Journal of Mathematical Research with Applications
May, 2020, Vol. 40, No. 3, pp. 305–312
DOI:10.3770/j.issn:2095-2651.2020.03.007
Http://jmre.dlut.edu.cn

The Open-Point and Compact-Open Topology on $C(X)$

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Abstract In this note we define a new topology on $C(X)$, the set of all real-valued continuous functions on a Tychonoff space X. The new topology on $C(X)$ is the topology having subbase open sets of both kinds: $[f,C,\varepsilon]=\{g\in C(X):|f(x)-g(x)|<\varepsilon$ for every $x\in C\}$ and $[U,r]^-=\{g\in C(X):g^{-1}(r)\cap U\neq\emptyset\}$, where $f\in C(X)$, $C\in\mathcal{K}(X)=\{$ nonempty compact subsets of $X\}$, $\epsilon>0$, while U is an open subset of X and $r\in\mathbb{R}$. The space $C(X)$ equipped with the new topology $\mathcal{T}_{kh}$ which is stated above is denoted by $C_{kh}(X)$. Denote $X_0=\{x\in X: x$ is an isolated point of $X\}$ and $X_c=\{x\in X: x$ has a compact neighborhood in $X\}$.

We show that if X is a Tychonoff space such that $X_0=X_c$, then the following statements are equivalent:

(1) X_0 is G_δ-dense in X;
(2) $C_{kh}(X)$ is regular;
(3) $C_{kh}(X)$ is Tychonoff;
(4) $C_{kh}(X)$ is a topological group.

We also show that if X is a Tychonoff space such that $X_0=X_c$ and $C_{kh}(X)$ is regular space with countable pseudocharacter, then X is σ-compact. If X is a metrizable hemicompact countable space, then $C_{kh}(X)$ is first countable.

Keywords $C_p(X)$; $C_k(X)$; $C_{kh}(X)$; G_δ-dense

MR(2010) Subject Classification 54C30; 54C35; 54C40

1. Introduction

The set $C(X)$ of all real-valued continuous functions on a Tychonoff space X has a number of topologies. Two commonly used topology on $C(X)$ are the point-open topology $\mathcal{T}_p$ and the compact-open topology $\mathcal{T}_k$. The symbol $\mathbb{R}$ denotes the space of real numbers with the natural topology. The space $C(X)$ with the point-open topology $\mathcal{T}_p$ is denoted by $C_p(X)$. It has a subbase consisting of sets of the form $[x,V]^+=\{f\in C(X):f(x)\in V\}$, where $x\in X$ and V is open in $\mathbb{R}$. Given a subset A of a Tychonoff space X, $f\in C(X)$ and $\varepsilon>0$, define $[f,A,\varepsilon]=\{g\in C(X):|f(x)-g(x)|<\varepsilon$ for every $x\in A\}$. If $\mathcal{K}(X)$ is the collection of all nonempty compact subsets of X, then for each $f\in C(X)$, the family $\{[f,K,\varepsilon]:K\in\mathcal{K}(X),\varepsilon>0\}$ forms a neighborhood base at f in the compact-open topology $\mathcal{T}_k$ on $C(X)$. When $C(X)$ is equipped with the compact-open topology $\mathcal{T}_k$, we denote the corresponding space by $C_k(X)$.

Received June 3, 2019; Accepted October 9, 2019
Supported by the National Natural Science Foundation of China (Grant No. 11771029) and the Natural Science Foundation of Beijing City (Grant No. 1202003).
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In [1], Jindal, McCoy and Kundu introduced two new topologies on $C(X)$ that are called the open-point topology and the bi-point-open topology. The open-point topology on $C(X)$ has a subbase consisting of sets of the form $[U,r]^- = \{f \in C(X) : f^{-1}(r) \cap U \neq \emptyset\}$, where U is an open subset of X and $r \in \mathbb{R}$. The open-point topology on $C(X)$ is denoted by $\mathcal{T}_h$ and the space $C(X)$ equipped with the open-point topology $\mathcal{T}_h$ is denoted by $C_h(X)$.

The bi-point-open topology on $C(X)$ is the topology having subbase open sets of both kinds: $[x,V]^+$ and $[U,r]^-$, where $x \in X$ and V is an open subset of $\mathbb{R}$, while U is an open subset of X and $r \in \mathbb{R}$. The space $C(X)$ equipped with the bi-point-open topology $\mathcal{T}_{ph}$ is denoted by $C_{ph}(X)$. Some properties of $C_h(X)$ and $C_{ph}(X)$ are discussed in [1]. A nonempty subset of a space X is said to be G_δ-dense provided that it meets every nonempty G_δ-subset of X (see [1]). Recall that a paratopological group is a group with a topology such that multiplication on the group is jointly continuous. A topological group G is a paratopological group with a topology such that the inverse mapping of G into itself associating x^{-1} with $x \in G$ is continuous [2]. In [1], it was proved that the space $C_h(X)$ is Hausdorff if and only if the set of isolated points in X is dense in X. In [1], it was also proved that the spaces $C_h(X)$ and $C_{ph}(X)$ are completely regular if and only if they are topological groups if and only if the set of isolated points in X is G_δ-dense in X.

In this note, we introduce a new topology on $C(X)$. The new topology on $C(X)$ is the topology having subbase open sets of both kinds: $[f,C,\varepsilon]$ and $[U,r]^-$, where $f \in C(X)$, $C \in \mathcal{K}(X)$, $\epsilon > 0$, while U is an open subset of X and $r \in \mathbb{R}$. The space $C(X)$ equipped with the new topology $\mathcal{T}_{kh}$ which is stated above is denoted by $C_{kh}(X)$. Denote $X_0 = \{x \in X : x$ is an isolated point of $X\}$ and $X_c = \{x \in X : x$ has a compact neighborhood in $X\}$. We show that if X is a Tychonoff space such that $X_0 = X_c$, then the following statements are equivalent:

(1) X_0 is G_δ-dense in X;

(2) $C_{kh}(X)$ is regular;

(3) $C_{kh}(X)$ is Tychonoff;

(4) $C_{kh}(X)$ is a topological group.

We also show that if X is a Tychonoff space such that $X_0 = X_c$ and $C_{kh}(X)$ is regular space with countable pseudocharacter, then X is σ-compact. If X is a metrizable hemicompact countable space, then $C_{kh}(X)$ is first countable.

The set of all positive integers is denoted by $\mathbb{N}$ and ω is $\mathbb{N} \cup \{0\}$. Notations and terminology we follow [3] and [4]. Let X be a topological space. Denote $X_0 = \{x \in X : x$ is an isolated point of $X\}$, $X_c = \{x \in X : x$ has a compact neighborhood in $X\}$. Thus $X_0 \subset X_c$. If $X = \mathbb{R}$, then $X_0 = \emptyset$ and $X_c = \mathbb{R}$. Denote by $\mathcal{K}(X)$ the collection of all nonempty compact subsets of X.

2. Main results

We firstly study some basic properties of $C_{kh}(X)$.

Lemma 2.1 *If x_0 is an isolated point of a space X and V is an open subset of $\mathbb{R}$ such that $r \in V$, then $[\{x_0\},r]^- \subset [x_0,V]^+$.*

Proof Let $f \in [\{x_0\}, r]^-$. Then $f(x_0) = r \in V$. Thus $f \in [x_0, V]^+$. □

Proposition 2.2 *$C_{kh}(X)$ is a Hausdorff space.*

Proof Since $C_p(X)$ is Hausdorff and $\mathcal{T}_p \subset \mathcal{T}_{kh}$, $C_{kh}(X)$ is a Hausdorff space. □

In Example 2.20, we show that there exists a Tychonoff space X such that $C_{kh}(X)$ is not regular.

The following conclusion is pointed out in the proof of [4, Proposition 0.4.1 (3)].

Proposition 2.3 ([4]) *If a subspace Y of a Tychonoff space X is compact, then for each real-valued continuous function g on Y there is a real-valued continuous function f on X such that $f|Y = g$.*

Theorem 2.4 *If X is a Tychonoff space such that $X_0 = X_c$, then the following statements are equivalent:*

(1) X_0 is G_δ-dense in X;

(2) $C_{kh}(X)$ is regular;

(3) $C_{kh}(X)$ is Tychonoff;

(4) $C_{kh}(X)$ is a topological group.

Proof (1)⇒(2). If X_0 is G_δ-dense in X, then $C_h(X)$ is regular [1, Theorem 3.7]. By [5, Theorem 1.1.5], we know that $C_k(X)$ is Tychonoff. Thus $C_k(X)$ is regular. Since $\mathcal{T}_{kh}$ is the joint of $\mathcal{T}_k$ and $\mathcal{T}_h$, $C_{kh}(X)$ is regular.

(2)⇒(1). Suppose X_0 is not G_δ-dense in X. Then there exists a nonempty G_δ-set A of X such that $A \cap X_0 = \emptyset$. Since X is a Tychonoff space and A is a G_δ-set of X, there exists a continuous function $f : X \to [0, 1]$ such that $f^{-1}(0) \neq \emptyset$ and $f^{-1}(0) \subset A$. Thus $[X, 0]^-$ is an open neighborhood of f in $C_{kh}(X)$. If $F = C_{kh}(X) \setminus [X, 0]^-$, then F is a closed subset of $C_{kh}(X)$ such that $f \notin F$. Since $C_{kh}(X)$ is regular, there exist open subsets U and W of $C_{kh}(X)$ such that $f \in U$, $F \subset W$ and $U \cap W = \emptyset$. Since U is open in $C_{kh}(X)$ and $f \in U$, there exist a compact subset C of X, some $\varepsilon > 0$, some $m \in \mathbb{N}$ and an open subset U_i of X and a point $r_i \in [0, 1]$ for each $i \le m$ such that $f \in [f, C, \varepsilon] \cap (\bigcap_{i \le m} [U_i, r_i]^-) \subset U$.

Let $E = \{i \le m : f^{-1}(r_i) \cap U_i \cap X_0 \neq \emptyset\}$ and $G = \{1, 2, \dots, m\} \setminus E$. For each $i \in E$, there exists some $x_i \in X_0 \cap U_i$ such that $f(x_i) = r_i$. Since $f^{-1}(0) \subset A$ and $A \cap X_0 = \emptyset$, $f(x_i) = r_i > 0$ for each $i \in E$. For each $i \in G$, $f^{-1}(r_i) \cap U_i \cap X_0 = \emptyset$ and $f^{-1}(r_i) \cap U_i \neq \emptyset$. Thus the set $U_i \not\subset X_0$ for each $i \in G$. Since $X_0 = X_c$ and $U_i \not\subset X_0$, the set U_i cannot be included in any compact subset of X for each $i \in G$. If $i \in E$, then $f \in [\{x_i\}, r_i]^- \subset [x_i, (f(x_i) - \varepsilon, f(x_i) + \varepsilon)]^+ \cap [U_i, r_i]^-$. Denote $C^* = C \setminus \{x_i : i \in E\}$. Since the set C is compact in X and $\{x_i\}$ is open in X for each $i \in E$, the set C^* is compact in X and $f \in [f, C^*, \varepsilon] \cap (\bigcap_{i \in E} [\{x_i\}, r_i]^-) \cap (\bigcap_{i \in G} [U_i, r_i]^-) \subset [f, C, \varepsilon] \cap (\bigcap_{i \le m} [U_i, r_i]^-) \subset U$.

For each $i \in E$, the point $x_i \in X_0 \cap U_i$ and $f(x_i) = r_i > 0$. If $\varepsilon_1 = \min\{\frac{\varepsilon}{4}, \min\{r_i : i \in E\}\}$, then $0 < \varepsilon_1 \le \frac{\varepsilon}{4}$ and $\varepsilon_1 \le r_i$ for each $i \in E$. If $f_1 = f|(C^* \cup \{x_i : i \in E\})$, then $f_1 : C^* \cup \{x_i : i \in E\} \to \mathbb{R}$ is continuous satisfying $f_1(x) = f(x) \ge 0$ for each $x \in C^*$ and $f_1(x_i) = r_i > 0$ for each $i \in E$. Let f_2 be a real-valued function on $C^* \cup \{x_i : i \in E\}$ such that

$$f_2(x) = \begin{cases} f_1(x) + \frac{\varepsilon}{4}, & \text{if } x \in C^*; \\ f_1(x), & \text{if } x \in \{x_i : i \in E\}. \end{cases}$$

Since $C^* \cap \{x_i : i \in E\} = \emptyset$ and $\{x_i\}$ is clopen in $C^* \cup \{x_i : i \in E\}$ for each $i \in E$, the real-valued function f_2 is continuous on $C^* \cup \{x_i : i \in E\}$. Since $C^* \cup \{x_i : i \in E\}$ is compact in X and X is a Tychonoff space, there exists a continuous real-valued function $f_3 : X \to \mathbb{R}$ such that $f_3|(C^* \cup \{x_i : i \in E\}) = f_2$ by Proposition 2.3. If g is a real-valued function on X such that $g(x) = \max\{\frac{\varepsilon_1}{2}, f_3(x)\}$ for each $x \in X$, then the function g is continuous on X and $g^{-1}(0) = \emptyset$. Since $g^{-1}(0) = \emptyset$, the function $g \notin [X, 0]^-$. Thus $g \in F \subset W$. If $x \in C^*$, then $f_3(x) = f_2(x) = f_1(x) + \frac{\varepsilon}{4} = f(x) + \frac{\varepsilon}{4} \geq \frac{\varepsilon}{4} \geq \varepsilon_1 > \frac{\varepsilon_1}{2}$. Thus $g(x) = f(x) + \frac{\varepsilon}{4}$ for each $x \in C^*$. If $i \in E$, then $f_3(x_i) = f_2(x_i) = f_1(x_i) = f(x_i) = r_i \geq \varepsilon_1 > \frac{\varepsilon_1}{2}$. So $g(x_i) = f(x_i) = r_i$ for each $i \in E$. Thus $g|(C^* \cup \{x_i : i \in E\}) = f_3|(C^* \cup \{x_i : i \in E\})$ and $g(x_i) = f(x_i) = r_i$ for each $i \in E$.

Since W is open in $C_{kh}(X)$ and the function $g \in W$, there exist some compact subset B of X, some $\delta > 0$ ($\delta < \frac{\varepsilon}{4}$), some $l \in \mathbb{N}$, an open subset W_j of X and $h_j \in \mathbb{R}$ for each $j \leq l$ such that $g \in [g, B, \delta] \cap (\bigcap_{i \in E}[\{x_i\}, r_i]^-) \cap (\bigcap_{j \leq l}[W_j, h_j]^-) \subset W$. By Lemma 2.1, we can assume that $B \cap \{x_i : i \in E\} = \emptyset$. We can also assume that $C^* \subset B$ and $\{x_i\} \cap W_j = \emptyset$ for each $i \in E$ and for each $j \leq l$. For each $j \leq l$, there exists some $y_j \in g^{-1}(h_j) \cap W_j$. If $B_1 = B \cup \{x_i : i \in E\} \cup \{y_j : j \leq l\}$, then B_1 is compact subset of X.

Denote $G = \{i_1, i_2, \ldots, i_p\}$, where $p \leq m$ and $i_t \neq i_q$ if $t, q \leq p$ and $t \neq q$. Since $U_{i_1} \not\subset X_0$ and $X_0 = X_c$, the set $U_{i_1} \setminus B_1 \neq \emptyset$. Take a point $z_{i_1} \in U_{i_1} \setminus B_1$. Let $q < p$. Assume that we have a point z_{i_t} for each $t \leq q$ such that $z_{i_t} \in U_{i_t} \setminus (B_1 \cup \{z_{i_k} : k < t\})$. Since $B_1 \cup \{z_{i_t} : t \leq q\}$ is compact and $U_{i_{q+1}} \not\subset X_c$, we know that $U_{i_{q+1}} \setminus (B_1 \cup \{z_{i_t} : t \leq q\}) \neq \emptyset$. Take a point $z_{i_{q+1}} \in U_{i_{q+1}} \setminus (B_1 \cup \{z_{i_t} : t \leq q\})$. Thus the set $\{z_{i_q} : 1 \leq q \leq p\}$ is disjoint from B_1 and satisfies that $z_{i_q} \in U_{i_q}$, $z_{i_q} \neq z_{i_t}$ if $q \neq t$ and $q, t \leq p$.

Define a function $g_1 : B_1 \cup \{z_{i_q} : 1 \leq q \leq p\} \to \mathbb{R}$ such that $g_1|B_1 = g|B_1$ and $g_1(z_{i_q}) = r_{i_q}$ for each $q \leq p$. Since $B_1 \cap \{z_{i_q} : 1 \leq q \leq p\} = \emptyset$ and $\{z_{i_q}\}$ is clopen in $B_1 \cup \{z_{i_q} : 1 \leq q \leq p\}$ for each $1 \leq q \leq p$, the function g_1 is continuous. Since $B_1 \cup \{z_{i_q} : 1 \leq q \leq p\}$ is compact in X and X is a Tychonoff space, there exists a continuous function $g^* : X \to \mathbb{R}$ such that $g^*|(B_1 \cup \{z_{i_q} : 1 \leq q \leq p\}) = g_1$ by Proposition 2.3. Since $g^* \in [g, B, \delta] \cap (\bigcap_{i \in E}[\{x_i\}, r_i]^-) \cap (\bigcap_{j \leq l}[W_j, h_j]^-)$ and $[g, B, \delta] \cap (\bigcap_{i \in E}[\{x_i\}, r_i]^-) \cap (\bigcap_{j \leq l}[W_j, h_j]^-) \subset W$, the function $g^* \in W$.

Since $g^*|B_1 = g_1|B_1 = g|B_1$ and $C^* \subset B \subset B_1$, for each $x \in C^*$ we have $|g^*(x) - f(x)| = |g(x) - f(x)| = |(f(x) + \frac{\varepsilon}{4}) - f(x)| = \frac{\varepsilon}{4} < \varepsilon$. If $i \in E$, then the point $x_i \in B_1 \cap U_i$. Thus $g^*(x_i) = g(x_i) = f(x_i) = r_i$. So ${g^*}^{-1}(r_i) \cap U_i \neq \emptyset$ for each $i \in E$. If $j \in G$, then there exists some $q \leq p$ such that $i_q = j$. Thus $r_{i_q} = r_j$. Since $z_{i_q} \in U_{i_q}$ and $g^*(z_{i_q}) = g_1(z_{i_q}) = r_{i_q}$, ${g^*}^{-1}(r_{i_q}) \cap U_{i_q} \neq \emptyset$. So ${g^*}^{-1}(r_j) \cap U_j \neq \emptyset$ for each $j \in G$. Thus ${g^*}^{-1}(r_i) \cap U_i \neq \emptyset$ for each $i \leq m$. So $g^* \in [f, C^*, \varepsilon] \cap (\bigcap_{i \in E}[\{x_i\}, r_i]^-) \cap (\bigcap_{i \in G}[U_i, r_i]^-)$. Since $[f, C^*, \varepsilon] \cap (\bigcap_{i \in E}[\{x_i\}, r_i]^-) \cap (\bigcap_{i \in G}[U_i, r_i]^-) \subset [f, C, \varepsilon] \cap (\bigcap_{i \leq m}[U_i, r_i]^-) \subset U$, the function $g^* \in U$.

Thus the continuous function $g^* \in U \cap W$. This is a contradiction with $U \cap W = \emptyset$. So X_0 is G_δ-dense in X. We have proved that (2) implies (1).

Thus (1) and (2) are equivalent.

Now we prove that (1), (3) and (4) are equivalent. Assume that (1) holds. Then X_0 is G_δ-dense in X. Thus $C_h(X)$ is a topological group by [1, Theorem 3.7]. Since $C_k(X)$ is also a topological group by [5, Theorem 1.1.7], the space $C_{kh}(X)$ is a topological group. Thus (4) holds. So (1) implies (4).

Now we assume that (4) holds. Then $C_{kh}(X)$ is a topological group. Thus $C_{kh}(X)$ is a Hausdorff topological group by Proposition 2.2. Since any Hausdorff topological group is Tychonoff [6, Theorem 2.4], $C_{kh}(X)$ is Tychonoff. So (3) holds. Hence $C_{kh}(X)$ is regular. Thus X_0 is G_δ-dense in X by (2). So (1) holds.

Thus we complete the proof. □

Lemma 2.5 *Let X be a Tychonoff space. If $C_{kh}(X)$ has countable pseudocharacter, then there is a σ-compact subset D of X such that $\overline{D} = X$.*

Proof Since $C_{kh}(X)$ has countable pseudocharacter, there exists a countable family $\mathcal{U}$ of open sets in $C_{kh}(X)$ such that $\{0_X\} = \bigcap\mathcal{U}$, where 0_X is the constant zero function in $C(X)$. Assume that $\mathcal{U} = \{W_n : n \in \mathbb{N}\}$. For each $n \in \omega$, we can assume that $W_n = [0_X, C_n, \varepsilon_n] \cap (\bigcap_{i \leq m_n} [V_{in}, 0]^-)$, where V_{in} is open in X for each $i \leq m_n$. Denote $0_X = f$. Let $n \in \mathbb{N}$. For each $i \leq m_n$, there is some $y_{in} \in f^{-1}(0) \cap V_{in}$. If $B_n = C_n \cup \{y_{in} : i \leq m_n\}$, then B_n is a compact subset of X. In what follows, we show that $\overline{\bigcup\{B_n : n \in \mathbb{N}\}} = X$.

Suppose $\overline{\bigcup\{B_n : n \in \mathbb{N}\}} \neq X$. There is some $x_0 \in X \setminus \overline{\bigcup\{B_n : n \in \mathbb{N}\}}$. Since X is a Tychonoff space, there exists a continuous function $g : X \to [0,1]$ such that $g(\overline{\bigcup\{B_n : n \in \mathbb{N}\}}) \subset \{0\}$ and $g(x_0) = 1$. For each $n \in \mathbb{N}$, $g|C_n = f|C_n$ and $g(y_{in}) = 0$ for each $i \leq m_n$. So $g \in W_n$ for each $n \in \mathbb{N}$. Thus $g \in \bigcap_{n\in\mathbb{N}} W_n$. So $g = f = 0_X$. This is a contradiction. So we have proved that $\overline{D} = X$, where $D = \bigcup\{B_n : n \in \mathbb{N}\}$ is a σ-compact subset of X. □

Theorem 2.6 *Let X be a Tychonoff space such that $X_0 = X_c$. If $C_{kh}(X)$ is a regular space with countable pseudocharacter, then X is σ-compact.*

Proof Since $C_{kh}(X)$ is regular and $X_0 = X_c$, the set X_0 is G_δ-dense in X by Theorem 2.4. Since $C_{kh}(X)$ has countable pseudocharacter, there exists a σ-compact set D of X such that $\overline{D} = X$ by Lemma 2.5. Suppose $D \neq X$. Then $X \setminus D$ is a nonempty G_δ-set of X. Since X_0 is G_δ-dense in X, the set $(X \setminus D) \cap X_0 \neq \emptyset$. Take a point $x_0 \in (X \setminus D) \cap X_0$. Then $\{x_0\}$ is open in X and $\{x_0\} \cap D = \emptyset$. This contradicts that $\overline{D} = X$. Thus $D = X$. Hence X is σ-compact. □

Corollary 2.7 *Let X be a Tychonoff space such that $X_0 = X_c$. If $C_{kh}(X)$ is a regular first-countable (metrizable) space, then X is σ-compact.*

Proposition 2.8 *Let X be a Tychonoff space such that X_0 is G_δ-dense in X. Then for every $f \in C(X)$ and for any open neighborhood O of f in $C_{kh}(X)$ there exist some compact subset C of X, some $\varepsilon > 0$, some $m \in \mathbb{N}$ and $x_i \in X_0$ for each $i \leq m$ such that $f \in [f, C, \varepsilon] \cap (\bigcap_{i\leq m}[\{x_i\}, r_i]^-) \subset O$ and $C \cap \{x_1, \ldots, x_m\} = \emptyset$.*

Proof Let f be any element of $C(X)$ and let O be any open neighborhood of f in $C_{kh}(X)$. There exist some compact subset C_1 of X, some $\varepsilon > 0$, some $m \in \mathbb{N}$ and an open subset U_i of X and $r_i \in \mathbb{R}$ for each $i \leq m$ such that $f \in [f, C_1, \varepsilon] \cap (\bigcap_{i \leq m}[U_i, r_i]^-) \subset O$. For each $i \leq m$, the set $f^{-1}(r_i) \cap U_i$ is a nonempty G_δ-set of X. Since X_0 is G_δ-dense in X, there exists some $x_i \in X_0 \cap (f^{-1}(r_i) \cap U_i)$ for each $i \leq m$. Thus $\{x_i : i \leq m\}$ is open in X.

If $C = C_1 \setminus \{x_1, x_2, \ldots, x_m\}$, then C is compact in X and $f \in [f, C, \varepsilon] \cap (\bigcap_{i \leq m}[\{x_i\}, r_i]^-) \subset [f, C_1, \varepsilon] \cap (\bigcap_{i \leq m}[U_i, r_i]^-) \subset O$ and satisfying $C \cap \{x_1, \ldots, x_m\} = \emptyset$. $\square$

Proposition 2.9 *If X is a Tychonoff space such that every compact subset of X is finite, then $C_{kh}(X) = C_{ph}(X)$.*

Corollary 2.10 *If X is a discrete space, then $C_{kh}(X) = C_{ph}(X)$.*

Proposition 2.11 *If X is a singleton point topological space, then $C_h(X)$ is not second-countable.*

So we have:

Proposition 2.12 *If X is a singleton point topological space, then $C_{kh}(X)$ and $C_{ph}(X)$ are not second-countable.*

In [1, Theorem 4.10], it was proved that for any Tychonoff space X, the following statements are equivalent:

(a) $C_h(X)$ is metrizable;

(b) X is a countable discrete space;

(c) $C_{ph}(X)$ is first-countable and X_0 is G_δ-dense in X;

(d) $C_{ph}(X)$ is metrizable.

Thus we have:

Theorem 2.13 *For any discrete space, the following statements are equivalent:*

(1) X is a countable space;

(2) $C_h(X)$ is metrizable;

(3) $C_{ph}(X)$ is metrizable;

(4) $C_{kh}(X)$ is metrizable;

(5) $C_{kh}(X)$ is first-countable;

(6) $C_{kh}(X)$ has countable pseudocharacter.

Proof By [1, Theorem 4.10], we know that (1) implies (2) and (2) implies (3). By Corollary 2.10, (3) implies (4). It is obvious that (4) implies (5).

Now we assume that (5) holds. By Proposition 2.2, $C_{kh}(X)$ is a Hausdorff space. Thus $C_{kh}(X)$ is a first-countable Hausdorff space. So $C_{kh}(X)$ has countable pseudocharacter. Hence (6) holds.

Now we assume that (6) holds. Since X is discrete, $X_0 = X_c = X$ is G_δ-dense in X. Thus $C_{kh}(X)$ is regular by Theorem 2.4. So $C_{kh}(X)$ is a regular space with countable pseudocharacter.

Thus X is σ-compact by Theorem 2.6. Since X is discrete, the space X is countable. Thus (1) holds. □

If X is an uncountable discrete space, then $X_0 = X_c = X$ is G_δ-dense in X. Thus $C_{kh}(X)$ is a topological group by Theorem 2.4. Since X is not countable, $C_{kh}(X)$ does not have countable pseudocharacter by Theorem 2.13.

Question 2.14 Let X be a Tychonoff space. Suppose $C_{kh}(X)$ is metrizable. Is X a discrete space?

Recall that a space X is hemicompact if in the family of all compact subsets of X ordered by $\subset$ there exists a countable cofinal subfamily.

Theorem 2.15 *If X is a metrizable hemicompact countable space, then $C_{kh}(X)$ is first countable.*

Proof Let f be any element of $C(X)$. Since X is countable, $f(X)$ is a countable subset of $\mathbb{R}$. Since X is metrizable and countable, X is second-countable. Let $\mathcal{B}_X$ be a countable base of X. Since X is hemicompact, there exists a countable family $\{C_n : n \in \mathbb{N}\} \subset \mathcal{K}(X)$ such that for each $F \in \mathcal{K}(X)$, there exists some $n \in \mathbb{N}$ such that $F \subset C_n$. If $\mathcal{B}_f = \{[f, C_n, \frac{1}{m}] \cap (\bigcap_{i \le k}[U_i, r_i]^-) : n \in \mathbb{N}, m \in \mathbb{N}, k \in \mathbb{N}, U_i \in \mathcal{B}_X, r_i \in f(X)\}$, then $|\mathcal{B}_f| \le \omega$. Let O be any open neighborhood of f in $C_{kh}(X)$. Then there exist some compact subset F of X, some $m \in \mathbb{N}$, some $k \in \mathbb{N}$, and an open subset W_i of X and a point $r_i \in \mathbb{R}$ for each $i \le k$ such that $f \in [f, F, \frac{1}{m}] \cap (\bigcap_{i \le k}[W_i, r_i]^-) \subset O$. For each $i \le k$, there exists some $x_i \in W_i$ such that $f(x_i) = r_i$. So there exists some $U_i \in \mathcal{B}_X$ such that $x_i \in U_i \subset W_i$ for each $i \le k$. Thus $f \in [U_i, r_i]^- \subset [W_i, r_i]^-$ for each $i \le k$. Since F is a compact subset of X, there exists some $n \in \mathbb{N}$ such that $F \subset C_n$. Thus $[f, F, \frac{1}{m}] \subset [f, C_n, \frac{1}{m}]$. So $f \in [f, C_n, \frac{1}{m}] \cap (\bigcap_{i \le k}[U_i, r_i]^-) \subset O$ and $[f, C_n, \frac{1}{m}] \cap (\bigcap_{i \le k}[U_i, r_i]^-) \in \mathcal{B}_f$. Thus the family $\mathcal{B}_f$ is a countable base of neighborhoods of f in $C_{kh}(X)$. So $C_{kh}(X)$ is first-countable. □

Let $\mathbb{Q}$ be the set of rational numbers with the natural topology. In [7, Proposition 30], it is pointed out that $\mathbb{Q}$ is not hemicompact. This shows that a metrizable and countable space can fail to be hemicompact.

Corollary 2.16 *Let X be a compact metrizable countable space. Then $C_{kh}(X)$ is first countable.*

Proposition 2.17 *Let X be a Tychonoff space. Then $C_{ph}(X)$ and $C_{kh}(X)$ are always Hausdorff.*

Proof By Proposition 2.2, $C_{kh}(X)$ is Hausdorff. Since $C_p(X)$ is Hausdorff and $\mathcal{T}_p \subset \mathcal{T}_{ph}$, $C_{ph}(X)$ is Hausdorff. □

Similarly to Lemma 2.5, we have:

Theorem 2.18 *Let X be a Tychonoff space. If $C_{ph}(X)$ has countable pseudocharacter, then X is separable.*

Proof The proof is similar to the proof of Lemma 2.5. So we omit it. □

Example 2.19 Let $X = \omega_1 + 1$ be a space such that $Y = \omega_1$ is a discrete subspace of

X and $\mathcal{B}(\omega_1) = \{(\alpha, \omega_1] : \alpha \in \omega_1\}$ is a base of neighborhoods of the point ω_1 in X. Then $C_{ph}(X) = C_{kh}(X)$ is a topological group and $C_{kh}(X)$ ($C_{ph}(X)$, $C_h(X)$) does not have countable pseudocharacter.

Proof Since every compact subset of X is finite, $C_{ph}(X) = C_{kh}(X)$ by Proposition 2.9. Since $X_0 = X_c = \omega_1$ and ω_1 is G_δ-dense in X, $C_{kh}(X)$ is a regular topological group by Theorem 2.4. Thus $C_{ph}(X)$ is a regular topological group. Since X is not σ-compact, $C_{kh}(X)$ does not have countable pseudocharacter by Theorem 2.6. Thus $C_{ph}(X)$ does not have countable pseudocharacter. Since every open subset of $C_h(X)$ is open in $C_{ph}(X)$, the space $C_h(X)$ also does not have countable pseudocharacter. □

The following example shows that there exists a Tychonoff space X such that $C_{kh}(X)$ is not regular.

Example 2.20 If $X = \omega_\omega + 1$ is a topological space such that $Y = \omega_\omega$ is a discrete subspace of X and $\mathcal{B}(\omega_\omega) = \{(\alpha, \omega_\omega] : \alpha \in \omega_\omega\}$ is a base of open neighborhoods of the point ω_ω in X, then $C_{kh}(X)$ is not regular.

Proof Since $Y = \omega_\omega$ is a discrete subspace of X, it is obvious that $X_0 = Y$. Let $\omega_0 = \omega$. Thus $\{(\omega_i, \omega_\omega] : i \in \omega\}$ is also a base of open neighborhoods of the point ω_ω in X, where ω_1 is the first uncountable limit ordinal and ω_{i+1} is the first cardinal which is larger than ω_i for each $i \in \omega$. Suppose that the point ω_ω has a compact neighborhood W in X. Then there exists some $i \in \omega$ such that $(\omega_i, \omega_\omega] \subset W$. So $(\omega_{i+1}, \omega_\omega] \subset (\omega_i, \omega_\omega] \subset W$. Denote $F = (\omega_i, \omega_\omega] \setminus (\omega_{i+1}, \omega_\omega]$. Then $F = (\omega_i, \omega_{i+1}]$ is an uncountable closed discrete subspace of X and $F \subset W$. Since W is compact, the set F is compact. Thus F is an infinite compact discrete space. This contradicts that every compact discrete space is finite. Thus the point ω_ω has no compact neighborhood in X. So $X_0 = X_c = Y$. Since $\{\omega_\omega\} = \bigcap\{(\omega_i, \omega_\omega] : i \in \omega\}$ is a G_δ-set in X and $\{\omega_\omega\} \cap Y = \emptyset$, the set Y is not G_δ-dense in X. Thus $C_{kh}(X)$ is not regular by Theorem 2.4. □

Acknowledgements We thank the referees for their time and comments.

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