAP) if and only if $B$ is an rIAP (resp., IAP).

Let $A = [a_{ij}]$ be an $n \times n$ sign pattern. We say that $A$ contains a negative 2-cycle (resp., positive 2-cycle) if $a_{ij}a_{ji} = -$ (resp., $a_{ij}a_{ji} = +$) for some $i \neq j$.

Recently, Kim et al. [4] have obtained the minimal critical sets of inertias for irreducible zero-nonzero patterns of order $n = 2, 3, 4$ and for irreducible sign patterns of orders $n = 2, 3$. Yu et al. [5] have given all the minimal critical sets of refined inertias and inertias for irreducible
zero-nonzero patterns of order 2 and 3. Also, Yu [6] has identified all the minimal critical sets of refined inertias for irreducible sign patterns of orders 2. All minimal critical sets of refined inertias for full sign patterns of order 3 have been identified [7]. Identifying all minimal critical sets of refined inertias and inertias for irreducible sign patterns that have at least one zero entry has been posed as an open question in [7]. The minimum cardinality of such a set is also open. In this paper, the minimal critical sets of refined inertias and the minimal critical sets of inertias for irreducible sign patterns of order 3 with at least one zero entry are identified.

The main results are the following two theorems.

**Theorem 1.1** The only minimal critical sets of refined inertias for $3 \times 3$ irreducible sign patterns with at least one zero entry are the following sets and their reversals.

\[
\{(3,0,0,0),(0,3,0,0)\}, \{(3,0,0,0),(0,2,1,0)\}, \{(3,0,0,0),(0,1,2,0)\}, \{(3,0,0,0),(0,1,0,2)\}, \\
\{(3,0,0,0),(0,0,3,0)\}, \{(3,0,0,0),(0,0,1,2)\}, \{(2,0,1,0),(0,2,1,0)\}, \{(2,0,1,0),(0,1,2,0)\}, \\
\{(2,0,1,0),(0,1,0,2)\}, \{(1,0,2,0),(0,0,3,0)\}, \{(1,0,2,0),(0,0,1,2)\}, \{(1,0,2,0),(0,1,2,0)\}, \\
\{(1,0,2,0),(0,1,0,2)\}, \{(1,0,2,0),(0,0,3,0)\}, \{(1,0,2,0),(0,0,1,2)\}, \{(1,0,2,0),(0,1,2,0)\}, \{(1,0,2,0),(0,1,0,2)\}.
\]

**Theorem 1.2** The only minimal critical sets of inertias for $3 \times 3$ irreducible sign patterns with at least one zero entry are the following sets and their reversals.

\[
\{(3,0,0),(0,3,0)\}, \{(3,0,0),(0,2,1)\}, \{(3,0,0),(0,1,2)\}, \\
\{(3,0,0),(0,0,3)\}, \{(2,0,1),(0,2,1)\}, \{(2,0,1),(0,1,2)\}, \\
\{(2,0,1),(0,0,3)\}, \{(1,0,2),(0,1,2)\}, \{(1,0,2),(0,0,3)\}.
\]

The followings are immediate from Theorems 1.1, 1.2 and results of the reference [7].

**Theorem 1.3** The only minimal critical sets of refined inertias for $3 \times 3$ irreducible sign patterns are the following sets and their reversals.

\[
\{(3,0,0,0),(0,3,0,0)\}, \{(3,0,0,0),(0,2,1,0)\}, \{(3,0,0,0),(0,1,2,0)\}, \\
\{(3,0,0,0),(0,1,0,2)\}, \{(2,0,1,0),(0,2,1,0)\}, \{(2,0,1,0),(0,1,2,0)\}, \\
\{(2,0,1,0),(0,1,0,2)\}, \{(1,0,2,0),(0,1,2,0)\}, \{(1,0,2,0),(0,1,0,2)\}.
\]

**Theorem 1.4** The only minimal critical sets of inertias for $3 \times 3$ irreducible sign patterns are the following sets and their reversals.

\[
\{(3,0,0),(0,3,0)\}, \{(3,0,0),(0,2,1)\}, \{(3,0,0),(0,1,2)\}, \\
\{(2,0,1),(0,2,1)\}, \{(2,0,1),(0,1,2)\}, \{(1,0,2),(0,1,2)\}.
\]

We will give the proofs of Theorems 1.1 and 1.2 in Sections 3 and 4, respectively.

2. Preliminaries

In this section, we outline some results which are well known for the characterization of $3 \times 3$
Minimal critical sets of refined inertias for irreducible sign patterns of order 3

Lemma 2.1 ([8]) If \( A \) is a sign pattern of order 3, then the following statements are equivalent:

1. \( A \) is spectrally arbitrary.
2. \( A \) is inertially arbitrary.
3. \( A \) is refined inertially arbitrary.
4. Up to equivalence, \( A \) is a superpattern of one of the following sign pattern:

\[
D_3 = \begin{bmatrix}
- & + & 0 \\
0 & - & + \\
0 & 0 & +
\end{bmatrix},
\quad D_2 = \begin{bmatrix}
- & + & 0 \\
0 & - & 0 \\
0 & 0 & +
\end{bmatrix},
\quad U = \begin{bmatrix}
- & + & 0 \\
- & 0 & + \\
0 & 0 & -
\end{bmatrix},
\quad V = \begin{bmatrix}
- & 0 & + \\
0 & - & 0 \\
0 & 0 & -
\end{bmatrix}.
\]

Lemma 2.2 ([8, 9]) Let

\[
G = \begin{bmatrix}
- & + & + & - \\
- & + & - & +
\end{bmatrix}.
\]

Then \( G \) requires a positive eigenvalue.

Lemma 2.3 ([8]) If \( \varphi \) is a subpattern of \( G \), then \( \varphi \) requires a nonnegative eigenvalue.

Lemma 2.4 ([3]) Let \( m \) be the maximum number of distinct refined inertias allowed by any sign pattern of order 3. Then \( m = 13 \).

3. The minimal critical sets of refined inertias for irreducible sign patterns of order 3

In this section, we identify the minimal critical sets of refined inertias for \( 3 \times 3 \) irreducible sign patterns with at least one zero entry.

By Lemma 2.4, there are 13 possible distinct refined inertias for a sign pattern of order 3. We use \( R \) to denote the set of these 13 possible distinct refined inertias, that is,

\[
R = \{(3,0,0,0),(2,1,0,0),(2,0,1,0),(1,2,0,0),(1,1,1,0),(1,0,2,0),
\quad (1,0,0,2),(0,3,0,0),(0,2,1,0),(0,1,2,0),(0,1,0,2),(0,0,3,0),(0,0,1,2)\}.
\]

Let \( A \) be an \( n \times n \) sign pattern which is not an rIAP. We use \( R(A) \) to denote the set of all possible refined inertias that are not in \( ri(A) \), that is,

\[
R(A) = R \setminus ri(A) = \{(n_+,n_-,n_z,2n_p) \in \mathbb{Z}_+^4|n_++n_++n_z+2n_p = n_+n_-n_z2n_p \not\in ri(A)\},
\]

where \( \mathbb{Z}_+ \) is the set of all nonnegative integers.

For convenience, write

\[
R_0 = \{(0,0,3,0),(0,0,1,2)\},
\quad R_1 = \{(3,0,0,0),(2,0,1,0),(1,0,2,0),(1,0,0,2)\},
\quad R'_1 = \{(0,3,0,0),(0,2,1,0),(0,1,2,0),(0,1,0,2)\}.
\]
where $R'_1$ is the reversal of $R_1$. Let

$$\mathcal{P}_{13} = \begin{bmatrix} 0 & 0 & + \\ 0 & + & 0 \\ + & 0 & 0 \end{bmatrix}, \mathcal{P}_{23} = \begin{bmatrix} + & 0 & 0 \\ 0 & 0 & + \\ 0 & + & 0 \end{bmatrix}, \mathcal{P}_{12} = \begin{bmatrix} 0 & + & 0 \\ + & 0 & 0 \\ 0 & 0 & + \end{bmatrix},$$

and $\mathcal{D}_1 = \text{diag}(-, +, +), \mathcal{D}_2 = \text{diag}(+, +, +), \mathcal{D}_3 = \text{diag}(+, +, -)$.

**Lemma 3.1** Let $\mathcal{A}$ be a $3 \times 3$ irreducible sign pattern with at least one zero entry. Suppose all diagonal entries of $\mathcal{A}$ are nonzero, and the diagonal entries of $\mathcal{A}$ have different signs. If $\mathcal{A}$ is not an rIAP, then one of the following conditions holds.

1. $R_0 \cup R_1 \subseteq R(\mathcal{A})$;
2. $R_0 \cup R'_1 \subseteq R(\mathcal{A})$.

**Proof** Up to equivalence, we can assume that $a_{11} = -$, $a_{22} = +$ and $a_{13} = +$. Note that $\mathcal{A}$ is irreducible (this means $\mathcal{A}$ has at most three zero entries), and has at least one zero entry. We consider the following three cases.

Case 1. Exactly one off-diagonal entry of $\mathcal{A}$ is zero.

Up to equivalence, $\mathcal{A}$ has the following forms.

$$\begin{align*}
(1.1) & \quad \begin{bmatrix} - & * & * \\ * & + & * \\ 0 & * & + \end{bmatrix}, \\
(1.2) & \quad \begin{bmatrix} - & * & * \\ * & + & * \\ 0 & 0 & + \end{bmatrix}, \\
(1.1)' & \quad \begin{bmatrix} - & * & * \\ 0 & + & * \\ * & * & + \end{bmatrix},
\end{align*}$$

where $* \in \{+, -\}$. Noting that $\mathcal{P}_{23}(1.1)' \mathcal{P}_{23} = (1.1)$, so $(1.1)'$ and $(1.1)$ are equivalent.

Let $\mathcal{A} = [a_{ij}]$ have form $(1.1)$. If $a_{12} < 0$, taking $\mathcal{A}' = \mathcal{D}_2 \mathcal{A} \mathcal{D}_2$, then $\mathcal{A}'$ and $\mathcal{A}$ are equivalent and the $(1, 2)$ entry of $\mathcal{A}'$ is positive. If $a_{13} < 0$, taking $\mathcal{A}'' = \mathcal{D}_3 \mathcal{A} \mathcal{D}_3$, then $\mathcal{A}''$ and $\mathcal{A}$ are equivalent and the $(1, 3)$ entry of $\mathcal{A}''$ is positive. So, without loss of generality, we can take $a_{12} = a_{13} = +$.

According to the number of negative 2-cycles, $\mathcal{A}$ is possibly one of the following sign patterns.

$$\begin{align*}
\mathcal{A}_1 & = \begin{bmatrix} - & + & + \\ + & + & + \\ 0 & + & + \end{bmatrix}, \\
\mathcal{A}_2 & = \begin{bmatrix} - & + & + \\ + & + & - \\ 0 & - & + \end{bmatrix}, \\
\mathcal{A}_3 & = \begin{bmatrix} - & + & + \\ - & + & + \\ 0 & + & + \end{bmatrix}, \\
\mathcal{A}_4 & = \begin{bmatrix} - & + & + \\ - & - & - \\ 0 & - & - \end{bmatrix}, \\
\mathcal{A}_5 & = \begin{bmatrix} - & + & + \\ + & + & + \\ 0 & - & + \end{bmatrix}, \\
\mathcal{A}_6 & = \begin{bmatrix} - & + & + \\ + & + & - \\ 0 & + & - \end{bmatrix}, \\
\mathcal{A}_7 & = \begin{bmatrix} - & + & + \\ - & - & + \\ 0 & + & + \end{bmatrix}, \\
\mathcal{A}_8 & = \begin{bmatrix} - & + & + \\ - & + & - \\ 0 & + & + \end{bmatrix}, \\
\mathcal{A}_9 & = \begin{bmatrix} - & + & + \\ + & + & + \\ + & + & + \end{bmatrix}, \\
\mathcal{A}_{10} & = \begin{bmatrix} - & + & + \\ + & - & + \\ + & + & + \end{bmatrix}, \\
\mathcal{A}_{11} & = \begin{bmatrix} - & + & + \\ - & + & + \\ + & + & + \end{bmatrix}, \\
\mathcal{A}_{12} & = \begin{bmatrix} - & + & + \\ - & - & + \\ + & + & + \end{bmatrix}.
\end{align*}$$

Let $\mathcal{A} = [a_{ij}]$ have form $(1.2)$. Without loss of generality, we can take $a_{12} = a_{13} = +$. According to the number of negative 2-cycles, $\mathcal{A}$ is possibly one of the following sign patterns.

$$\begin{align*}
\mathcal{A}_9 & = \begin{bmatrix} - & + & + \\ + & + & + \\ + & + & + \end{bmatrix}, \\
\mathcal{A}_{10} & = \begin{bmatrix} - & + & + \\ + & + & - \\ + & + & + \end{bmatrix}, \\
\mathcal{A}_{11} & = \begin{bmatrix} - & + & + \\ - & + & + \\ + & 0 & + \end{bmatrix}, \\
\mathcal{A}_{12} & = \begin{bmatrix} - & + & + \\ - & - & + \\ + & 0 & + \end{bmatrix}.
\end{align*}$$
We may also assume that 

\[ a, b, c, d, e > 0. \]

Then the characteristic polynomial of \( A \) is

\[ p_A(x) = x^3 + (a - d - 1)x^2 + (-ad - a - c + d - e)x + ad - ae + c - bce. \]

By the relationship between the coefficients of the characteristic polynomial and eigenvalues, if 

\[ ri(A) \in R_0 \cup R'_1 = \{(0, 3, 0, 0), (0, 2, 1, 0), (0, 1, 2, 0), (0, 0, 3, 0), (0, 0, 0, 1, 2)\}, \]

then

\[ \begin{cases} a - d - 1 \geq 0, \\ -ad - a - c + d - e \geq 0. \end{cases} \]

Adding both sides of above two inequalities, respectively, we have \(-ad - c - e - 1 \geq 0\). It is a contradiction. Hence \( R_0 \cup R'_1 \subseteq R(A_i) \).

For \( A_4 \), without loss of generality, let

\[ A = \begin{bmatrix} -a & 1 & b \\ c & d & 1 \\ 0 & -e & 1 \end{bmatrix} \in Q(A_4), \]

where \( a, b, c, d, e > 0. \) Then the characteristic polynomial of \( A \) is

\[ p_A(x) = x^3 + (a - d - 1)x^2 + (-ad - a + c + d - e)x + ad - ae - c - bce. \]

If \( ri(A) \in R_0 \cup R'_1 \), then

\[ \begin{cases} a - d - 1 \geq 0, \\ -ad - a + c + d - e \geq 0, \\ ad - ae - c - bce \geq 0. \end{cases} \]

Adding both sides of above three inequalities, respectively, we have \(-1 - e - ae - bce \geq 0\). It is a contradiction. Hence, \( R_0 \cup R'_1 \subseteq R(A_5) \).

Noting that \( A_5 \) requires negative determinant, we get \( R_0 \cup R_1 \subseteq R(A_5) \).
For $A_{16}$, noting that $A_{16}$ is a subpattern of $G$, by Lemma 2.3, we know that $(0,3,0,0)$ and $(0,1,0,2)$ are not the refined inertias of $A_{16}$. In the following, we show that $(0,2,1,0)$, $(0,1,2,0)$, $(0,0,3,0)$ and $(0,0,1,2)$ are not the refined inertias of $A_{16}$.

For any $A \in Q(A_{16})$, without loss of generality, let

$$A = \begin{bmatrix} -a & 1 & b \\ -c & d & -1 \\ -e & 0 & 1 \end{bmatrix},$$

where $a, b, c, d, e > 0$. Then the characteristic polynomial of $A$ is $p_A(x) = x^3 + p_1x^2 + p_2x + p_3$, where

$$\begin{cases} p_1 = -1 + a - d, \\ p_2 = -a + c + d - ad + be, \\ p_3 = -c + ad - e - bde. \end{cases}$$

If $ri(A) \in \{(0,2,1,0),(0,1,2,0),(0,0,3,0),(0,0,1,2)\}$, then $p_3 = 0$, $p_1 \geq 0$ and $p_2 \geq 0$. By $p_3 = 0$, we have $a = \frac{-c + ad - e - bde}{d}$. Then

$$\begin{cases} dp_1 = -d + c + e + bde - d^2 \geq 0, \\ dp_2 = -(c + e + bde) + cd + d^2 - (c + e + bde)d + bed \\ = -c - e + d^2 - ed - bed^2 \geq 0. \end{cases}$$

If $bc \geq 1$, then $dp_2 = -c - e + d^2 - ed - bed^2 \leq -c - e - ed < 0$.
If $c + e > d^2$, then $dp_2 = -c - e + d^2 - ed - bed^2 < -ed - bed^2 < 0$.
If $be < 1$ and $c + e \leq d^2$, then $dp_1 = -d + c + e + bde - d^2 < 0$.

All of above are contradictions. Hence, $(0,2,1,0)$, $(0,1,2,0)$, $(0,0,3,0)$ and $(0,0,1,2)$ are not the refined inertias of $A_{16}$ and so $R_0 \cup R_1^{\prime} \subseteq R(A_{16})$.

Case 2. Exactly two off-diagonal entries of $A$ are zero.

According to whether the two zero entries are in one 2-cycle or not, up to equivalence, $A$ has the following forms

$$(2.1) \begin{bmatrix} - & \ast & 0 \\ \ast & + & \ast \\ 0 & \ast & + \end{bmatrix}, \quad (2.2) \begin{bmatrix} - & \ast & \ast \\ \ast & 0 & + \\ \ast & 0 & + \end{bmatrix}, \quad (2.1)^{\prime} \begin{bmatrix} - & 0 & \ast \\ 0 & + & \ast \\ \ast & + & \ast \end{bmatrix},$$

$$(2.3) \begin{bmatrix} - & 0 & \ast \\ \ast & + & \ast \\ 0 & \ast & + \end{bmatrix}, \quad (2.4) \begin{bmatrix} - & \ast & \ast \\ \ast & 0 & + \\ 0 & \ast & + \end{bmatrix}, \quad (2.4)^{\prime} \begin{bmatrix} - & 0 & \ast \\ \ast & + & 0 \\ \ast & + & \ast \end{bmatrix},$$

where $\ast \in \{+, -\}$. Noting that $P_{23}(2.1)^{\prime}P_{23} = (2.1)$, $P_{23}(2.4)^{\prime}P_{23} = (2.4)$, so $(2.1)^{\prime}$ and $(2.1)$, $(2.4)^{\prime}$ and $(2.4)$ are equivalent, respectively.

Let $A = [a_{ij}]$ have form (2.1). If $a_{12} < 0$, taking $A^\prime = D_1AD_1$, then $A^\prime$ and $A$ are equivalent and the $(1,2)$ entry of $A^\prime$ is positive. If $a_{23} < 0$, taking $A'' = D_3AD_3$, then $A''$ and $A$ are equivalent and the $(2,3)$ entry of $A''$ is positive. So, without loss of generality, we can take $a_{12} = a_{23} = +$. 
According to the number of the negative 2-cycles, \( \mathcal{A} \) is possibly one of the following sign patterns.

\[
\mathcal{A}_{17} = \begin{bmatrix} - & + & 0 \\ + & + & + \\ 0 & + & + \end{bmatrix}, \mathcal{A}_{18} = \begin{bmatrix} - & + & 0 \\ - & + & + \\ 0 & - & + \end{bmatrix}, \mathcal{A}_{19} = \begin{bmatrix} + & + & + \\ - & + & + \\ 0 & 0 & + \end{bmatrix}, \mathcal{A}_{20} = \begin{bmatrix} - & + & 0 \\ 0 & + & + \\ 0 & 0 & - \end{bmatrix}.
\]

Let \( \mathcal{A} \) have form (2.2). Without loss of generality, we can let \( a_{12} = a_{14} = + \). According to the number of the negative 2-cycles, \( \mathcal{A} \) is possibly one of the following sign patterns.

\[
\mathcal{A}_{21} = \begin{bmatrix} - & + & + \\ + & 0 & 0 \\ + & + & 0 \end{bmatrix}, \mathcal{A}_{22} = \begin{bmatrix} - & + & + \\ - & + & 0 \\ + & + & 0 \end{bmatrix}, \mathcal{A}_{23} = \begin{bmatrix} - & + & + \\ + & + & 0 \\ - & 0 & + \end{bmatrix}, \mathcal{A}_{24} = \begin{bmatrix} - & + & + \\ - & + & 0 \\ 0 & - & 0 \end{bmatrix}.
\]

Let \( \mathcal{A} = [a_{ij}] \) have form (2.3). If \( a_{13} < 0 \), taking \( \mathcal{A}' = D_1 \mathcal{A} D_1 \), then \( \mathcal{A}' \) and \( \mathcal{A} \) are equivalent and the \((1,3)\) entry of \( \mathcal{A}' \) is positive. If \( a_{23} < 0 \), taking \( \mathcal{A}'' = D_2 \mathcal{A} D_2 \), then \( \mathcal{A}'' \) and \( \mathcal{A} \) are equivalent and the \((2,3)\) entry of \( \mathcal{A}'' \) is positive. So, without loss of generality, we can take \( a_{13} = a_{23} = + \).

According to the number of the negative 2-cycles, \( \mathcal{A} \) is possibly one of the following sign patterns.

\[
\mathcal{A}_{25} = \begin{bmatrix} - & 0 & + \\ + & + & + \\ 0 & + & + \end{bmatrix}, \mathcal{A}_{26} = \begin{bmatrix} - & 0 & + \\ - & + & + \\ 0 & + & + \end{bmatrix}, \mathcal{A}_{27} = \begin{bmatrix} - & 0 & + \\ + & + & + \\ 0 & 0 & + \end{bmatrix}, \mathcal{A}_{28} = \begin{bmatrix} - & 0 & + \\ 0 & + & + \\ 0 & 0 & - \end{bmatrix}.
\]

Assume that \( \mathcal{A} \) has form (2.4). Without loss of generality, we can let \( a_{12} = a_{13} = + \). According to the number of the negative 2-cycles, \( \mathcal{A} \) is possibly one of the following sign patterns.

\[
\mathcal{A}_{29} = \begin{bmatrix} - & + & + \\ + & 0 & 0 \\ 0 & + & + \end{bmatrix}, \mathcal{A}_{30} = \begin{bmatrix} - & + & + \\ - & + & 0 \\ 0 & - & + \end{bmatrix}, \mathcal{A}_{31} = \begin{bmatrix} - & + & + \\ - & + & 0 \\ 0 & + & + \end{bmatrix}, \mathcal{A}_{32} = \begin{bmatrix} - & + & + \\ - & + & 0 \\ 0 & 0 & + \end{bmatrix}.
\]

Firstly, let us notice the following facts.

1. \( \mathcal{P}_{13}(\mathcal{A}_{28}) \mathcal{P}_{13}, \mathcal{D}_1 \mathcal{A}_{10}^T \mathcal{D}_1 \) are superpatterns of \( \mathcal{D}_{3,3} \).
2. \( \mathcal{A}_{20} \) is a superpattern of \( \mathcal{D}_{3,2} \).
3. \( \mathcal{D}_3 \mathcal{P}_{12}(\mathcal{A}_{22}) \mathcal{P}_{12} \mathcal{D}_3 = \mathcal{U} \).
4. \( \mathcal{P}_{23} \mathcal{A}_{23} \mathcal{P}_{23} = \mathcal{A}_{22} \).

Then by Lemma 2.1, \( \mathcal{A}_{20}, \mathcal{A}_{22}, \mathcal{A}_{23}, \mathcal{A}_{28} \) and \( \mathcal{A}_{31} \) are rIAPs.

Thus, \( \mathcal{A} \) is equivalent to one of patterns in Case 2 except for \( \mathcal{A}_{20}, \mathcal{A}_{22}, \mathcal{A}_{23}, \mathcal{A}_{28} \) and \( \mathcal{A}_{31} \).

By similar argument to \( \mathcal{A}_1 \) in Case 1, we can get \( R_0 \cup R'_1 \subseteq R(A_i) \) for \( i = 17, 21, 25, 26, 29, 30 \).

By similar argument to \( \mathcal{A}_1 \) in Case 1, we can get \( R_0 \cup R'_1 \subseteq R(A_i) \) for \( i = 18, 32 \).

Noting that \( \mathcal{A}_{19} \) and \( \mathcal{A}_{27} \) require negative determinants, we get \( R_0 \cup R'_1 \subseteq R(A_i) \) for \( i = 19, 27 \).

For \( \mathcal{A}_{24} \), noting that it is a subpattern of \( \mathcal{G} \), by Lemma 2.3, we know that \((0,3,0,0)\) and \((0,1,0,2)\) do not belong to the refined inertias of \( \mathcal{A}_{24} \). In the following, we prove that \((0,2,1,0)\), \((0,1,2,0)\), \((0,0,3,0)\) and \((0,0,1,2)\) do not belong to the refined inertias of \( \mathcal{A}_{24} \) as well.
For any $A \in Q(A_{24})$, without loss of generality, let

$$A = \begin{bmatrix}
-a & 1 & b \\
-c & d & 0 \\
-e & 0 & 1
\end{bmatrix},$$

where $a, b, c, d, e > 0$. Then the characteristic polynomial of $A$ is $p_A(x) = x^3 + p_1x^2 + p_2x + p_3$, where

\[
\begin{cases}
p_1 = -1 + a - d, \\
p_2 = -a + c + d - ad + be, \\
p_3 = -c + ad - bde.
\end{cases}
\]

If $ri(A) \in \{(0, 2, 1, 0), (0, 1, 2, 0), (0, 0, 3, 0), (0, 0, 1, 2)\}$, then $p_3 = 0$, $p_1 \geq 0$ and $p_2 \geq 0$. By $p_3 = 0$, we have $a = \frac{c + bde}{d}$. Then

\[
\begin{cases}
dp_1 = -d + c + bde - d^2 \geq 0, \\
dp_2 = -(c + bde) + cd + d^2 - (c + bde)d + bed = -c + d^2 - bed^2 \geq 0.
\end{cases}
\]

If $be \geq 1$, then $dp_2 = -c + d^2 - bed^2 \leq -c < 0$.

If $c > d^2$, then $dp_2 = -c + d^2 - bed^2 < -bed^2 < 0$.

If $be < 1$ and $c \leq d^2$, then $dp_1 = -d + c + bde - d^2 < 0$.

All of above are contradictions. Hence, $(0, 2, 1, 0)$, $(0, 1, 2, 0)$, $(0, 0, 3, 0)$ and $(0, 0, 1, 2)$ do not belong to the refined inertias of $A_{24}$. Thus $R_0 \cup R_1^l \subseteq R(A_{24})$.

Case 3. Exactly three off-diagonal entries of $A$ are zero.

Up to equivalence, $A$ has the following unique form

$$(3.1) \begin{bmatrix}
- & 0 & * \\
* & + & 0 \\
0 & * & +
\end{bmatrix},$$

where $* \in \{+, -\}$.

Let $A = [a_{ij}]$ have form (3.1). If $a_{13} < 0$, taking $A' = D_1AD_1$, then $A'$ and $A$ are equivalent and the $(1, 3)$ entry of $A'$ is positive. If $a_{32} < 0$, taking $A'' = D_2AD_2$, then $A''$ and $A$ are equivalent and the $(3, 2)$ entry of $A''$ is positive. So, without loss of generality, we can take $a_{13} = a_{32} = +$.

Then $A$ is possibly one of the following sign patterns.

$$A_{33} = \begin{bmatrix}
- & 0 & + \\
+ & + & 0
\end{bmatrix}, \quad A_{34} = \begin{bmatrix}
- & 0 & + \\
0 & + & +
\end{bmatrix}.$$

By similar argument to $A_1$ in Case 1, we can get $R_0 \cup R_1^l \subseteq R(A_{33})$.

Noting that $A_{34}$ requires negative determinant, we get $R_0 \cup R_1 \subseteq R(A_{34})$. □

**Lemma 3.2** Let $A$ be a $3 \times 3$ irreducible sign pattern with at least one zero entry. Suppose $A$ has one zero diagonal entry, and two nonzero diagonal entries have different signs. If $A$ is not an rIAP, then one of the following conditions holds.
(1) \( R_0 \cup R_1 \subseteq R(A) \);
(2) \( R_0 \cup R'_1 \subseteq R(A) \).

**Proof** Up to equivalence, we can assume \( a_{11} = - \), \( a_{22} = 0 \) and \( a_{33} = + \). Note that \( A \) is irreducible and has at least one zero entry. We consider the following three cases.

Case 1. All off-diagonal entries of \( A \) are nonzero.

Without loss of generality, we can take \( a_{12} = a_{13} = + \). According to the number of negative 2-cycles, \( A \) is possibly one of the following sign patterns.

\[
A_1 = \begin{bmatrix} - & + & + \\ + & 0 & + \\ + & + & + \end{bmatrix}, A_2 = \begin{bmatrix} - & + & + \\ + & 0 & - \\ + & + & + \end{bmatrix}, A_3 = \begin{bmatrix} - & + & + \\ - & 0 & + \\ + & + & + \end{bmatrix}, A_4 = \begin{bmatrix} - & + & + \\ - & 0 & - \\ + & + & + \end{bmatrix},
\]

\[
A_5 = \begin{bmatrix} - & + & + \\ + & 0 & + \\ - & + & + \end{bmatrix}, A_6 = \begin{bmatrix} - & + & + \\ + & 0 & - \\ - & + & + \end{bmatrix}, A_7 = \begin{bmatrix} - & + & + \\ + & 0 & + \\ - & + & + \end{bmatrix}, A_8 = \begin{bmatrix} - & + & + \\ + & 0 & - \\ - & + & + \end{bmatrix},
\]

\[
A_9 = \begin{bmatrix} - & + & + \\ + & 0 & + \\ - & + & + \end{bmatrix}, A_{10} = \begin{bmatrix} - & + & + \\ - & 0 & - \\ - & + & + \end{bmatrix}, A_{11} = \begin{bmatrix} - & + & + \\ - & 0 & + \\ - & + & + \end{bmatrix}, A_{12} = \begin{bmatrix} - & + & + \\ - & 0 & - \\ - & + & + \end{bmatrix},
\]

\[
A_{13} = \begin{bmatrix} - & + & + \\ + & 0 & + \\ - & + & + \end{bmatrix}, A_{14} = \begin{bmatrix} - & + & + \\ + & 0 & - \\ - & + & + \end{bmatrix}, A_{15} = \begin{bmatrix} - & + & + \\ - & 0 & + \\ - & + & + \end{bmatrix}, A_{16} = \begin{bmatrix} - & + & + \\ - & 0 & - \\ - & + & + \end{bmatrix}.
\]

Firstly, let us notice the following facts.

(1) \( A_{11}, A_{15}, D_3A_{12}D_3, D_3A_{16}D_3 \) are superpatterns of \( D_{3,2} \).

(2) \( D_3A_{14}D_3, D_3A_{13}D_2D_3, D_3A_{15}D_2 \) are superpatterns of \( D_{3,3} \).

(3) \( A_9, D_1A_7^T, D_2A_6D_2, D_2P_{13}(-A_{14})P_{13}D_2 \) are superpatterns of \( V \).

(4) \( A_9^T = A_7, D_3P_{13}(-A_7)P_{13}D_3 = A_3 \).

Then by Lemma 2.1, \( A_3, A_4, A_6, A_7, A_8, A_9, A_{11}, A_{12}, A_{14}, A_{15} \) and \( A_{16} \) are rIAPs.

Thus, in this case, \( A \) is equivalent to one of \( A_1, A_2, A_{10} \) and \( A_{13} \).

For \( A_1 \), without loss of generality, let

\[
A = \begin{bmatrix} -a & 1 & b \\ c & 0 & 1 \\ d & e & 1 \end{bmatrix} \in Q(A_1),
\]

where \( a, b, c, d, e > 0 \). Then the characteristic polynomial of \( A \) is

\[
p_A(x) = x^3 + (a - 1)x^2 + (-c - bd - e - a)x + c - d - ac - bee.
\]

Since \(-c - bd - e - a < 0\), we have \( n_+(A) \geq 1 \) and \( n_-(A) \geq 1 \). Then \( R_0 \cup R'_1 \subseteq R(A_1) \).

By similar argument to \( A_1 \), we can get \( R_0 \cup R'_1 \subseteq R(A_2) \).

Noting that \( A_{10} \) requires positive determinant, we get \( R_0 \cup R'_1 \subseteq R(A_{10}) \).

Noting that \( A_{13} \) requires negative determinant, we get \( R_0 \cup R_1 \subseteq R(A_{13}) \).
Case 2. Exactly one off-diagonal entry of $A$ is zero.

Up to equivalence, $A$ has the following forms

\[
\begin{align*}
\text{(2.1)} & \quad \begin{bmatrix} - & * & * \\ * & 0 & * \\ 0 & * & + \end{bmatrix}, \quad \text{(2.2)} & \quad \begin{bmatrix} * & 0 & * \\ 0 & 0 & * \\ * & 0 & + \end{bmatrix},
\end{align*}
\]

where $* \in \{+, -\}$. Noting that $P_{13}(- (2.2)')^T P_{13} = (2.2)$, so $(2.2)'$ and (2.2) are equivalent.

Let $A$ have form (2.1). Without loss of generality, let $a_{12} = a_{23} = +$. According to the number of the negative 2-cycles, $A$ is possibly one of the following sign patterns.

\[
\begin{align*}
A_{17} &= \begin{bmatrix} - & + & + \\ + & 0 & + \\ 0 & + & + \end{bmatrix}, A_{18} = \begin{bmatrix} - & + & + \\ + & 0 & + \\ 0 & + & + \end{bmatrix}, A_{19} = \begin{bmatrix} - & 0 & + \\ 0 & 0 & + \\ 0 & + & + \end{bmatrix}, A_{20} = \begin{bmatrix} - & 0 & + \\ 0 & 0 & + \\ 0 & + & + \end{bmatrix},
\end{align*}
\]

\[
\begin{align*}
A_{21} &= \begin{bmatrix} - & + & + \\ + & 0 & + \\ 0 & - & + \end{bmatrix}, A_{22} = \begin{bmatrix} - & + & + \\ + & 0 & + \\ 0 & - & + \end{bmatrix}, A_{23} = \begin{bmatrix} - & 0 & + \\ 0 & 0 & + \\ 0 & - & + \end{bmatrix}, A_{24} = \begin{bmatrix} - & 0 & + \\ 0 & 0 & + \\ 0 & - & + \end{bmatrix}.
\end{align*}
\]

Let $A$ have form (2.2). Without loss of generality, let $a_{12} = a_{13} = +$. According to the number of the negative 2-cycles, $A$ is possibly one of the following sign patterns.

\[
\begin{align*}
A_{25} &= \begin{bmatrix} - & + & + \\ + & 0 & + \\ + & 0 & + \end{bmatrix}, A_{26} = \begin{bmatrix} - & + & + \\ + & 0 & - \\ + & 0 & + \end{bmatrix}, A_{27} = \begin{bmatrix} - & 0 & + \\ + & 0 & + \\ + & 0 & + \end{bmatrix}, A_{28} = \begin{bmatrix} - & 0 & + \\ + & 0 & + \\ + & 0 & + \end{bmatrix},
\end{align*}
\]

\[
\begin{align*}
A_{29} &= \begin{bmatrix} - & + & + \\ + & 0 & - \\ - & 0 & + \end{bmatrix}, A_{30} = \begin{bmatrix} - & + & + \\ + & 0 & - \\ - & 0 & + \end{bmatrix}, A_{31} = \begin{bmatrix} - & + & + \\ - & 0 & + \\ - & 0 & + \end{bmatrix}, A_{32} = \begin{bmatrix} - & + & + \\ - & 0 & + \\ - & 0 & + \end{bmatrix}.
\end{align*}
\]

Firstly, let us notice the following facts.

(1) $A_{31}, D_1A_{19}^T D_1, D_3P_{13}(-A_{22})P_{13}D_3, D_3A_{28}D_3$ are the superpatterns of $D_{3,3}$.  

(2) $A_{23}, A_{24}$ are the superpatterns of $D_{3,2}$. 

(3) $P_{13}D_2(-A_{30})D_2P_{13} = \mathcal{V}$.

Then by Lemma 2.1, $A_{19}, A_{22}, A_{23}, A_{24}, A_{28}, A_{30}$ and $A_{31}$ are rIAPs.

Thus, in this case, $A$ is equivalent to one of patterns in Case 2 except for $A_{19}, A_{22}, A_{23}, A_{24}, A_{28}, A_{30}$ and $A_{31}$.  

By similar argument to $A_1$ in Case 1, we can get $R_0 \cup R_i^I \subseteq R(A_i)$ for $i = 17, 18, 25$.  

Noting that $A_{20}, A_{27}$ and $A_{32}$ require positive determinants, we get $R_0 \cup R_i^I \subseteq R(A_i)$ for $i = 20, 27, 32$.  

Noting that $A_{21}, A_{26}$ and $A_{29}$ require negative determinants, we get $R_0 \cup R_i \subseteq R(A_i)$ for $i = 21, 26, 29$.  

Case 3. Exactly two off-diagonal entries of $A$ are zero.
According to whether the two zero entries are in one 2-cycle or not, up to equivalence, \( A \) has the following forms

\[
\begin{align*}
(3.1) & \quad \begin{bmatrix} - & * & 0 \\ * & 0 & * \\ 0 & * & + \end{bmatrix}, \\
(3.2) & \quad \begin{bmatrix} * & 0 & 0 \\ * & 0 & + \\ 0 & * & + \end{bmatrix}, \\
(3.3) & \quad \begin{bmatrix} - & 0 & * \\ * & 0 & * \\ 0 & * & + \end{bmatrix}, \quad (3.2)' \quad \begin{bmatrix} * & 0 & 0 \\ * & 0 & + \\ 0 & * & + \end{bmatrix}, \\
\end{align*}
\]

where \( * \in \{+, -\} \). Noting that \( P_{13}(-3.2)' P_{13} = (3.2), P_{13}(-3.3)' P_{13} = (3.3) \), so \( (3.2)' \) and \( (3.2), (3.3)' \) and \( (3.3) \) are equivalent, respectively.

Let \( A \) have form \((3.1)\). Without loss of generality, let \( a_{12} = a_{23} = + \). According to the number of the negative 2-cycles, \( A \) is possibly one of the following sign patterns.

\[
A_{31} = \begin{bmatrix} - & + & 0 \\ + & 0 & + \\ 0 & + & + \end{bmatrix}, A_{34} = \begin{bmatrix} - & + & 0 \\ + & 0 & + \\ 0 & - & + \end{bmatrix}, A_{35} = \begin{bmatrix} - & + & 0 \\ 0 & - & + \\ 0 & + & + \end{bmatrix}, A_{36} = \begin{bmatrix} - & - & 0 \\ 0 & - & + \\ 0 & + & + \end{bmatrix}.
\]

Let \( A \) have form \((3.2)\). Without loss of generality, let \( a_{12} = a_{13} = + \). According to the number of the negative 2-cycles, \( A \) is possibly one of the following sign patterns.

\[
A_{37} = \begin{bmatrix} - & + & + \\ + & 0 & 0 \\ + & 0 & + \end{bmatrix}, A_{38} = \begin{bmatrix} - & + & + \\ - & 0 & 0 \\ + & 0 & + \end{bmatrix}, A_{39} = \begin{bmatrix} - & + & + \\ + & 0 & 0 \\ - & 0 & + \end{bmatrix}, A_{40} = \begin{bmatrix} - & + & + \\ - & 0 & 0 \\ - & 0 & + \end{bmatrix}.
\]

Let \( A \) have form \((3.3)\). Without loss of generality, let \( a_{13} = a_{23} = + \). According to the number of the negative 2-cycles, \( A \) is possibly one of the following sign patterns.

\[
A_{41} = \begin{bmatrix} - & 0 & + \\ + & 0 & + \\ 0 & + & + \end{bmatrix}, A_{42} = \begin{bmatrix} - & 0 & + \\ - & 0 & + \\ 0 & + & + \end{bmatrix}, A_{43} = \begin{bmatrix} - & 0 & + \\ + & 0 & + \\ 0 & - & + \end{bmatrix}, A_{44} = \begin{bmatrix} - & 0 & + \\ - & 0 & + \\ 0 & + & + \end{bmatrix}.
\]

Let \( A \) have form \((3.4)\). Without loss of generality, let \( a_{12} = a_{13} = + \). According to the number of the negative 2-cycles, \( A \) is possibly one of the following sign patterns.

\[
A_{45} = \begin{bmatrix} - & + & + \\ 0 & 0 & + \\ + & 0 & + \end{bmatrix}, A_{46} = \begin{bmatrix} - & + & + \\ 0 & 0 & - \\ + & 0 & + \end{bmatrix}, A_{47} = \begin{bmatrix} - & + & + \\ 0 & 0 & + \\ - & 0 & + \end{bmatrix}, A_{48} = \begin{bmatrix} - & + & + \\ 0 & 0 & - \\ - & 0 & + \end{bmatrix}.
\]

Firstly, let us notice the following facts.

1. \( P_{13}(-A_{44}) P_{13} \) is a superpattern of \( D_{3.3} \).
2. \( A_{36} \) is \( D_{3.2} \).

Then by Lemma 2.1, \( A_{36} \) and \( A_{44} \) are rIAPs.

Thus in this case, \( A \) is equivalent to one pattern in Case 3 except for \( A_{36} \) and \( A_{44} \).

By similar argument to \( A_1 \) in Case 1, we can get \( R_0 \cup R'_i \subseteq R(A_i) \) for \( i = 33, 42, \).
Noting that $A_{34}, A_{37}, A_{39}, A_{43}, A_{46}$ and $A_{47}$ require negative determinants, we get $R_0 \cup R_1 \subseteq R(A_i)$ for $i = 34, 37, 39, 43, 46, 47$.

Noting that $A_{35}, A_{38}, A_{40}, A_{41}, A_{45}$ and $A_{48}$ require positive determinants, we get $R_0 \cup R'_1 \subseteq R(A_i)$ for $i = 35, 38, 40, 41, 45, 48$.

**Case 4.** Exactly three off-diagonal entries of $A$ are zero.

Up to equivalence, $A$ has the following unique form

\[
(4.1) \begin{bmatrix}
- & 0 & *\\
* & 0 & 0 \\
0 & * & +
\end{bmatrix},
\]

where $* \in \{+, -\}$.

It is easy to see that $A$ is sign nonsingular. If $A$ requires positive determinant, then $R_0 \cup R'_1 \subseteq R(A)$. If $A$ requires negative determinant, then $R_0 \cup R_1 \subseteq R(A)$. \( \square \)

**Theorem 3.3** Let $A$ be a $3 \times 3$ irreducible sign pattern with at least one zero entry. If $A$ is not an rIAP, then one of the following conditions holds:

1. $R_0 \cup R_1 \subseteq R(A)$;
2. $R_0 \cup R'_1 \subseteq R(A)$;
3. $R_1 \cup R'_1 \subseteq R(A)$.

**Proof** Let $A$ be a $3 \times 3$ irreducible sign pattern with at least one zero entry. Suppose $A$ is not an rIAP. We consider the following cases.

**Case 1.** All diagonal entries of $A$ are zero.

Since $A$ requires the zero trace, $(3, 0, 0, 0), (2, 0, 1, 0), (0, 1, 0, 2), (0, 3, 0, 0), (0, 2, 1, 0), (0, 1, 2, 0), \text{ and } (0, 1, 0, 2)$ do not belong to the refined inertias of $A$, and so $R_1 \cup R'_1 \subseteq R(A)$.

**Case 2.** Sign pattern $A$ has at least one nonzero diagonal entry, and all nonzero diagonal entries of $A$ have the same sign.

If all nonzero diagonal entries of $A$ are negative, then $A$ requires the negative trace, and so $R_0 \cup R_1 \subseteq R(A)$. If all nonzero diagonal entries of $A$ are positive, then $A$ requires the positive trace, and so $R_0 \cup R'_1 \subseteq R(A)$.

**Case 3.** Sign pattern $A$ has at least two nonzero diagonal entries, and the nonzero diagonal entries of $A$ have different signs.

By Lemmas 3.1 and 3.2, we know the result holds. \( \square \)

**Theorem 3.4** There exists a $3 \times 3$ irreducible sign pattern $A$ with at least one zero entry such that $A$ is not an rIAP, and $R(A) = R_1 \cup R'_1$.

**Proof** Let

\[
S_1 = \begin{bmatrix}
0 & + & + \\
- & 0 & + \\
+ & + & 0
\end{bmatrix}.
\]

It is easy to see that $S_1$ is not an rIAP. Since $S_1$ requires the zero trace, so $R_1 \cup R'_1 \subseteq R(S_1)$.
On the other hand, for any $A \in Q(S_1)$, we may assume that $a_{12} = a_{13} = 1$ (otherwise they can be 1 by suitable similarities). Thus, without loss of generality, assume

$$A = \begin{bmatrix} 0 & 1 & 1 \\
-a & 0 & b \\
c & d & 0 \end{bmatrix},$$

where $a, b, c, d > 0$.

By taking suitable values of $a, b, c, d$ shown in Table 1, we can find real matrices in $Q(S_1)$ with each refined inertia in $R \setminus (R_0 \cup R_1')$.

<table>
<thead>
<tr>
<th>refined inertia</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2, 1, 0, 0)</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(1, 2, 0, 0)</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(1, 1, 1, 0)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(0, 0, 3, 0)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(0, 0, 1, 2)</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1 Realization of each refined inertia in $R \setminus (R_0 \cup R_1')$

**Theorem 3.5** There exists a $3 \times 3$ irreducible sign pattern $A$ with at least one zero entry such that $A$ is not an rIAP, and $R(A) = R_0 \cup R_1'$.

**Proof** Let

$$S_2 = \begin{bmatrix} - & + & + \\
- & + & - \\
0 & - & + \end{bmatrix}.$$

Noting that $S_2$ is the sign pattern $A_4$ in the proof of Lemma 3.1, by Lemma 3.1, $S_2$ is not an rIAP and $R_0 \cup R_1' \subseteq R(S_2)$.

On the other hand, take

$$A = \begin{bmatrix} -a & 1 & b \\
-c & d & -1 \\
0 & -e & 1 \end{bmatrix} \in Q(S_2),$$

where $a, b, c, d, e > 0$. By taking suitable values of $a, b, c, d, e$ shown in Table 2, we can find real matrices in $Q(S_2)$ with each refined inertia in $R \setminus (R_0 \cup R_1')$. □

**Theorem 3.6** There exists a $3 \times 3$ irreducible sign pattern $A$ with at least one zero entry such that $A$ is not an rIAP, and $R(A) = R_0 \cup R_1$.

**Proof** Let $S_3 = -S_2$. By Theorem 3.5, the result follows. □

**Lemma 3.7 ([7])** Let $H$ be a proper subset of set of all possible refined inertias of real matrices of order $n$. Then $H$ is a critical set of refined inertias for a family $F$ of sign pattern of order $n$ if and only if every $n \times n$ sign pattern $A$ in $F$ that is not an rIAP, $H \cap R(A) \neq \emptyset$. 

Proof Let $F$ be the set of all irreducible sign patterns of order 3 with at least one zero entry that are not rAPs. By Lemma 3.7, we only need to prove $H \cap R(A) \neq \emptyset$ for every $A$ in $F$ if and only if one of the following conditions holds:

(1) $H \cap R_1 \neq \emptyset$ and $H \cap R'_1 \neq \emptyset$;
(2) $H \cap R_0 \neq \emptyset$ and $H \cap R_1 \neq \emptyset$;
(3) $H \cap R_0 \neq \emptyset$ and $H \cap R'_1 \neq \emptyset$.

By Theorem 3.3, the sufficiency is clear. For the necessity, let $H \cap R(A) \neq \emptyset$ for every $A$ in $F$. Then by Theorems 3.4–3.6, $H \cap (R_1 \cup R'_1) \neq \emptyset$, $H \cap (R_0 \cup R_1) \neq \emptyset$, and $H \cap (R_0 \cup R'_1) \neq \emptyset$. So the necessity holds. □

Proof of Theorem 1.1 Let $H$ be a proper subset of the set of all possible refined inertias for irreducible sign patterns of order 3 with at least one zero entry. By Theorem 3.8, $H$ is critical set of refined inertias if and only if $H \cap R_1 \neq \emptyset$ and $H \cap R'_1 \neq \emptyset$, or $H \cap R_0 \neq \emptyset$ and $H \cap R'_1 \neq \emptyset$.

To make $H$ a minimal critical set, then one of the following conditions holds:

(1) $|H \cap R_1| = 1$ and $|H \cap R'_1| = 1$;
(2) $|H \cap R_0| = 1$ and $|H \cap R_1| = 1$;
(3) $|H \cap R_0| = 1$ and $|H \cap R'_1| = 1$.

We pick up exactly one refined inertia from $R_1$ and one refined inertia from $R_0 \cup R'_1$, or one refined inertia from $R'_1$ and one refined inertia from $R_0$, and let them form new sets as follows.

$\{(3, 0, 0, 0), (0, 3, 0, 0), (0, 2, 1, 0)\}$, $\{(3, 0, 0, 0), (0, 1, 2, 0)\}$, $\{(3, 0, 0, 0), (0, 1, 0, 2)\}$, $\{(3, 0, 0, 0), (0, 0, 3, 0)\}$, $\{(3, 0, 0, 0), (0, 0, 1, 2)\}$, $\{(2, 0, 1, 0), (0, 3, 0, 0)\}$, $\{(2, 0, 1, 0), (0, 3, 0, 0)\}$, $\{(2, 0, 1, 0), (0, 1, 2, 0)\}$, $\{(2, 0, 1, 0), (0, 1, 2, 0)\}$, $\{(2, 0, 1, 0), (0, 1, 2, 0)\}$, $\{(2, 0, 1, 0), (0, 1, 2, 0)\}$.
Minimal critical sets of refined inertias for irreducible sign patterns of order 3

\{(1,0,2,0),(3,0,0)\}, \{(1,0,2,0),(2,1,0)\}, \{(1,0,2,0),(1,2,0)\}, \{(1,0,2,0),(1,0,2)\}, \{(1,0,2,0),(0,1,0)\}, \\
\{(1,0,2,0),(0,3,0)\}, \{(1,0,2,0),(0,3,0)\}, \{(1,0,2,0),(0,1,2)\}, \{(1,0,2,0),(1,1,0)\}, \\
\{(0,3,0,0),(0,3,0)\}, \{(0,3,0,0),(0,3,0)\}, \{(0,3,0,0),(0,3,0)\}, \{(0,3,0,0),(0,3,0)\}, \\
\{(0,1,2,0),(0,0,3)\}, \{(0,1,2,0),(0,0,3)\}, \{(0,1,2,0),(0,0,3)\}, \{(0,1,2,0),(0,0,3)\}, \\
\{(0,1,0,2),(0,0,3)\}, \{(0,1,0,2),(0,0,3)\}, \{(0,1,0,2),(0,0,3)\}, \{(0,1,0,2),(0,0,3)\}.

Note that
\{(2,0,1,0),(0,3,0)\} is the reversal of \{(3,0,0,0),(0,2,1,0)\},
\{(1,0,2,0),(0,3,0)\} is the reversal of \{(3,0,0,0),(0,1,2,0)\},
\{(0,3,0,0),(0,2,1)\} is the reversal of \{(2,0,1,0),(0,1,2,0)\},
\{(1,0,2,0),(3,0,0)\} is the reversal of \{(3,0,0,0),(0,1,2,0)\},
\{(1,0,0,2),(2,1,0)\} is the reversal of \{(2,0,1,0),(0,1,0,2)\},
\{(1,0,0,2),(0,2,1)\} is the reversal of \{(2,0,1,0),(0,1,0,2)\},
\{(0,3,0,0),(0,0,3)\} is the reversal of \{(3,0,0,0),(0,0,3)\},
\{(0,3,0,0),(0,0,1,2)\} is the reversal of \{(3,0,0,0),(0,0,1,2)\},
\{(0,2,1,0),(0,0,3)\} is the reversal of \{(2,0,1,0),(0,0,3)\},
\{(0,2,1,0),(0,0,1,2)\} is the reversal of \{(2,0,1,0),(0,0,1,2)\},
\{(0,1,2,0),(0,0,3)\} is the reversal of \{(1,0,2,0),(0,0,3)\},
\{(0,1,2,0),(0,0,1,2)\} is the reversal of \{(1,0,2,0),(0,0,1,2)\},
\{(0,1,0,2),(0,0,3)\} is the reversal of \{(1,0,0,2),(0,0,3)\},
\{(0,1,0,2),(0,0,1,2)\} is the reversal of \{(1,0,0,2),(0,0,1,2)\}.

So we drop them out.

Theorem 1.1 now follows.$\square$

By Theorem 1.1, it is clear that the maximum cardinality of a minimum critical set of refined inertias for $3 \times 3$ irreducible sign patterns with at least one zero entry is 2.

4. The minimal critical sets of inertias for irreducible sign patterns of order 3

Using the same method as in the proof of Theorem 1.2 in [7], we can get the result of Theorem 1.2.

5. Summary and conclusions

In this paper, we obtained all the minimum critical sets of refined inertias and inertias for $3 \times 3$ irreducible sign patterns with at least one zero entry. Based on these conclusions and results from the reference [7], we identified all the minimum critical sets of refined inertias and inertias for $3 \times 3$ irreducible sign patterns.

Further topics of interest for future research include the investigation of all the minimal critical sets of refined inertias and inertias for sign patterns of order $n$ ($n \geq 4$) and other parameters for sign patterns (see for instance the recent results in [10]).
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References