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Recollements, Tilting Homological Dimensions and Higher-Dimensional Auslander-Reiten Theory

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Abstract In this paper we mainly investigate the behavior of tilting homological dimensions of the categories involved in the recollement of abelian categories $(\mathscr{A},\mathscr{B},\mathscr{C})$. In particular, when abelian category $\mathscr{B}$ is hereditary, we give the connections between n-almost split sequences in the categories of $(\mathscr{A},\mathscr{B},\mathscr{C})$.

Keywords recollement; tilting homological dimension; n-almost split sequence

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1. Introduction

Throughout, we denote by $\mathbb{N}$, K and Id the set of nonnegative integers, a fixed field and the identity functor, respectively. Recall that a recollement situation between abelian categories $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ is a diagram

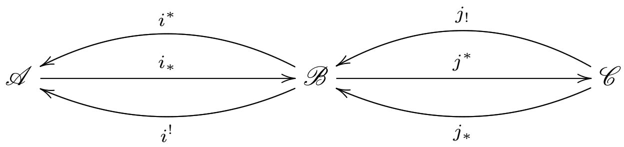

Diagram 1 The recollement of abelian categories

satisfying the following conditions:

(r1) $(i^*,i_*,i^!)$ and $(j_!,j^*,j_*)$ are adjoint triples;

(r2) the functors $i_*,j_!$ and j_* are fully faithful;

(r3) $\mathrm{Im}i_*=\mathrm{Ker}j^*$, which plays an important role in algebraic geometry, representation theory, polynomial functor theory, ring theory and so on. The readers may refer to [1–6] and references therein.

In analogy to the theories of tilting and almost split sequence for artin algebras [3, 7, 8], the corresponding version of abelian categories were also studied by many authors [1, 9]. Happel, Beligiannis and Reiten, and recently Hügel, Koenig and Liu, studied connections between recollements of triangulated categories in connection with tilting theory, homological conjectures and

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stratifications of derived categories of rings, see for example [1, 10, 11]. In 2014, Psaroudakis [6] investigated global, finitistic, and representation dimensions of recollements of abelian categories.

Let $\mathscr{B}$ be an abelian category with enough projectives. Then each object has a projective resolution. It follows that every object has a tilting projective resolution. As a generalization of the usual projective dimension of $M \in \mathscr{B}$, we now give the notion of tilting projective dimension as follows. t.proj.dim(M) is defined to be the least number n such that there is a tilting projective resolution

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where all P_i's are tilting projective. If there is no such n, we say that the tilting projective dimension of M is infinite, denoted by t.proj.dim$(M) = \infty$. Hence it is natural to define the tilting global dimension of $\mathscr{B}$ as

$$\text{t.gl.dim}(\mathscr{B}) = \sup\{\text{t.proj.dim}(M) \mid \forall M \in \mathscr{B}\}.$$

The tilting projective dimension is a generalization of projective dimension in the category of modules. Moreover, tilting objects in an abelian category is also a generalization of canonical tilting modules. Hence motivated by [6], we study the connections between the tilting global dimension of the categories involved in a recollement $(\mathscr{A}, \mathscr{B}, \mathscr{C})$.

The organization and the main results of the paper are as follows. In Section 2, we focus on tilting global dimensions of abelian categories involved in a recollement, which can be viewed as a generation of global dimension (compare with [6, Theorem 4.1]).

Theorem 1.1 *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories such that $\mathscr{B}$ and $\mathscr{C}$ have enough projective and injective objects. Then we have an upper bound for the tilting global dimension of $\mathscr{B}$*

$$\text{t.gl.dim}\mathscr{B} \leq \text{t.gl.dim}\mathscr{A} + \text{t.gl.dim}\mathscr{C} + \sup\{\text{t.proj.dim}_{\mathscr{B}} i_*(P) \mid P \in \text{Tproj}(\mathscr{A})\} + 1,$$

where Tproj$(\mathscr{A})$ *is the tilting projective subcategory.*

Recently, in the context of higher dimensional Auslander-Reiten theory, n-almost split sequences have attracted considerable attention as a generation of the classical almost split sequence. Guo [12] found a necessary and sufficient condition for the quadratic dual of n-translation algebras to have n-almost split sequences in the category of its projective modules. Recall from [9, Chapter I.4] and [13, Section IV.1] that a short exact sequence

$$0 \longrightarrow X \xrightarrow{\mu} E \xrightarrow{\pi} Y \longrightarrow 0$$

in an abelian category $\mathscr{B}$ is called almost split if it is non-split, X and Y are indecomposable and for $f \in \text{Hom}_{\mathscr{B}}(W, Y)$ which is not split epimorphism there is $g \in \text{Hom}_{\mathscr{B}}(W, E)$ such that $f = \pi \circ g$. Then we say that an abelian category $\mathscr{B}$ has almost split sequences if for all indecomposable non-projective objects B there is an exact sequence

$$0 \longrightarrow B'' \longrightarrow B' \longrightarrow B \longrightarrow 0$$

which satisfies the above conditions. When $\mathscr{B}$ is hereditary, we know from [6] that $\mathscr{A}$ and $\mathscr{C}$ are also hereditary. So it is natural for us to consider the properties of almost split sequences in a recollement $(\mathscr{A}, \mathscr{B}, \mathscr{C})$.

In Section 3, we aim to provide a criterion to decide when n-almost split sequences in $\mathscr{B}$ can be preserved in $\mathscr{A}$ and $\mathscr{C}$.

Theorem 1.2 *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories with tilting hereditary abelian category $\mathscr{B}$. If $\mathscr{B}$ has n-almost split sequences, and the functor i^* is exact, then $\mathscr{A}$ and $\mathscr{C}$ have (at most) n-almost split sequences.*

2. Recollements related to tilting theory and the tilting global dimension

Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories. Some properties of a recollement are listed as follows. The readers may refer to [3, 5, 11], [6, Remarks 2.2-2.5] and references therein.

(i) The functors $j^* : \mathscr{B} \to \mathscr{C}$ and $i_* : \mathscr{A} \to \mathscr{B}$ are exact. Moreover, $i^* i_* \simeq \mathrm{Id}_{\mathscr{A}}$, $\mathrm{Id}_{\mathscr{A}} \simeq i^! i_*$, $j^* j_* \simeq \mathrm{Id}_{\mathscr{C}}$ and $\mathrm{Id}_{\mathscr{C}} \simeq j^* j_!$.

(ii) If the pair $(j_!, j^*)$ is an adjoint functor pair and the functor j^* is exact, then the left adjoint functor $j_!$ preserves projective objects.

(iii) If the pair (j^*, j_*) is an adjoint functor pair and the functor j_* is exact, then the left adjoint functor j^* preserves projective objects.

(iv) If the pair $(j_!, j^*)$ is an adjoint functor pair and the functor $j_!$ is exact, then the right adjoint functor j^* preserves injective objects.

(v) If the pair (j^*, j_*) is an adjoint functor pair and the functor j^* is exact, then the right adjoint functor j_* preserves injective objects.

(vi) For any adjoint functor pair, the left adjoint functor preserves the right exactness and commutes with any direct sums; the right adjoint functor preserves the left exactness and commutes with any direct products, such as for the adjoint pair $(j_!, j^*)$, we have that $\mathrm{Add}(j_!(M)) = j_!(\mathrm{Add}(M))$ and $\mathrm{Prod}(j^*(N)) = j^*(\mathrm{Prod}(N))$.

Inspired by [13–15], we introduce the following notion.

Definition 2.1 *Let T be a tilting object in $\mathscr{B}$ and $\mathcal{T}(T)$ be a torsion class of the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$. An object M in $\mathscr{B}$ is called tilting projective if $\mathrm{Hom}_{\mathscr{B}}(M, -)$ preserves the exactness of sequences in $\mathcal{T}(T)$.*

Remark 2.2 (1) Each projective object is tilting projective; but the converse is not true. In [13, Example 1.2(d)], the tilting object $T = 100 \oplus 111 \oplus 001$ is a tilting projective but not projective.

(2) An object $M \in \mathscr{B}$ is tilting projective if and only if $\mathrm{Ext}^1_{\mathscr{B}}(M, L) = 0$ for any $L \in \mathcal{T}(T)$.

From now on we always suppose that $\mathscr{B}$ has enough projective and injective objects. Thus we have the derived functors $\mathrm{Ext}^n_{\mathscr{B}}(M, -)$ for $\mathrm{Hom}^n_{\mathscr{B}}(M, -)$ and $\mathrm{Ext}^n_{\mathscr{B}}(-, N)$ for $\mathrm{Hom}^n_{\mathscr{B}}(-, N)$.

Lemma 2.3 *Let T be a tilting object in $\mathscr{B}$. An object M is tilting projective if and only if $\mathrm{Ext}^1_{\mathscr{B}}(M,L)=0$ for any $L\in\mathcal{T}(T)$.*

Proof $\Rightarrow$. For any $L\in\mathcal{T}(T)$, there exists an exact sequence

$$0\longrightarrow L\longrightarrow E\longrightarrow N\longrightarrow 0$$

with E injective. This sequence is in $\mathcal{T}(T)$ by our assumption. Applying the Hom functor $\mathrm{Hom}_{\mathscr{B}}(M,-)$, we have the following long sequence

$$0\longrightarrow \mathrm{Hom}_{\mathscr{B}}(M,L)\longrightarrow \mathrm{Hom}_{\mathscr{B}}(M,E)\longrightarrow \mathrm{Hom}_{\mathscr{B}}(M,N)\longrightarrow \mathrm{Ext}^1_{\mathscr{B}}(M,L)\longrightarrow\cdots.$$

By Definition 2.1, we obtain $\mathrm{Ext}^1_{\mathscr{B}}(M,L)=0$ for any $L\in\mathcal{T}(T)$.

$\Leftarrow$. Since $\mathrm{Ext}^1_{\mathscr{B}}(M,L)=0$ for any $L\in\mathcal{T}(T)$, it follows that for any short exact sequence

$$0\longrightarrow N_1\longrightarrow N\stackrel{h}{\longrightarrow} N_2\longrightarrow 0$$

in $\mathcal{T}(T)$ and any homomorphism $f:M\to N_2$, there exists a morphism $g:M\to N$ such that $f=g\circ h$. Applying the functor $\mathrm{Hom}_{\mathscr{B}}(M,-)$, we have that the sequence

$$0\longrightarrow \mathrm{Hom}_{\mathscr{B}}(M,N_1)\longrightarrow \mathrm{Hom}_{\mathscr{B}}(M,N)\longrightarrow \mathrm{Hom}_{\mathscr{B}}(M,N_2)\longrightarrow 0$$

is exact. Hence M is tilting projective. $\Box$

Lemma 2.4 *Let T be a tilting object in $\mathscr{B}$ and $M\in\mathscr{B}$. Then $\mathrm{t.proj.dim}M\le n$ if and only if $\mathrm{Ext}^{n+1}_{\mathscr{B}}(M,N)=0$ for any $N\in\mathcal{T}(T)$.*

Proof $\Leftarrow$. By the definition of the tilting projective dimension, if there exists an exact sequence

$$0\longrightarrow X\longrightarrow P_{n-1}\longrightarrow\cdots\longrightarrow P_1\longrightarrow P_0\longrightarrow M\longrightarrow 0$$

where all P_i's are tilting projective. Now we only need to prove that X is also tilting projective. By the Dimension-Shift, we have the following isomorphism

$$\mathrm{Ext}^{n+1}_{\mathscr{B}}(M,N)\cong \mathrm{Ext}^1_{\mathscr{B}}(X,N)=0.$$

Using Lemma 2.3, we obtain that X is tilting projective.

$\Rightarrow$. We will prove the necessity by using induction on n: If $\mathrm{t.proj.dim}M\le 1$, then there is an exact sequence

$$0\longrightarrow P_1\longrightarrow P_0\longrightarrow M\longrightarrow 0$$

with tilting projectives P_0 and P_1. By applying $\mathrm{Hom}_{\mathscr{B}}(-,N)$, we obtain that

$$0\longrightarrow \mathrm{Hom}_{\mathscr{B}}(M,N)\longrightarrow \mathrm{Hom}_{\mathscr{B}}(P_0,N)\longrightarrow \mathrm{Hom}_{\mathscr{B}}(P_1,N)\longrightarrow \mathrm{Ext}^1_{\mathscr{B}}(M,N)$$
$$\longrightarrow \mathrm{Ext}^1_{\mathscr{B}}(P_0,N)\longrightarrow\cdots.$$

Thus, $\mathrm{Ext}^2_{\mathscr{B}}(M,N)\cong \mathrm{Ext}^1_{\mathscr{B}}(P_1,N)=0$ for any $N\in\mathcal{T}(T)$. We now suppose the result holds for $\mathrm{t.proj.dim}M\le n-1$, then there exists an exact sequence

$$0\longrightarrow P_n\longrightarrow P_{n-1}\longrightarrow\cdots\longrightarrow P_1\longrightarrow P_0\longrightarrow M\longrightarrow 0\ .$$

However, by the assumption that $\mathscr{B}$ has enough injective objects, so we can obtain an exact sequence

$$0 \longrightarrow N \longrightarrow E \longrightarrow N' \longrightarrow 0$$

with E injective. It follows that N' and E are in $\mathcal{T}(T)$ since T is tilting. Thus we have

$$\mathrm{Ext}^{n+1}_{\mathscr{B}}(M,N) \cong \mathrm{Ext}^{n}_{\mathscr{B}}(M,N') = 0.$$

By induction assumption the necessity holds. □

Lemma 2.5 *Let* $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow M_4 \longrightarrow 0$ *be an exact sequence in an abelian category* $\mathscr{B}$ *with enough projective and injective objects.*

(1) If $M_4 = 0$*, then we have*

(i) if $\mathrm{t.proj.dim}M_1 < \mathrm{t.proj.dim}M_2$*, then* $\mathrm{t.proj.dim}M_3 = \mathrm{t.proj.dim}M_2$*;*

(ii) if $\mathrm{t.proj.dim}M_1 > \mathrm{t.proj.dim}M_2$*, then* $\mathrm{t.proj.dim}M_3 = \mathrm{t.proj.dim}M_1 + 1$*;*

(iii) if $\mathrm{t.proj.dim}M_1 = \mathrm{t.proj.dim}M_2$*, then* $\mathrm{t.proj.dim}M_3 \leq \mathrm{t.proj.dim}M_1 + 1$*.*

(2) If $M_4 \neq 0$*, then*

$$\mathrm{t.proj.dim}M_3 \leq \max\{\mathrm{t.proj.dim}M_1 + 1, \mathrm{t.proj.dim}M_2, \mathrm{t.proj.dim}M_4\}.$$

Proof When $M_4 = 0$, for any $N \in \mathcal{T}(T)$ and $n \geq 0$, there exists a long exact sequence as follows

$$\cdots \longrightarrow \mathrm{Ext}^{n}_{\mathscr{B}}(M_3,N) \longrightarrow \mathrm{Ext}^{n}_{\mathscr{B}}(M_2,N) \longrightarrow \mathrm{Ext}^{n}_{\mathscr{B}}(M_1,N) \longrightarrow \mathrm{Ext}^{n+1}_{\mathscr{B}}(M_3,N)$$

$$\longrightarrow \mathrm{Ext}^{n+1}_{\mathscr{B}}(M_2,N) \longrightarrow \mathrm{Ext}^{n+1}_{\mathscr{B}}(M_1,N) \longrightarrow \cdots$$

Case 1. If $m \geq n$, and $\mathrm{Ext}^{m}_{\mathscr{B}}(M_1,N) = 0$ but $\mathrm{Ext}^{n}_{\mathscr{B}}(M_2,N) \neq 0$, then $\mathrm{Ext}^{n}_{\mathscr{B}}(M_3,N) \neq 0$. So for $j > 0$ we have the isomorphism

$$\mathrm{Ext}^{n+j}_{\mathscr{B}}(M_3,N) \cong \mathrm{Ext}^{n+j}_{\mathscr{B}}(M_2,N).$$

Thus, $\mathrm{t.proj.dim}M_3 = \mathrm{t.proj.dim}M_2$.

Case 2. If $m \geq n$, and $\mathrm{Ext}^{m}_{\mathscr{B}}(M_2,N) = 0$ but $\mathrm{Ext}^{n}_{\mathscr{B}}(M_1,N) \neq 0$, then $\mathrm{Ext}^{n+1}_{\mathscr{B}}(M_3,N) \neq 0$ and for any $j = 1,2,\ldots, \mathrm{Ext}^{n+j}_{\mathscr{B}}(M_3,N) \cong \mathrm{Ext}^{n+j-1}_{\mathscr{B}}(M_1,N)$. Hence

$$\mathrm{t.proj.dim}M_3 = \mathrm{t.proj.dim}M_1 + 1.$$

Case 3. If $m \geq n$, and $\mathrm{Ext}^{m}_{\mathscr{B}}(M_2,N) = \mathrm{Ext}^{m}_{\mathscr{B}}(M_1,N) = 0$, then

$$\mathrm{Ext}^{n+1}_{\mathscr{B}}(M_3,N) = 0 = \mathrm{Ext}^{n+2}_{\mathscr{B}}(M_3,N).$$

So $\mathrm{t.proj.dim}M_3 \leq \mathrm{t.proj.dim}M_1 + 1$.

The assertion for $M_4 \neq 0$ follows directly from the above result. □

For convenience, we define the $\mathscr{A}$-relative tilting global dimension of $\mathscr{B}$ by

$$\mathrm{t.gl.dim}_{\mathscr{A}}\mathscr{B} := \sup\{\ \mathrm{t.proj.dim}_{\mathscr{B}} i_*(A) \mid \forall A \in \mathscr{A}\ \}.$$

Lemma 2.6 *Let* $(\mathscr{A},\mathscr{B},\mathscr{C})$ *be a recollement of abelian categories such that* $\mathscr{C}$ *has enough projective objects. Then*

$$\mathrm{t.proj.dim}_{\mathscr{B}} j_!(C) \leq \mathrm{t.proj.dim}_{\mathscr{C}} C + \mathrm{t.gl.dim}_{\mathscr{A}}\mathscr{B} + 1.$$

Proof If $\mathrm{t.proj.dim}_{\mathscr{C}}C=\infty$ or $\mathrm{t.gl.dim}_{\mathscr{A}}\mathscr{B}=\infty$, then the assertion is obvious. We only have to consider the case finite dimension. We will prove by using induction on $\mathrm{t.proj.dim}_{\mathscr{C}}C$. Write $\mathrm{t.gl.dim}_{\mathscr{A}}\mathscr{B}=n$. Firstly we suppose that C is a tilting projective object in $\mathscr{C}$, then it follows from Lemma 2.3 that $j_!(C)$ is a tilting projective object in $\mathscr{B}$ since $j_!$ is fully faithful. And so the result holds. Secondly we assume that $\mathrm{t.proj.dim}_{\mathscr{C}}C=m$ and that the result also holds for any object of $\mathscr{C}$ with the tilting projective dimension less than m, i.e., $\mathrm{t.proj.dim}_{\mathscr{B}}j_!(C')\le \mathrm{t.proj.dim}_{\mathscr{C}}C'+n+1$ for any object $C'\in\mathscr{C}$ and $\mathrm{t.proj.dim}_{\mathscr{C}}C'<m$. Since $\mathrm{t.proj.dim}_{\mathscr{C}}C<m$, it follows that there exists an exact sequence as follows

$$0\longrightarrow T_m\xrightarrow{t_m}T_{m-1}\longrightarrow\cdots\longrightarrow T_1\xrightarrow{t_1}T_0\xrightarrow{t_0}C\longrightarrow 0$$

with $T_i\in\mathrm{Tproj}\mathscr{C}$ (the subcategory of all tilting projective objects of $\mathscr{C}$). If we take $K_0=\mathrm{Ker}t_0$, then it is easy to see that $\mathrm{t.proj.dim}_{\mathscr{C}}K_0<m$. By the induction hypothesis, we can obtain that $\mathrm{t.proj.dim}_{\mathscr{B}}j_!(K_0)\le\mathrm{t.proj.dim}_{\mathscr{C}}K_0+n+1$. Applying the right exact functor $j_!$ to the short exact sequence

$$0\longrightarrow K_0\xrightarrow{i_0}T_0\xrightarrow{a_0}C\longrightarrow 0\ ,$$

we have that

$$0\longrightarrow L_1(j_!C)\longrightarrow j_!(K_0)\xrightarrow{j_!i_0}j_!(T_0)\xrightarrow{j_!a_0}j_!(C)\longrightarrow 0 \tag{2.1}$$

with $\mathrm{Ker}(j_!a_0)=K_0'$. However, since $j^*:\mathscr{B}\to\mathscr{C}$ is exact and $\mathrm{Id}_{\mathscr{C}}\simeq j^*j_!$ it follows that $j^*\mathrm{Ker}(j_!i_0)\cong\mathrm{Ker}(i_0)$. Thus, $j^*\mathrm{Ker}(j_!i_0)=0$, and hence $\mathrm{t.proj.dim}_{\mathscr{B}}L_1(j_!C)\le n$. Thus from Lemma 2.3 and the short exact sequence

$$0\longrightarrow L_1j_!(C)\longrightarrow j_!(K_0)\longrightarrow \mathrm{Ker}j_!(a_0)\longrightarrow 0\ ,$$

we obtain that $\mathrm{t.proj.dim}_{\mathscr{B}}\mathrm{Ker}(j_!a_0)\le m+n$. Therefore, it follows from (2.1) that

$$\mathrm{t.proj.dim}_{\mathscr{B}}j_!(C)\le m+n+1$$

since $j_!(T_0)\in\mathrm{Tproj}\mathscr{B}$. □

Theorem 2.7 *Let $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of abelian categories such that $\mathscr{B}$ and $\mathscr{C}$ have enough projective and injective objects. Then we have an upper bound for the tilting global dimension of $\mathscr{B}$*

$$\mathrm{t.gl.dim}\mathscr{B}\le\mathrm{t.gl.dim}\mathscr{A}+\mathrm{t.gl.dim}\mathscr{C}+\sup\{\mathrm{t.proj.dim}_{\mathscr{B}}i_*(P)\mid P\in\mathrm{Tproj}(\mathscr{A})\}+1$$

where $\mathrm{Tproj}(\mathscr{A})$ *is the tilting projective subcategory of* $\mathscr{A}$*.*

Proof Let B be an object in $\mathscr{B}$. Suppose that $\mathrm{t.gl.dim}_{\mathscr{A}}\mathscr{B}=n<\infty$ and $\mathrm{t.gl.dim}\mathscr{C}=m<\infty$. From [6, Proposition 2.6] there exists the following exact sequence

$$0\longrightarrow\mathrm{Ker}\mu_B\longrightarrow j_!j^*(B)\xrightarrow{\mu_B}B\longrightarrow\mathrm{Coker}\mu_B\longrightarrow 0\ ,$$

with $\mathrm{Ker}\mu_B\in i_*(\mathscr{A})$ and $\mathrm{Coker}\mu_B\in i(\mathscr{A})$. So we have

$$\mathrm{t.proj.dim}_{\mathscr{B}}\mathrm{Ker}\mu_B\le n\ \text{and}\ \mathrm{t.proj.dim}_{\mathscr{B}}\mathrm{Coker}\mu_B\le n.$$

By Lemmas 2.5 and 2.6, it is easy to see

$$\begin{aligned}\text{t.proj.dim}_{\mathscr{B}}B &\leq \max\{n+1, \text{t.proj.dim}_{\mathscr{B}} j_! j^*(B)\}\\ &\leq \max\{n+1, \text{t.proj.dim}_{\mathscr{C}} j^*(B)+n+1\}\\ &= \text{t.proj.dim}_{\mathscr{C}} j^*(B)+n+1.\end{aligned}$$

Since $j^*(B)$ is an object of $\mathscr{C}$, we infer that $\text{t.proj.dim}_{\mathscr{B}}B \leq m+n+1$. Hence, $\text{t.gl.dim}\mathscr{B} \leq \text{t.gl.dim}_{\mathscr{A}}\mathscr{B} + \text{t.gl.dim}\mathscr{C} + 1$. Furthermore, for any $A \in \mathscr{A}$ we assume that

$$\sup\{\text{t.proj.dim}_{\mathscr{B}} i_*(P) \mid P \in \text{Tproj}(\mathscr{A})\} = n < \infty.$$

In order to prove

$$\text{t.gl.dim}\mathscr{B} \leq \text{t.gl.dim}\mathscr{A} + \text{t.gl.dim}\mathscr{C} + \sup\{\text{t.proj.dim}_{\mathscr{B}} i_*(P) \mid P \in \text{Tproj}(\mathscr{A})\} + 1,$$

it suffices to show that

$$\text{t.gl.dim}_{\mathscr{A}}\mathscr{B} \leq \text{t.gl.dim}\mathscr{A} + \sup\{\text{t.proj.dim}_{\mathscr{B}} i_*(P) \mid P \in \text{Tproj}(\mathscr{A})\}.$$

So we only need to check that $\text{t.proj.dim}_{\mathscr{B}} i_*(A) \leq \text{t.proj.dim}_{\mathscr{A}} A + n$. If A is a tilting projective object of $\mathscr{A}$, then $\text{t.proj.dim}_{\mathscr{B}} i_*(\mathscr{A}) \leq n$ and so our result holds. Now suppose that $\text{t.proj.dim}_{\mathscr{A}} A = m$, then we have the exact sequence

$$0 \longrightarrow T_m \longrightarrow T_{m-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow A \longrightarrow 0$$

with $T_i \in \text{Tproj}\mathscr{A}$ for $0 \leq i \leq m$. So we know that $\text{t.proj.dim}_{\mathscr{B}} i_*(T_i) \leq m+n$. Therefore, $\text{t.gl.dim}_{\mathscr{A}}\mathscr{B} \leq \text{t.gl.dim}\mathscr{A} + \sup\{\text{t.proj.dim}_{\mathscr{B}} i_*(P) \mid P \in \text{Tproj}(\mathscr{A})\}$. We conclude that

$$\text{t.gl.dim}\mathscr{B} \leq \text{t.gl.dim}\mathscr{A} + \text{t.gl.dim}\mathscr{C} + \sup\{\text{t.proj.dim}_{\mathscr{B}} i_*(P) \mid P \in \text{Tproj}(\mathscr{A})\} + 1. \quad \square$$

Here is a well-known example of recollements of abelian categories, which can be refereed to [5, Example 2.10], [6, Example 2.7], [16, Proposition 2.7] for more details.

Example 2.8 Let $\Lambda = \left[\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right]$ be the triangular matrix algebra defined above. Then there exists a recollement as follows

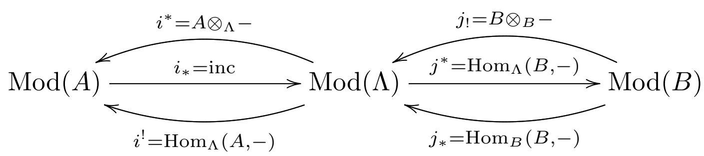

Diagram 2 The recollement of module categories over the triangular matrix algebra

Clearly,

$$\begin{aligned}\text{t.gl.dimMod}(\Lambda) \leq &\text{t.gl.dimMod}(A) + \text{t.gl.dimMod}(B) +\\ &\sup\{\text{t.proj.dim}_{\Lambda} i_*(P) \mid P \in \text{Tproj}(\text{Mod}(A))\} + 1.\end{aligned}$$

We say an abelian category $\mathscr{B}$ is tilting hereditary if $\text{t.gl.dim}\mathscr{B} \leq 1$. That is, if T is a tilting object in $\mathscr{B}$, we always have $\text{Ext}^2_{\mathscr{B}}(B, L) = 0$ for all $B \in \mathscr{B}$ and $L \in \mathcal{T}(T)$. As a corollary of Theorem 2.7, the following result shows the properties of tilting hereditary in a recollement $(\mathscr{A},\mathscr{B},\mathscr{C})$, which plays a crucial role in studying almost split sequences in the categories involved in a recollement $(\mathscr{A},\mathscr{B},\mathscr{C})$.

Corollary 2.9 *Let $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of abelian categories such that $\mathscr{B}$ and $\mathscr{C}$ have enough projective objects. If $\mathscr{B}$ is tilting hereditary, then $\mathscr{A}$ and $\mathscr{C}$ are also tilting hereditary.*

3. The Auslander-Reiten theory

Now it is convenient to recall the following notions [6]. Let $\mathscr{A}$ be an abelian category and $\mathcal{K}(\mathscr{A})$ the homotopy category of complexes over $\mathscr{A}$. Then there exists a triangulated category $\mathcal{D}(\mathscr{A})$, which is the derived category of $\mathscr{A}$. Denote by $\mathcal{D}^b(\mathscr{A})$ the full subcategory of $\mathcal{D}(\mathscr{A})$ with objects being those complexes which have bound cohomology. In particular, there is a canonical embedding of $\mathscr{A}$ into $\mathcal{D}(\mathscr{A})$.

Theorem 3.1 *Let $(\mathscr{A},\mathscr{B},\mathscr{C})$ be a recollement of abelian categories and $\mathscr{B}$ be a hereditary abelian category with a tilting object T, and suppose that $\mathscr{B}$ and $\mathscr{C}$ have enough projectives. If the functors $j_!$ and j_* are exact, then both $\mathscr{A}$ and $\mathscr{C}$ have almost split sequences.*

Proof Firstly, it is easy to see from [6, Theorem 4.8] that $\mathscr{A}$ and $\mathscr{C}$ are also hereditary. Secondly, we also know that i^*T and j^*T are tilting objects in $\mathscr{A}$ and $\mathscr{C}$, respectively. If we take $\Lambda_{\mathscr{A}} = \mathrm{End}(i^*T)^{op}$, $\Lambda_{\mathscr{C}} = \mathrm{End}(j^*T)^{op}$ and $\Lambda_{\mathscr{B}} = \mathrm{End}(T)^{op}$, then $\mathcal{D}^b(\mathscr{A})$ and $\mathcal{D}^b(\Lambda_{\mathscr{A}})$, $\mathcal{D}^b(\mathscr{B})$ and $\mathcal{D}^b(\Lambda_{\mathscr{B}})$, $\mathcal{D}^b(\mathscr{C})$ and $\mathcal{D}^b(\Lambda_{\mathscr{C}})$ are derived equivalent. Finally, we conclude by [9, Proposition 4.8] that both $\mathscr{A}$ and $\mathscr{C}$ have almost split sequences. Now we will give a proof by using the definition of the almost split sequence directly. We only prove that $\mathscr{C}$ has almost split sequences, it is similar for $\mathscr{A}$. For any indecomposable non-projective object C in $\mathscr{C}$, it suffices to show that there exists an exact sequence

$$0 \longrightarrow C'' \xrightarrow{f} C' \xrightarrow{g} C \longrightarrow 0 \tag{3.1}$$

satisfying the following conditions

(i) C and C'' are indecomposable in $\mathscr{C}$;

(ii) It is non-split;

(iii) Any morphism $h : W \to C$ which is not a split epimorphism factors through g.

Now we give the proof in three steps:

Step 1. We have that the sequence

$$0 \longrightarrow j_!C'' \xrightarrow{j_!f} j_!C' \xrightarrow{j_!g} j_!C \longrightarrow 0 \tag{3.2}$$

is an almost split sequence in $\mathscr{B}$ by applying the exact functor $j_!$ to (3.1). According to the definition of almost split sequences, we only need to verify that $j_!C$ is indecomposable non-projective. Firstly, we claim that $j_!C$ is indecomposable. Otherwise, there is an isomorphism $j_!C \cong B_1 \oplus B_2$ with nonzero objects B_1 and B_2 in $\mathscr{B}$. Since $\mathrm{Id}_{\mathscr{C}} \simeq j^*j_!$ and j^* commutes with any direct sums, it follows that

$$C \cong j^*j_!C \cong j^*(B_1 \oplus B_2) \cong j^*B_1 \oplus j^*B_2.$$

So we have $C \cong j^*B_1 \oplus j^*B_2$, which is a contradiction. Hence $j_!C$ is indecomposable. Similarly, we can show that $j_!C''$ is also indecomposable. Secondly, we claim that $j_!C$ is non-projective. Otherwise, the sequence (3.2) is split. After applying the exact functor j^* it derives the exact sequence (3.1) since $\mathrm{Id}_{\mathscr{C}} \simeq j^*j_!$, which contradicts the hypothesis that C is non-projective. Thus, $j_!C$ is non-projective. So $j_!C$ is indecomposable non-projective.

Step 2. We now claim that C'' is indecomposable. It is known from the hypothesis that C is already indecomposable. If C'' is not indecomposable, then there are two nonzero objects C_1 and C_2 in $\mathscr{C}$ such that $C'' \cong C_1 \oplus C_2$. it deduces that $j_!C'' \cong j_!(C_1 \oplus C_2) \cong j_!C_1 \oplus j_!C_2$ by applying the exact functor $j_!$. This is a contradiction with the indecomposable object $j_!C''$.

Step 3. We next prove that the assertion for condition (iii) holds. For any morphism $h : W \to C$ which is not a split epimorphism, then we have that $j_!h : j_!W \to j_!C$ is also not a split epimorphism in $\mathscr{B}$ since $j_!$ is a right exact functor. Thus for the sequence (3.2), there exists a morphism $j_!t : j_!W \to j_!C'$ such that $j_!h = j_!g \circ j_!t$. Applying the exact functor j^* again, we get a morphism $t : W \to C'$ such that $h = g \circ t$, this means that h factors through g.

Finally, the condition (ii) can be verified easily by reduction to absurdity. This shows that for any indecomposable non-projective object C in $\mathscr{C}$, there exists an almost split sequence. Consequently, $\mathscr{C}$ has almost split sequences. □

The final main result of this section is to show that the above theorem holds for the situation of n-almost split sequences [12, 17]. Now let us give the definition of the n-almost split sequence in a Krull-Schmidt abelian category $\mathscr{B}$. It is easy to see from [6, Section 6] that $\mathscr{A}$ and $\mathscr{C}$ involved in a recollement $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ are Krull-Schmidt abelian categories, and if $\mathscr{B}$ is of finite representation type, then it follows that $\mathscr{A}$ and $\mathscr{C}$ are of finite representation type.

Definition 3.2 ([18]) *Let $\mathscr{B}$ be a representation finite abelian category and let $n \in \mathbb{Z}_{>0}$. An n-cluster tilting object M in $\mathscr{B}$ is an object such that*

$$\begin{aligned}\mathrm{add}M &= \{X \in \mathscr{B} \mid \mathrm{Ext}^i_{\mathscr{B}}(M, X) = 0, \forall\ 0 < i < n\} \\ &= \{X \in \mathscr{B} \mid \mathrm{Ext}^i_{\mathscr{B}}(X, M) = 0, \forall\ 0 < i < n\}\end{aligned}$$

Lemma 3.3 *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories with the exact functor i^*. If M is an n-cluster tilting object in $\mathscr{B}$, then j^*M and i^*M are n-cluster tilting in $\mathscr{C}$ and $\mathscr{A}$, respectively.*

Proof We only prove that j^*M is an n-cluster tilting object in $\mathscr{C}$. It can be proved similarly that i^*M is n-cluster tilting in $\mathscr{A}$. By Definition 3.2, we have to check that the following equality holds

$$\begin{aligned}\mathrm{add}(j^*M) &= \{Y \in \mathscr{C} \mid \mathrm{Ext}^i_{\mathscr{C}}(j^*M, Y) = 0, \forall\ 0 < i < n\} \\ &= \{Y \in \mathscr{C} \mid \mathrm{Ext}^i_{\mathscr{C}}(Y, j^*M) = 0, \forall\ 0 < i < n\}.\end{aligned}$$

Since the exact functor j^* commutes with any direct sums, it follows that

$$\begin{aligned}\mathrm{add}(j^*M) = j^*(\mathrm{add}M) &= j^*\{X \in \mathscr{B} \mid \mathrm{Ext}^i_{\mathscr{B}}(M,X) = 0, \forall\, 0<i<n\} \\ &= j^*\{X \in \mathscr{B} \mid \mathrm{Ext}^i_{\mathscr{B}}(X,M) = 0, \forall\, 0<i<n\} \\ &= \{j^*X \in \mathscr{C} \mid \mathrm{Ext}^i_{\mathscr{C}}(j^*M, j^*X) = 0, \forall\, 0<i<n\} \\ &= \{j^*X \in \mathscr{C} \mid \mathrm{Ext}^i_{\mathscr{C}}(j^*X, j^*M) = 0, \forall\, 0<i<n\}.\end{aligned}$$

Therefore, all objects Y are actually j^*X with X in add$M$. □

Definition 3.4 *Let $\mathscr{B}$ be a representation finite abelian category, and M be the n-cluster tilting object in $\mathscr{B}$. An exact sequence*

$$0 \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{f_1} C_0 \longrightarrow 0$$

with $C_i \in \mathrm{add}M$ is said to be an n-almost split sequences if the following holds

(i) *For every i, we have $f_i \in \mathrm{rad}(C_i, C_{i-1})$.*

(ii) *The objects C_{n+1} and C_0 are indecomposable.*

(iii) *The sequence*

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{B}}(X, C_{n+1}) \xrightarrow{f^*_{n+1}} \mathrm{Hom}_{\mathscr{B}}(X, C_n) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_{\mathscr{B}}(X, C_1) \xrightarrow{f^*_1} \mathrm{rad}_{\mathscr{B}}(X, C_0) \longrightarrow 0$$

is exact for any $X \in \mathrm{add}M$.

Remark 3.5 (i) $\mathrm{rad}_{\mathscr{B}}(-,-)$ is the subfunctor of $\mathrm{Hom}_{\mathscr{B}}(-,-)$, which is defined by

$$\mathrm{rad}_{\mathscr{B}}(X,Y) = \{f \in \mathrm{Hom}_{\mathscr{B}}(X,Y) \mid hfg \text{ is not an isomorphism for any } g: A \to X \text{ and } h: Y \to A \text{ with } A\,\mathrm{ind}(\mathscr{B})\},$$

where $\mathrm{ind}(\mathscr{B})$ denotes the subcategory of $\mathscr{B}$ consisting of all indecomposable objects of $\mathscr{B}$.

(ii) An abelian category $\mathscr{B}$ is n-representation-finite if gl.dim$\mathscr{B} \leq n$ and there exists an n-cluster tilting object M in $\mathscr{B}$. Note from [6] that if $\mathscr{B}$ is an representation-finite abelian category, then it follows that $\mathscr{A}$ and $\mathscr{C}$ are representation-finite abelian categories since $\mathscr{A}$ and $\mathscr{C}$ are fully embedded in $\mathscr{B}$. In fact, if the functor i^* and j^* are exact, for an n-cluster tilting object M, then j^*M and i^*M are (at most) n-cluster tilting in $\mathscr{C}$ and $\mathscr{A}$, respectively.

Theorem 3.6 *Let $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories, and $\mathscr{B}$ be the representation-finite abelian category with an n-cluster tilting object M. If $\mathscr{B}$ has n-almost split sequences, $\mathrm{add}j_!j^*(\mathscr{B}) \subseteq \mathrm{add}\mathscr{B}$, the functor i^* is exact, then $\mathscr{A}$ and $\mathscr{C}$ have (at most) n-almost split sequences.*

Proof Here we only show that $\mathscr{C}$ has (at most) n-almost split sequence. It is similar to the proof of $\mathscr{A}$. Let

$$0 \longrightarrow B_{n+1} \xrightarrow{b_{n+1}} B_n \longrightarrow \cdots \longrightarrow B_1 \xrightarrow{b_1} B_0 \longrightarrow 0$$

be an n-almost split sequence in $\mathscr{B}$ with $B_i \in \mathrm{add}M$. Applying the exact functor j^*, we get the following exact sequence

$$0 \longrightarrow j^*(B_{n+1}) \xrightarrow{j^*(b_{n+1})} j^*(B_n) \longrightarrow \cdots \longrightarrow j^*(B_1) \xrightarrow{j^*(b_1)} j^*(B_0) \longrightarrow 0$$

with $j^*(B_i) \in \mathrm{add} j^*M$. It suffices to prove that it is an n-almost split sequence in $\mathscr{C}$.

Step 1. We firstly show that $j^*(B_{n+1})$ and $j^*(B_0)$ are indecomposable in $\mathscr{C}$. Here we only prove that $j^*(B_0)$ is indecomposable, it is similar for $j^*(B_{n+1})$. Now we assume that $j^*(B_0)$ is decomposable, that is, there exist two non-zero objects C_0 and C_1 in $\mathscr{C}$ such that $j^*(B_0) \cong C_0 \oplus C_1$. It follows from [4, Lemmas 3.1(4) and 3.2(4)] that the exactness of the functor i^* is equivalent to the exactness of the functor $j_!$. So after applying the functor $j_!$ we obtain that $j_!j^*(B_0) \cong j_!C_0 \oplus j_!C_1$ with non-zero objects $j_!C_0$ and $j_!C_1$. Otherwise, if we suppose that $j_!C_0$ is a zero object, then we find that $j^*j_!(C_0) \cong C_0$ since $j^*j_! \simeq \mathrm{Id}_{\mathscr{C}}$, a contradiction with non-zero object C_0. Since $\mathrm{add} j_!j^*(\mathscr{B}) \subseteq \mathrm{add}\mathscr{B}$, it follows that $j_!C_0 \in \mathrm{add} j_!j^*(B_0) \subseteq \mathrm{add}B_0$. So there exists an object $B' \in \mathscr{B}$ such that $B_0 \cong B' \oplus j_!C_0$, which is a contradiction with the indecomposable object B_0. Similarly, one can show that $j^*(B_{n+1})$ is also indecomposable.

Step 2. We will verify that $j^*(b_i) \in \mathrm{rad}(j^*(B_i), j^*(B_{i-1}))$ for every i, that is, $h \circ j^*(b_i) \circ g$ is not an isomorphism for any $h : C \to j^*(B_i)$ and $g : j^*(B_{i-1}) \to C$ with indecomposable object C of $\mathscr{C}$.

$$\begin{array}{ccc} & & C \\ & \swarrow^{h} & \uparrow g \\ j^*(B_i) & \xrightarrow{j^*(b_i)} & j^*(B_{i-1}) \end{array}$$

Now we assume that $h \circ j^*(b_i) \circ g$ is an isomorphism. Then we can obtain the following diagram

$$\begin{array}{ccc} & & j_!(C) \\ & \swarrow^{j_!h} & \uparrow j_!g \\ j_!j^*(B_i) & \xrightarrow{j_!j^*(b_i)} & j_!j^*(B_{i-1}) \end{array}$$

since $j_!$ is fully faithful and $\mathrm{add} j_!j^*(\mathscr{B}) \subseteq \mathrm{add}\mathscr{B}$ by applying the functor $j_!$. So we find that $j_!h \circ j_!j^*(b_i) \circ j_!g$ is also an isomorphism, a contradiction with $j_!j^*(b_i) \in \mathrm{rad}(j_!j^*(B_i), j_!j^*(B_{i-1})) \subseteq \mathrm{rad}(B_i, B_{i-1})$.

Step 3. We claim that the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{C}}(X, j^*(B_{n+1})) \xrightarrow{j^*(b_{n+1})^*} \mathrm{Hom}_{\mathscr{C}}(X, j^*(B_n)) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_{\mathscr{C}}(X, j^*(B_1))$$

$$\xrightarrow{j^*(b_1)^*} \mathrm{rad}_{\mathscr{C}}(X, j^*(B_0)) \longrightarrow 0$$

is exact for any $X \in \mathrm{add}(j^*M)$. Since $X \in \mathrm{add}(j^*M)$ it follows that $lX \in \mathrm{add}(j_!j^*M) \subseteq \mathrm{add}M$. So we have the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{B}}(j_!X, B_{n+1}) \xrightarrow{b_{n+1}^*} \mathrm{Hom}_{\mathscr{B}}(j_!X, B_n) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_{\mathscr{B}}(j_!X, B_1)$$

$$\xrightarrow{b_1^*} \mathrm{rad}_{\mathscr{B}}(j_!X, B_0) \longrightarrow 0$$

Actually, $\mathrm{rad}_{\mathscr{C}}(X, j^*(B_0)) = \mathrm{Hom}_{\mathscr{C}}(X, j^*(B_0))$ and $\mathrm{rad}_{\mathscr{B}}(j_!X, B_0) = \mathrm{Hom}_{\mathscr{B}}(j_!X, B_0)$. The adjoint isomorphisms $\mathrm{Hom}_{\mathscr{B}}(j_!X, B_i) \cong \mathrm{Hom}_{\mathscr{C}}(X, j^*(B_i))$ $(\forall\, 0 < i < n+1)$ ensure that the claim holds naturally. Consequently, $\mathscr{C}$ has (at most) n-almost split sequence. □

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