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Normal Bipolar Fuzzy Ideals in Non-involutive Residuated Lattices

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Abstract In this paper, the problem of bipolar fuzzy ideal is further studied in non-involutive residuated lattices. The notion of normal bipolar fuzzy ideal is introduced, some important properties and equivalent characterizations of normal bipolar fuzzy ideals are obtained. In addition, two special types of normal bipolar fuzzy ideals are defined, which are called maxima and completely normal bipolar fuzzy ideals, respectively, and their relationships are discussed. This work further expands the way for revealing the structural characteristics of non-involutive residuated lattices.

Keywords non-classical logic; non-involutive residuated lattice; bipolar fuzzy set; normal bipolar fuzzy ideal

MR(2010) Subject Classification 03B50; 03G25; 03E72

1. Introduction

It is well-known that non-classical mathematical logic [1] has become a formal and useful tool for computer science to deal with fuzzy information and uncertain information. Various logical algebras have been proposed as the semantical systems of non-classical mathematical logic systems. Among these logical algebras, residuated lattices introduced by Ward and Dilworth [2] are very basic and important algebraic structures. Some other logical algebras such as MTL-algebra, BL-algebra, MV-algebras [3] and NM-algebra, which is also called R_0-algebra [4] are all able to be considered as particular classes of residuated lattices. Filter and ideal are two important concepts for studying of logical algebraic structures. The related research work has attracted much attention from scholars [5, 6]. It is noteworthy that in logical algebras with negative involutive (regular) properties, considering that filters and ideals are mutually dual, most people focus their attention on the problem of filters. However, when the negation operation in logical algebra loses its involution, the dual relationship between ideal and filter is also broken. Therefore, it will be a meaningful work to explore ideal and its application in the framework of non-involutive logic algebras. In view of this, the concepts of ideal and fuzzy ideal have been introduced in BL-algebra and non-involutive residuated lattices in [7–9], and some results with theoretical significance and application prospect have been obtained.

The concept of fuzzy sets, a remarkable idea in mathematics, was proposed by Zadeh [10] in 1965. At present, fuzzy sets have been extremely used to deal with the many problems in applied

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mathematics, control engineering, information sciences, expert systems and theory of automata etc. However, in traditional fuzzy sets, the membership degrees of elements are all restricted to the interval $[0, 1]$, which leads to a great difficulty in expressing the difference of the irrelevant elements from the contrary elements in fuzzy sets. If a set representation could express this kind of difference, it would be more informative than the traditional fuzzy set representation. Based on these observations, Zhang [11] introduced the notion of bipolar fuzzy sets, abbreviated as BF-sets, which is an extension of the traditional fuzzy sets. In the past few decades, more and more researchers have devoted themselves to applying BF-sets theory to various algebraic structures [12–18].

In order to further expand the way for revealing the structural characteristics of non-involutive residuated lattices, the present author applied BF-sets theory to study the problem of ideals in non-involutive residuated lattices in [19] and [20]. A new notion of bipolar fuzzy ideal (BF-ideal for short) was introduced and some important results were obtained. As the continuation and deepening of these work, in this paper, we introduce the notion of normal bipolar fuzzy ideal and investigate its properties in non-involutive residuated lattices. some interesting and significative results are obtained.

2. Preliminaries

In this section, we review related basic knowledge about non-involutive residuated lattices [2, 3, 8, 9], BF-sets [11–13] and BF-ideals [19, 20].

Definition 2.1 ([2]) *A residuated lattice is an algebra* $(L, \vee, \wedge, \otimes, \to, 0, 1)$ *of type* $(2, 2, 2, 2, 0, 0)$ *such that:*

(R1) $(L, \vee, \wedge, 0, 1)$ *is a bounded lattice with the greatest element* 1 *and the least element* 0*;*

(R2) $(L, \otimes, 1)$ *is a commutative monoid;*

(R3) $(\otimes, \to)$ *is an adjoint pair on* L*, i.e., for all* $x, y, z \in L$, $x \otimes y \leqslant z$ *if and only if* $x \leqslant y \to z$.

A non-involutive residuated lattice is a residuated lattice $(L, \vee, \wedge, \otimes, \to, 0, 1)$ *which satisfies that: there exists* $x \in L$ *such that* $x \neq x''$*, where* $x' = x \to 0$ *for all* $x \in L$.

In the sequel, a residuated lattice $(L, \vee, \wedge, \otimes, \to, 0, 1)$ will be denoted by L in short.

Lemma 2.2 ([2, 3]) *Let* L *be a non-involutive residuated lattice. Then for all* $x, y, z \in L$,

(P1) $x \leqslant y$ *if and only if* $x \to y = 1$*;*

(P2) $x \to x = 1$ *and* $x \to 1 = 1$ *and* $1 \to x = x$*;*

(P3) $x \leqslant y$ *implies* $x \otimes z \leqslant y \otimes z$ *and* $z \to x \leqslant z \to y$ *and* $y \to z \leqslant x \to z$*;*

(P4) $y \to z \leqslant (x \to y) \to (x \to z) \leqslant x \to (y \to z)$ *and* $x \to y \leqslant (y \to z) \to (x \to z)$*;*

(P5) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$ *and* $(y \vee z) \to x = (y \to x) \wedge (z \to x)$*;*

(P6) $(x \to y) \otimes (y \to z) \leqslant x \to z$ *and* $x \to (y \to z) = (x \otimes y) \to z = y \to (x \to z)$*;*

(P7) $x \leqslant y \Longrightarrow y' \leqslant x' \Longrightarrow x'' \leqslant y''$ *and* $x \leqslant x''$ *and* $x''' = x'$ *and* $x \otimes x' = 0$*;*

(P8) $x \to y' = y \to x' = (x \otimes y)'$ *and* $(x \to y')'' = x \to y'$ *and* $(x \to y'')'' = x \to y''$*;*

(P9) $(x \vee y)' = x' \wedge y'$ *and* $(x \wedge y)' \geqslant x' \vee y'$ *and* $x \to y \leqslant y' \to x'$.

Definition 2.3 ([8]) *Let L be a non-involutive residuated lattice. An ideal I is a non-empty subset of L such that $0 \in I$ and for all $x, y \in L$, $x \in I$ and $(x' \to y')' \in I$ imply $y \in I$. The set of all ideals of L is denoted by* $\mathrm{Id}(L)$.

In the unit interval $[0,1]$ equipped with the natural order, $\vee = \max$ and $\wedge = \min$. Let $X \neq \emptyset$, denote $J_{[0,1]} = \{\mu^P | \mu^P : X \to [0,1]\}$ and $J_{[-1,0]} = \{\mu^N | \mu^N : X \to [-1,0]\}$. For every $\mu_A^P \in J_{[0,1]}$ and $\mu_A^N \in J_{[-1,0]}$, we call $A = \{(x, \mu_A^P(x), \mu_A^N(x)) | x \in X\}$ a bipolar fuzzy set on X, and abbreviate A is a BF-set on X, where $\mu_A^P(x)$ is called a positive membership degree which denotes the satisfaction degree of an element x to some specific property about the BF-set A, and $\mu_A^N(x)$ is called a negative membership degree which denotes the satisfaction degree of x to some implicit counter-property about the BF-set A. For the sake of simplicity, we shall use the symbol $A = (\mu_A^P, \mu_A^N)$ for the BF-set $A = \{(x, \mu_A^P(x), \mu_A^N(x)) | x \in X\}$. The set of all BF-sets on X is denoted by $\mathrm{BFS}(X)$. Let $\{A_\lambda = (\mu_{A_\lambda}^P, \mu_{A_\lambda}^N) | \lambda \in \Lambda\} \subseteq \mathrm{BFS}(X)$. We define the BF-intersection $\bigsqcap_{\lambda \in \Lambda} A_\lambda$ and BF-union $\bigsqcup\limits_{\lambda \in \Lambda} A_\lambda$ of $\{A_\lambda | \lambda \in \Lambda\}$ as follows: for all $x \in X$,

(i) $\Big(\bigsqcap\limits_{\lambda \in \Lambda} A_\lambda\Big)(x) = \Big(\mu^P_{\bigsqcap\limits_{\lambda \in \Lambda} A_\lambda}(x), \mu^N_{\bigsqcap\limits_{\lambda \in \Lambda} A_\lambda}(x)\Big) = \Big(\bigwedge\limits_{\lambda \in \Lambda} \mu_{A_\lambda}^P(x), \bigvee\limits_{\lambda \in \Lambda} \mu_{A_\lambda}^N(x)\Big)$;

(ii) $\Big(\bigsqcup\limits_{\lambda \in \Lambda} A_\lambda\Big)(x) = \Big(\mu^P_{\bigsqcup\limits_{\lambda \in \Lambda} A_\lambda}(x), \mu^N_{\bigsqcup\limits_{\lambda \in \Lambda} A_\lambda}(x)\Big) = \Big(\bigvee\limits_{\lambda \in \Lambda} \mu_{A_\lambda}^P(x), \bigwedge\limits_{\lambda \in \Lambda} \mu_{A_\lambda}^N(x)\Big)$.

In particular, if $A = (\mu_A^P, \mu_A^N)$, $B = (\mu_B^P, \mu_B^N) \in \mathrm{BFS}(X)$, we define $A \sqcap B$ and $A \sqcup B$ as follows: for all $x \in X$,

(iii) $(A \sqcap B)(x) = (\mu_A^P(x) \wedge \mu_B^P(x), \mu_A^N(x) \vee \mu_B^N(x))$;

(iv) $(A \sqcup B)(x) = (\mu_A^P(x) \vee \mu_B^P(x), \mu_A^N(x) \wedge \mu_B^N(x))$.

And the binary relation $\sqsubseteq$ on $\mathrm{BFS}(X)$ is defined as follows:

$$A \sqsubseteq B \Longleftrightarrow \mu_A^P(x) \leqslant \mu_B^P(x) \text{ and } \mu_A^N(x) \geqslant \mu_B^N(x), \text{ for all } x \in X. \tag{2.1}$$

It is easy to see that $\sqsubseteq$ is a partial order on $\mathrm{BFS}(X)$, and we call it the BF-inclusion order.

Definition 2.4 ([19,20]) *Let L be a non-involutive residuated lattice. A BF-set $A = (\mu_A^P, \mu_A^N) \in$* $\mathrm{BFS}(L)$ *is called a BF-ideal of L if it satisfies the following conditions for all $x, y \in L$,*

(BFI1) $\mu_A^P(0) \geqslant \mu_A^P(x)$ *and* $\mu_A^N(0) \leqslant \mu_A^N(x)$;

(BFI2) $\mu_A^P(y) \geqslant \mu_A^P(x) \wedge \mu_A^P((x' \to y')')$ *and* $\mu_A^N(y) \leqslant \mu_A^N(x) \vee \mu_A^N((x' \to y')')$.

The set of all BF-ideals of L is denoted by $\mathrm{BFI}(L)$.

3. Normal bipolar fuzzy ideals

In this section, we introduce the notion of normal bipolar fuzzy ideal and discuss its properties in non-involutive residuated lattices.

Definition 3.1 *Let L be a non-involutive residuated lattice. A BF-ideal $A = (\mu_A^P, \mu_A^N) \in \mathrm{BFI}(L)$ is called a normal bipolar fuzzy ideal, normal BF-ideal for short, if there exists an element $x \in L$ such that $A(x) = (1, -1)$, i.e., $\mu_A^P(x) = 1$ and $\mu_A^N(x) = -1$. The set of all normal BF-ideals of L is denoted by* $\mathrm{NBFI}(L)$.

Remark 3.2 Let L be a non-involutive residuated lattice and $A=(\mu_A^P,\mu_A^N)\in \mathrm{NBFI}(L)$. According to Definition 3.1 and (BFI1) in Definition 2.4, it is obvious that $A(0)=(1,-1)$, i.e., $\mu_A^P(0)=1$ and $\mu_A^N(0)=-1$.

Example 3.3 Let $L=\{0,a,b,c,d,1\}$ and $0<a<b<c<d<1$, the operators $\to$ and $\otimes$ of L be defined as following Tables 1 and 2, respectively,

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	1	1	1	1
b	b	b	1	1	1	1
c	a	a	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

Table 1 Definition of the operator $\to$

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	a	a	a	a
b	0	a	b	b	b	b
c	0	a	b	c	c	c
c	0	a	b	c	d	d
1	0	a	b	c	d	1

Table 2 Definition of the operator $\otimes$

and $x'=x\to 0$ for all $x\in L$. Then $(L,\leqslant,\wedge,\vee,\otimes,\to,0,1)$ is a non-involutive residuated lattice. Define $A=(\mu_A^P,\mu_A^N)\in \mathrm{BFS}(L)$ as following Table 3.

x	0	a	b	c	d	1
$\mu_A^P(x)$	1	1	0.5	0.5	0.5	0.5
$\mu_A^N(x)$	-1	-1	-0.5	-0.5	-0.5	-0.5

Table 3 Definition of $A=(\mu_A^P,\mu_A^N)\in \mathrm{BFS}(L)$

By routine calculation, we know that $A=(\mu_A^P,\mu_A^N)\in \mathrm{NBFI}(L)$.

Definition 3.4 *Let L be a non-involutive residuated lattice and $A=(\mu_A^P,\mu_A^N)\in \mathrm{BFS}(L)$. An element $x_0\in L$ is called an extremal element of A if $\mu_A^P(x_0)\geqslant \mu_A^P(x)$ and $\mu_A^N(x_0)\leqslant \mu_A^N(x)$ for all $x\in L$. The set of all extremal element of $A=(\mu_A^P,\mu_A^N)$ is denoted by $\mathrm{Ext}(A)$.*

Theorem 3.5 *Let L be a non-involutive residuated lattice and $A=(\mu_A^P,\mu_A^N)\in \mathrm{BFI}(L)$. Then $A\in \mathrm{NBFI}(L)$ if and only if $A(x)=(1,-1)$ for all $x\in \mathrm{Ext}(A)$.*

Proof It is obvious by Definitions 3.1 and 3.4. □

Theorem 3.6 *Let L be a non-involutive residuated lattice, $A=(\mu_A^P,\mu_A^N)\in \mathrm{BFI}(L)$ and $x_0\in \mathrm{Ext}(A)$. Define a BF-set $A^*_{x_0}=(\mu^P_{A^*_{x_0}},\mu^N_{A^*_{x_0}})\in \mathrm{BFS}(L)$* as follows: for all $x\in L$,

$$\mu^P_{A^*_{x_0}}(x)=\mu_A^P(x)+1-\mu_A^P(x_0) \text{ and } \mu^N_{A^*_{x_0}}(x)=\mu_A^N(x)-1-\mu_A^N(x_0).$$

Then $A^*_{x_0}\in \mathrm{NBFI}(L)$ and $A\sqsubseteq A^*_{x_0}$.

Proof Firstly, we prove that $A^*_{x_0}$ is normal. In fact, for all $x\in L$, since $x_0\in \mathrm{Ext}(A)$, we have that $\mu_A^P(x_0)\geqslant \mu_A^P(x)$ and $\mu_A^N(x_0)\leqslant \mu_A^N(x)$ by Definition 3.4. It follows from $\mu^P_{A^*_{x_0}}(x)=\mu_A^P(x)+1-\mu_A^P(x_0)$ and $\mu^N_{A^*_{x_0}}(x)=\mu_A^N(x)-1-\mu_A^N(x_0)$ that $\mu^P_{A^*_{x_0}}(x)\in[0,1]$, $\mu^N_{A^*_{x_0}}(x)\in[-1,0]$, $\mu^P_{A^*_{x_0}}(x_0)=1$ and $\mu^N_{A^*_{x_0}}(x_0)=-1$. Thus $A^*_{x_0}=(\mu^P_{A^*_{x_0}},\mu^N_{A^*_{x_0}})$ is normal.

Secondly, we prove that $A^*_{x_0} \in \mathrm{BFI}(L)$. On the one hand, for all $x \in L$, it follows from $A = (\mu^P_A, \mu^N_A) \in \mathrm{BFI}(L)$ and (BFI1) that $\mu^P_{A^*_{x_0}}(0) = \mu^P_A(0) + 1 - \mu^P_A(x_0) \geqslant \mu^P_A(x) + 1 - \mu^P_A(x_0) = \mu^P_{A^*_{x_0}}(x)$ and $\mu^N_{A^*_{x_0}}(0) = \mu^N_A(0) - 1 - \mu^N_A(x_0) \leqslant \mu^N_A(x) - 1 - \mu^N_A(x_0) = \mu^N_{A^*_{x_0}}(x)$. Thus $A^*_{x_0}$ satisfies the condition (BFI1) in Definition 2.4. On the other hand, for all $x, y \in L$, it follows from $A = (\mu^P_A, \mu^N_A) \in \mathrm{BFI}(L)$ and (BFI2) that

$$\begin{aligned}\mu^P_{A^*_{x_0}}(y) &= \mu^P_A(y) + 1 - \mu^P_A(x_0) \geqslant [\mu^P_A(x) \wedge \mu^P_A((x' \to y')')] + 1 - \mu^P_A(x_0)\\ &= [\mu^P_A(x) + 1 - \mu^P_A(x_0)] \wedge [\mu^P_A((x' \to y')') + 1 - \mu^P_A(x_0)] = \mu^P_{A^*_{x_0}}(x) \wedge \mu^P_{A^*_{x_0}}((x' \to y')')\end{aligned}$$

and

$$\begin{aligned}\mu^N_{A^*_{x_0}}(y) &= \mu^N_A(y) - 1 - \mu^N_A(x_0) \leqslant [\mu^N_A(x) \vee \mu^N_A((x' \to y')')] - 1 - \mu^N_A(x_0)\\ &= [\mu^N_A(x) - 1 - \mu^N_A(x_0)] \vee [\mu^N_A((x' \to y')') - 1 - \mu^N_A(x_0)] = \mu^N_{A^*_{x_0}}(x) \vee \mu^N_{A^*_{x_0}}((x' \to y')').\end{aligned}$$

Thus $A^*_{x_0}$ also satisfies the condition (BFI2) in Definition 2.4. Hence $A^*_{x_0} \in \mathrm{BFI}(L)$.

Finally, it is obvious that $A \sqsubseteq A^*_{x_0}$. The proof is completed. □

Remark 3.7 Let L be a non-involutive residuated lattice. According to the definition of $A^*_{x_0} = (\mu^P_{A^*_{x_0}}, \mu^N_{A^*_{x_0}})$ in Theorem 3.6, we can obtain that $(A^*_{x_0})^*_{x_0} = A^*_{x_0}$ for all $A = (\mu^P_A, \mu^N_A) \in \mathrm{BFI}(L)$. In particular, if $A = (\mu^P_A, \mu^N_A) \in \mathrm{NBFI}(L)$, by Theorem 3.6 we have $A^*_{x_0} = A$.

Theorem 3.8 *Let L be a non-involutive residuated lattice and $A = (\mu^P_A, \mu^N_A) \in \mathrm{BFI}(L)$. Then the following conditions are equivalent:*

(i) $A \in \mathrm{NBFI}(L)$;

(ii) $A(0) = (1, -1)$, *i.e.*, $\mu^P_A(0) = 1$ *and* $\mu^N_A(0) = -1$;

(iii) $A = A^*_0$.

Proof (i)$\Longrightarrow$(ii). Let $A \in \mathrm{NBFI}(L)$, by Definition 3.1, there exists $x_0 \in L$ such that $A(x_0) = (1, -1)$, i.e., $\mu^P_A(x_0) = 1$ and $\mu^N_A(x_0) = -1$. It follows from $A \in \mathrm{BFI}(L)$ and (BFI1) that $1 = \mu^P_A(x_0) \leqslant \mu^P_A(0) \leqslant 1$ and $-1 = \mu^N_A(x_0) \geqslant \mu^N_A(0) \geqslant -1$. Thus $\mu^P_A(0) = 1$ and $\mu^N_A(0) = -1$, i.e., $A(0) = (1, -1)$.

(ii)$\Longrightarrow$(i). It is obvious by Definition 3.1.

(ii)$\Longrightarrow$(iii). Let $A(0) = (1, -1)$. Then $0 \in \mathrm{Ext}(A)$ and $A \in \mathrm{NBFI}(L)$. Hence $A = A^*_0$ by Remark 3.7.

(iii)$\Longrightarrow$(ii). Let $A = A^*_0$. For all $x \in L$, from the definition of A^*_0, we can get that

$$\mu^P_A(x) = \mu^P_{A^*_0}(x) = \mu^P_A(x) + 1 - \mu^P_A(0) \text{ and } \mu^N_A(x) = \mu^N_{A^*_0}(x) = \mu^N_A(x) - 1 - \mu^N_A(0).$$

Thus $\mu^P_A(0) = 1$ and $\mu^N_A(0) = -1$, i.e., $A(0) = (1, -1)$. □

Theorem 3.9 *Let L be a non-involutive residuated lattice and $I \in \mathrm{Id}(L)$. Define a BF-set $A(I) = (\mu^P_{A(I)}, \mu^N_{A(I)}) \in \mathrm{BFS}(L)$ as follows: for all $x \in L$,*

$$\mu^P_{A(I)}(x) = \begin{cases} 1, & x \in I, \\ 0, & x \notin I, \end{cases} \quad \text{and} \quad \mu^N_{A(I)}(x) = \begin{cases} -1, & x \in I, \\ 0, & x \notin I. \end{cases}$$

Then $A(I) \in \mathrm{NBFI}(L)$.

Proof Since $I \in \mathrm{Id}(L)$, by [19, Theorem 2.5], we can obtain that $A(I) \in \mathrm{BFI}(L)$. Next we claim that $A(I)$ is normal. In fact, it follows from $0 \in I \in \mathrm{Id}(L)$ that $\mu^P_{A(I)}(0) = 1$ and $\mu^N_{A(I)}(0) = -1$, i.e., $A(I)(0) = (1, -1)$. Thus $A(I)$ is normal and the proof is completed. □

Theorem 3.10 *Let L be a non-involutive residuated lattice, $A = (\mu^P_A, \mu^N_A) \in \mathrm{BFI}(L)$, $f : [0, \mu^P_A(0)] \to [0, 1]$ and $g : [\mu^N_A(0), 0] \to [-1, 0]$ two increasing functions. Define a BF-set $A_{(f,g)} = (\mu^P_{A_f}, \mu^N_{A_g}) \in \mathrm{BFS}(L)$ as follows: for all $x \in L$,*

$$\mu^P_{A_f} : L \to [0, 1], x \mapsto f(\mu^P_A(x)) \text{ and } \mu^N_{A_g} : L \to [-1, 0], x \mapsto g(\mu^N_A(x)).$$

Then the following conclusions are valid:

(i) $A_{(f,g)} \in \mathrm{BFI}(L)$;

(ii) If $f(\mu^P_A(0)) = 1$ *and* $g(\mu^N_A(0)) = -1$, *then* $A_{(f,g)} \in \mathrm{NBFI}(L)$;

(iii) If for all $(t, s) \in [0, \mu^P_A(0)] \times [\mu^N_A(0), 0]$, $f(t) \geqslant t$ *and* $g(s) \leqslant s$, *then* $A \sqsubseteq A_{(f,g)}$.

Proof (i) For all $x \in L$, by $A = (\mu^P_A, \mu^N_A) \in \mathrm{BFI}(L)$ and (BFI1), we have that $\mu^P_A(0) \geqslant \mu^P_A(x)$ and $\mu^N_A(0) \leqslant \mu^N_A(x)$. Since f and g are increasing, it follows that

$$\mu^P_{A_f}(0) = f(\mu^P_A(0)) \geqslant f(\mu^P_A(x)) = \mu^P_{A_f}(x) \text{ and } \mu^N_{A_g}(0) = g(\mu^N_A(0)) \leqslant g(\mu^N_A(x)) = \mu^N_{A_g}(x).$$

Thus $A_{(f,g)}$ satisfies the condition (BFI1) in Definition 2.4.

For all $x, y \in L$, by $A = (\mu^P_A, \mu^N_A) \in \mathrm{BFI}(L)$ and (BFI2), we have that

$$\mu^P_A(y) \geqslant \mu^P_A(x) \wedge \mu^P_A((x' \to y')') \text{ and } \mu^N_A(y) \leqslant \mu^N_A(x) \vee \mu^N_A((x' \to y')').$$

Since f and g are increasing, it follows that

$$\begin{aligned}\mu^P_{A_f}(y) =& f(\mu^P_A(y)) \geqslant f(\mu^P_A(x) \wedge \mu^P_A((x' \to y')')) \\ =& f(\mu^P_A(x)) \wedge f(\mu^P_A((x' \to y')')) = \mu^P_{A_f}(x) \wedge \mu^P_{A_f}((x' \to y')')\end{aligned}$$

and

$$\begin{aligned}\mu^N_{A_g}(y) =& g(\mu^N_A(y)) \leqslant g(\mu^N_A(x) \vee \mu^N_A((x' \to y')')) \\ =& g(\mu^N_A(x) \vee g(\mu^N_A((x' \to y')')) = \mu^N_{A_g}(x) \vee \mu^N_{A_g}((x' \to y')').\end{aligned}$$

Thus $A_{(f,g)}$ satisfies the condition (BFI2) in Definition 2.4. Hence $A_{(f,g)} \in \mathrm{BFI}(L)$.

(ii) Let $f(\mu^P_A(0)) = 1$ and $g(\mu^N_A(0)) = -1$. Then $\mu^P_{A_f}(0) = 1$ and $\mu^N_{A_g}(0) = -1$, i.e., $A_{(f,g)}(0) = (1, -1)$, it follows $A_{(f,g)} \in \mathrm{NBFI}(L)$ from (i) and Theorem 3.8.

(iii) Let for all $(t, s) \in [0, \mu^P_A(0)] \times [\mu^N_A(0), 0]$, $f(t) \geqslant t$ and $g(s) \leqslant s$. Then for all $x \in L$, we can obtain that $\mu^P_{A_f}(x) = f(\mu^P_A(x)) \geqslant \mu^P_A(x)$ and $\mu^N_{A_g}(x) = g(\mu^N_A(x)) \leqslant \mu^N_A(x)$. Hence $A \sqsubseteq A_{(f,g)}$. □

Definition 3.11 *Let L be a non-involutive residuated lattice and $A = (\mu^P_A, \mu^N_A) \in \mathrm{BFI}(L)$. Define two subsets L_A and Δ_A of L as follows:*

(i) $L_A = \{x \in L | \mu^P_A(x) = \mu^P_A(0) \text{ and } \mu^N_A(x) = \mu^N_A(0)\}$;

(ii) $\Delta_A = \{x \in L | \mu^P_A(x) = 1 \text{ and } \mu^N_A(x) = -1\}$.

Remark 3.12 Let L be a non-involutive residuated lattice. If $A = (\mu_A^P, \mu_A^N) \in \mathrm{NBFI}(L)$, from Theorem 3.8, we can easily obtain that $L_A = \Delta_A$.

Theorem 3.13 *Let L be a non-involutive residuated lattice and $A = (\mu_A^P, \mu_A^N) \in \mathrm{BFI}(L)$. Then the following conclusions are valid:*

(i) $L_A \in \mathrm{Id}(L)$;

(ii) $A \in \mathrm{NBFI}(L)$ *if and only if* $\Delta_A \in \mathrm{Id}(L)$.

Proof (i) It is the conclusion in [19, Corollary 2.2].

(ii) Let $A \in \mathrm{NBFI}(L)$. By Theorem 3.8 we have that $A(0) = (1, -1)$, i.e., $\mu_A^P(0) = 1$ and $\mu_A^N(0) = -1$, thus $0 \in \Delta_A$. For all $x, y \in L$, let $x \in \Delta_A$ and $(x' \to y')' \in \Delta_A$. It follows from the definition of Δ_A that $\mu_A^P(x) = \mu_A^P((x' \to y')') = 1$ and $\mu_A^N(x) = \mu_A^N((x' \to y')') = -1$. Since $A \in \mathrm{BFI}(L)$, by using (BFI2) we have that $\mu_A^P(y) \geqslant \mu_A^P(x) \wedge \mu_A^P((x' \to y')') = 1$ and $\mu_A^N(y) \leqslant \mu_A^N(x) \vee \mu_A^N((x' \to y')') = -1$, which shows that $\mu_A^P(y) = 1$ and $\mu_A^N(y) = -1$, thus $y \in \Delta A$. Hence $\Delta_A \in \mathrm{Id}(L)$. Conversely, let $\Delta_A \in \mathrm{Id}(L)$. We have $0 \in \Delta_A$ by Definition 2.3, and thus $\mu_A^P(0) = 1$ and $\mu_A^N(0) = -1$, i.e., $A(0) = (1, -1)$. It follows that $A \in \mathrm{NBFI}(L)$ from $A \in \mathrm{BFI}(L)$ and Theorem 3.8. □

Theorem 3.14 *Let L be a non-involutive residuated lattice, $A = (\mu_A^P, \mu_A^N) \in \mathrm{BFI}(L)$ and $B = (\mu_B^P, \mu_B^N) \in \mathrm{BFI}(L)$. Then the following conclusions are valid:*

(i) $A \sqsubseteq B$ *implies* $\Delta_A \sqsubseteq \Delta_B$;

(ii) If $A, B \in \mathrm{NBFI}(L)$, *then* $A \sqsubseteq B$ *implies* $L_A \sqsubseteq L_B$.

Proof (i) Let $A \sqsubseteq B$. For all $x \in \Delta_A$, we have that $1 = \mu_A^P(x) \leqslant \mu_B^P(x)$ and $-1 = \mu_A^N(x) \geqslant \mu_B^N(x)$. It shows that $\mu_B^P(x) = 1$ and $\mu_B^N(x) = -1$, thus $x \in \Delta_B$. Hence $\Delta_A \sqsubseteq \Delta_B$.

(ii) Let $A, B \in \mathrm{NBFI}(L)$ and $A \sqsubseteq B$. For all $x \in L_A$, we can obtain that $\mu_B^P(x) \geqslant \mu_A^P(x) = \mu_A^P(0) = 1$ and $\mu_B^N(x) \leqslant \mu_A^N(x) = \mu_A^N(0) = -1$ by Theorem 3.8, which shows that $\mu_B^P(x) = 1 = \mu_B^P(0)$ and $\mu_B^N(x) = -1 = \mu_B^N(0)$, thus $x \in L_B$. Hence $L_A \sqsubseteq L_B$. □

Theorem 3.15 (Extension Theorem) *Let L be a non-involutive residuated lattice, $A = (\mu_A^P, \mu_A^N) \in \mathrm{NBFI}(L)$ and $B = (\mu_B^P, \mu_B^N) \in \mathrm{BFI}(L)$. If $A \sqsubseteq B$, then $B \in \mathrm{NBFI}(L)$.*

Proof Since $A \in \mathrm{NBFI}(L)$, by Theorem 3.8, we have that $A(0) = (1, -1)$, i.e., $\mu_A^P(0) = 1$ and $\mu_A^N(0) = -1$. Let $A \sqsubseteq B$. We can obtain that $\mu_B^P(0) \geqslant \mu_A^P(0) = 1$ and $\mu_B^N(0) \leqslant \mu_A^N(0) = -1$. Thus $\mu_B^P(0) = 1$ and $\mu_B^N(0) = -1$, i.e., $B(0) = (1, -1)$. It follows from $B \in \mathrm{BFI}(L)$ and Theorem 3.8 that $B \in \mathrm{NBFI}(L)$. □

Remark 3.16 Let L be a non-involutive residuated lattice. It is obvious that $\mathrm{NBFI}(L)$ is a poset under the BF-inclusion order.

Theorem 3.17 *Let L be a non-involutive residuated lattice. Then any non-constant maximal element of $(\mathrm{NBFI}(L), \sqsubseteq)$ only takes a value among $(0, 0)$, $(1, -1)$ and $(1, 0)$.*

Proof Let $A = (\mu_A^P, \mu_A^N) \in \mathrm{NBFI}(L)$ be a non-constant maximal element of $(\mathrm{NBFI}(L), \sqsubseteq)$. Then there exists $x_0 \in L$ such that $\mu_A^P(x_0) = 1$ and $\mu_A^N(x_0) = -1$. Let $x \in L$ such that $\mu_A^P(x) \neq 1$ and $\mu_A^N(x) \neq -1$. Then $\mu_A^P(x) = 0$ and $\mu_A^N(x) = 0$. Otherwise, there exists $\varepsilon \in L$ such that $0 < \mu_A^P(\varepsilon) < 1$ and $-1 < \mu_A^N(\varepsilon) < 0$. Define $A_\varepsilon = (\mu_{A_\varepsilon}^P, \mu_{A_\varepsilon}^N) \in \mathrm{BFS}(L)$ as follows: $\forall x \in L$,

$$\mu_{A_\varepsilon}^P(x) = \frac{1}{2}[\mu_A^P(x) + \mu_A^P(\varepsilon)] \text{ and } \mu_{A_\varepsilon}^N(x) = \frac{1}{2}[\mu_A^N(x) + \mu_A^N(\varepsilon)].$$

Obviously, A_ε is well defined. We claim that $A_\varepsilon \in \mathrm{BFI}(L)$. In fact, for all $x \in L$, by $A = (\mu_A^P, \mu_A^N) \in \mathrm{BFI}(L)$ and (BFI1), we have that

$$\mu_{A_\varepsilon}^P(0) = \frac{1}{2}[\mu_A^P(0) + \mu_A^P(\varepsilon)] \geqslant \frac{1}{2}[\mu_A^P(x) + \mu_A^P(\varepsilon)] = \mu_{A_\varepsilon}^P(x)$$

and

$$\mu_{A_\varepsilon}^N(0) = \frac{1}{2}[\mu_A^N(0) + \mu_A^N(\varepsilon)] \leqslant \frac{1}{2}[\mu_A^N(x) + \mu_A^N(\varepsilon)] = \mu_{A_\varepsilon}^N(x),$$

thus A_ε satisfies the condition (BFI1) in Definition 2.4. For all $x, y \in L$, by $A = (\mu_A^P, \mu_A^N) \in \mathrm{BFI}(L)$ and (BFI2), we have that

$$\begin{aligned}\mu_{A_\varepsilon}^P(y) =& \frac{1}{2}[\mu_A^P(y) + \mu_A^P(\varepsilon)] \geqslant \frac{1}{2}[\mu_A^P(x) \wedge \mu_A^P((x' \to y')') + \mu_A^P(\varepsilon)] \\ =& \frac{1}{2}[\mu_A^P(x) + \mu_A^P(\varepsilon)] \wedge \frac{1}{2}[\mu_A^P((x' \to y')') + \mu_A^P(\varepsilon)] = \mu_{A_\varepsilon}^P(x) \wedge \mu_{A_\varepsilon}^P((x' \to y')')\end{aligned}$$

and

$$\begin{aligned}\mu_{A_\varepsilon}^N(y) =& \frac{1}{2}[\mu_A^N(y) + \mu_A^N(\varepsilon)] \leqslant \frac{1}{2}[\mu_A^N(x) \vee \mu_A^N((x' \to y')') + \mu_A^P(\varepsilon)] \\ =& \frac{1}{2}[\mu_A^N(x) + \mu_A^N(\varepsilon)] \vee \frac{1}{2}[\mu_A^N((x' \to y')') + \mu_A^N(\varepsilon)] = \mu_{A_\varepsilon}^N(x) \vee \mu_{A_\varepsilon}^N((x' \to y')'),\end{aligned}$$

thus A_ε also satisfies the condition (BFI2) in Definition 2.4. Hence $A_\varepsilon \in \mathrm{BFI}(L)$.

From $\mu_A^P(x_0) = 1$ and $\mu_A^N(x_0) = -1$, we can know that $x_0 \in \mathrm{Ext}(A)$. So, for all $x \in L$, we have that $\mu_A^P(x_0) \geqslant \mu_A^P(x)$ and $\mu_A^N(x_0) \leqslant \mu_A^N(x)$, and so we have that

$$\mu_{A_\varepsilon}^P(x_0) = \frac{1}{2}[\mu_A^P(x_0) + \mu_A^P(\varepsilon)] \geqslant \frac{1}{2}[\mu_A^P(x) + \mu_A^P(\varepsilon)] = \mu_{A_\varepsilon}^P(x)$$

and

$$\mu_{A_\varepsilon}^N(x_0) = \frac{1}{2}[\mu_A^N(x_0) + \mu_A^N(\varepsilon)] \leqslant \frac{1}{2}[\mu_A^N(x) + \mu_A^N(\varepsilon)] = \mu_{A_\varepsilon}^N(x).$$

Thus $x_0 \in \mathrm{Ext}(A_\varepsilon)$. Hence we can obtain that $(A_\varepsilon)_{x_0}^* = (\mu_{(A_\varepsilon)_{x_0}^*}^P, \mu_{(A_\varepsilon)_{x_0}^*}^P) \in \mathrm{NBFI}(L)$ by Theorem 3.6, where, for all $x \in L$,

$$\begin{aligned}\mu_{(A_\varepsilon)_{x_0}^*}^P(x) =& \mu_{A_\varepsilon}^P(x) + 1 - \mu_{A_\varepsilon}^P(x_0) = \frac{1}{2}[\mu_A^P(x) + \mu_A^P(\varepsilon)] + 1 - \frac{1}{2}[\mu_A^P(x_0) + \mu_A^P(\varepsilon)] \\ =& \frac{1}{2}[\mu_A^P(x) - \mu_A^P(x_0) + 2] = \frac{1}{2}[\mu_A^P(x) + 1]\end{aligned}$$

and

$$\begin{aligned}\mu_{(A_\varepsilon)_{x_0}^*}^N(x) =& \mu_{A_\varepsilon}^N(x) - 1 - \mu_{A_\varepsilon}^N(x_0) = \frac{1}{2}[\mu_A^N(x) + \mu_A^N(\varepsilon)] - 1 - \frac{1}{2}[\mu_A^N(x_0) + \mu_A^N(\varepsilon)] \\ =& \frac{1}{2}[\mu_A^N(x) - \mu_A^N(x_0) - 2] = \frac{1}{2}[\mu_A^N(x) - 1].\end{aligned}$$

Obviously, $A \sqsubseteq (A_\varepsilon)^*_{x_0}$. Since $\mu_A^P(x) \neq 1$ and $\mu_A^N(x) \neq -1$, we have that

$$\mu^P_{(A_\varepsilon)^*_{x_0}}(x) = \frac{1}{2}[\mu_A^P(x) + 1] > \mu_A^P(x) \text{ and } \mu^N_{(A_\varepsilon)^*_{x_0}}(x) = \frac{1}{2}[\mu_A^N(x) - 1] < \mu_A^N(x).$$

Thus $A \sqsubset (A_\varepsilon)^*_{x_0}$. On the other hand, by the definition of $(A_\varepsilon)^*_{x_0}$, we can get that

$$\mu^P_{(A_\varepsilon)^*_{x_0}}(\varepsilon) = \frac{1}{2}[\mu_A^P(\varepsilon) + 1] < 1 = \mu^P_{(A_\varepsilon)^*_{x_0}}(x_0),$$
$$\mu^N_{(A_\varepsilon)^*_{x_0}}(\varepsilon) = \frac{1}{2}[\mu_A^N(\varepsilon) - 1] > -1 = \mu^N_{(A_\varepsilon)^*_{x_0}}(x_0).$$

Therefore, $(A_\varepsilon)^*_{x_0}$ is non-constant, and A is not a maximal element of $(\mathrm{NBFI}(L), \sqsubseteq)$. This is a contradiction. Thus, we can obtain that $\mu_A^P(x) = 0$ and $\mu_A^N(x) = 0$. Hence μ_A^P only takes two possible values 0 and 1, μ_A^N only takes two possible values 0 and -1. This implies that all the possible values are $(0,0)$, $(1,0)$, $(0,-1)$ and $(1,-1)$. Further, if A takes a value from above four values, by using Corollary 3.1 in [19], we have that

$$V(0,0) = \{x \in L | \mu_A^P(x) \geqslant 0\} \cap \{x \in L | \mu_A^N(x) \leqslant 0\} = L,$$
$$V(1,0) = \{x \in L | \mu_A^P(x) \geqslant 1\} \cap \{x \in L | \mu_A^N(x) \leqslant 0\} = \{x \in L | \mu_A^P(x) = 1\},$$
$$V(0,-1) = \{x \in L | \mu_A^P(x) \geqslant 0\} \cap \{x \in L | \mu_A^N(x) \leqslant -1\} = \{x \in L | \mu_A^N(x) = -1\},$$
$$V(1,-1) = \{x \in L | \mu_A^P(x) \geqslant 1\} \cap \{x \in L | \mu_A^N(x) \leqslant -1\} = \{x \in L | \mu_A^P(x) = 1 \text{ and } \mu_A^N(x) = -1\}$$

are all non-empty ideals of L and satisfying

$$\text{(i) } V(1,-1) \subseteq V(0,-1) \subseteq V(0,0) \text{ and (ii) } V(1,-1) \subseteq V(1,0) \subseteq V(0,0).$$

For case (i), define a BF-set $B = (\mu_B^P, \mu_B^N) \in \mathrm{BFS}(L)$ as follows:

$$\mu_B^P(x) = \begin{cases} 1, & x \in V(0,-1), \\ 0, & \text{otherwise,} \end{cases} \text{ and } \mu_B^N(x) = \begin{cases} -1, & x \in V(0,-1), \\ 0, & \text{otherwise.} \end{cases}$$

Then according to Theorem 3.9, $B \in \mathrm{NBFI}(L)$. Now, for all $x \in V(0,-1)$, we have $\mu_B^P(x) = 1 \geqslant \mu_A^P(x)$ and $\mu_B^N(x) = -1 = \mu_A^N(x)$, i.e., $A \sqsubseteq B$. For all $x \in V(0,0) - V(0,-1)$, we have $\mu_B^P(x) = 0 = \mu_B^N(x)$. Since μ_A^P only takes two possible values 0 and 1, if $\mu_A^P(x) = 0$, then $\mu_A^P(x) = \mu_B^P(x) = 0$ and $\mu_A^N(x) \leqslant 0 = \mu_B^N(x)$, thus $B \sqsubseteq A$. Otherwise, if $\mu_A^P(x) = 1$, then $\mu_A^P(x) \geqslant \mu_B^P(x)$ and $\mu_A^N(x) \leqslant 0 = \mu_B^N(x)$. Whence, $B \sqsubseteq A$. In addition, for all $x \in V(0,-1) - V(1,-1)$, we have $\mu_A^P(x) = 0 < 1 = \mu_B^P(x)$ and $\mu_A^N(x) = -1 = \mu_B^N(x)$. Then $A \sqsubset B$, which contradicts the fact that A is a non-constant maximal element of $(\mathrm{NBFI}(L), \sqsubseteq)$. Therefore, $A(x) \neq (0,-1)$. For case (ii), we can show that $A(x) \neq (0,-1)$ similarly. Hence A only takes a value among $(0,0)$, $(1,-1)$ and $(1,0)$. □

Definition 3.18 *Let L be a non-involutive residuated lattice. A non-constant BF-ideal $A = (\mu_A^P, \mu_A^N) \in \mathrm{BFI}(L)$ is called a maximal BF-ideal, if there is no non-constant BF-ideal $B = (\mu_B^P, \mu_B^N) \in \mathrm{BFI}(L)$ such that $A \sqsubseteq B$. The set of all maximal BF-ideals of L is denoted by* $\mathrm{MBFI}(L)$.

Theorem 3.19 *Let L be a non-involutive residuated lattice and $A = (\mu_A^P, \mu_A^N) \in \mathrm{MBFI}(L)$. Then the following conclusions are valid:*

(i) $A \in \mathrm{NBFI}(L)$*;*

(ii) A *only takes a value among* $(0,0)$, $(1,-1)$ *and* $(1,0)$.

Proof (i) Let $A=(\mu_A^P,\mu_A^N)\in \mathrm{MBFI}(L)$. Then A is non-constant. We claim that $A_0^*=(\mu^P_{A_0^*},\mu^N_{A_0^*})$ is non-constant. Otherwise, for all $x\in L$, there exists $(s,t)\in[-1,0]\times[0,1]$ such that $t=\mu^P_{A_0^*}(x)=\mu_A^P(x)+1-\mu_A^P(0)$ and $s=\mu^N_{A_0^*}(x)=\mu_A^N(x)-1-\mu_A^N(0)$, thus $\mu_A^P(x)=t+\mu_A^P(0)-1$ and $\mu_A^N(x)=s+\mu_A^N(0)+1$, it contradicts that A is non-constant. Since $A\sqsubseteq A_0^*$ by Theorem 3.6, we have that $A=A_0^*$ by $A\in\mathrm{MBFI}(L)$. It follows that $A\in\mathrm{NBFI}(L)$ from Theorem 3.8.

(ii) Since $A\in\mathrm{NBFI}(L)$ by (i), according to Theorem 3.8, we have $A(0)=(1,-1)$. It follows from Theorem 3.17 and its proof that A_0^* is a non-constant maximal element of $(\mathrm{NBFI}(L),\sqsubseteq)$, thus A_0^* only takes a value among $(0,0)$, $(1,-1)$ and $(1,0)$. Hence, A also only takes a value among $(0,0)$, $(1,-1)$ and $(1,0)$ by $A\sqsubseteq A_0^*$. $\square$

Definition 3.20 *Let* L *be a non-involutive residuated lattice. A non-constant BF-ideal* $A=(\mu_A^P,\mu_A^N)\in\mathrm{BFI}(L)$ *is called completely normal if there exists* $x\in L$ *such that* $A(x)=(0,0)$*. The set of all completely normal BF-ideals of* L *is denoted by* $\mathrm{CBFI}(L)$.

Remark 3.21 Let L be a non-involutive residuated lattice. It is obvious that $\mathrm{CBFI}(L)\subseteq\mathrm{NBFI}(L)$. So we can obtain the following result.

Theorem 3.22 *Let* L *be a non-involutive residuated lattice. Then any non-constant maximal element of* $(\mathrm{NBFI}(L),\sqsubseteq)$ *is also a maximal element of* $(\mathrm{CBFI}(L),\sqsubseteq)$.

Proof Let $A=(\mu_A^P,\mu_A^N)$ be a non-constant maximal element of $(\mathrm{NBFI}(L),\sqsubseteq)$. By Theorem 3.17, A only takes a value among $(0,0)$, $(1,-1)$ and $(1,0)$. Then there exists $x_0,x_1,x_2\in L$ such that $A(x_0)=(0,0)$, $A(x_1)=(1,-1)$ and $A(x_2)=(1,0)$, Thus $A\in\mathrm{CBFI}(L)$. Further, assume that $B=(\mu_B^P,\mu_B^N)\in\mathrm{CBFI}(L)$ and $A\sqsubseteq B$, then $A\sqsubseteq B$ in $\mathrm{NBFI}(L)$. Since A is a maximal element of $(\mathrm{NBFI}(L),\sqsubseteq)$ and B is non-constant, we have that $A=B$. Hence A also a maximal element of $(\mathrm{CBFI}(L),\sqsubseteq)$. $\square$

From the above results, we can easily obtain the following corollary.

Corollary 3.23 *Let* L *be a non-involutive residuated lattice. Then* $\mathrm{MBFI}(L)\subseteq\mathrm{CBFI}(L)$.

Theorem 3.24 *Let* L *be a non-involutive residuated lattice and* $A=(\mu_A^P,\mu_A^N)\in\mathrm{BFI}(L)$. *Define a BF-set* $\hat{A}=(\mu_{\hat{A}}^P,\mu_{\hat{A}}^N)\in\mathrm{BFS}(L)$ *as follows: for all* $x\in L$,

$$\mu_{\hat{A}}^P(x)=\frac{\mu_A^P(x)-\mu_A^P(1)}{\mu_A^P(0)-\mu_A^P(1)} \text{ and } \mu_{\hat{A}}^N(x)=\frac{\mu_A^N(x)-\mu_A^N(1)}{\mu_A^N(1)-\mu_A^N(0)}.$$

Then $\hat{A}\in\mathrm{CBFI}(L)$.

Proof Firstly, from $\mu_{\hat{A}}^P(x)\in[0,1]$ and $\mu_{\hat{A}}^N(x)\in[-1,0]$, $\hat{A}$ is well defined.

Secondly, for all $x\in L$, since $\mu_{\hat{A}}^P(0)=1\geqslant\mu_{\hat{A}}^P(x)$ and $\mu_{\hat{A}}^N(0)=-1\leqslant\mu_{\hat{A}}^N(x)$, we can obtain that $\hat{A}(0)=(1,-1)$ and $\hat{A}$ satisfies the condition (BFI1) in Definition 2.4.

Thirdly, for all $x, y \in L$, it follows from $A = (\mu_A^P, \mu_A^N) \in \mathrm{BFI}(L)$ and (BFI2) that

$$\begin{aligned}\mu_{\hat{A}}^P(x) \wedge \mu_{\hat{A}}^P((x' \to y')') =& \frac{\mu_A^P(x) - \mu_A^P(1)}{\mu_A^P(0) - \mu_A^P(1)} \wedge \frac{\mu_A^P((x' \to y')') - \mu_A^P(1)}{\mu_A^P(0) - \mu_A^P(1)} \\ =& \frac{[\mu_A^P(x) - \mu_A^P(1)] \wedge [\mu_A^P((x' \to y')') - \mu_A^P(1)]}{\mu_A^P(0) - \mu_A^P(1)} \\ =& \frac{\mu_A^P(x) \wedge \mu_A^P((x' \to y')') - \mu_A^P(1)}{\mu_A^P(0) - \mu_A^P(1)} \\ \leqslant& \frac{\mu_A^P(y) - \mu_A^P(1)}{\mu_A^P(0) - \mu_A^P(1)} = \mu_{\hat{A}}^P(y)\end{aligned}$$

and

$$\begin{aligned}\mu_{\hat{A}}^N(x) \vee \mu_{\hat{A}}^N((x' \to y')') =& \frac{\mu_A^N(x) - \mu_A^N(1)}{\mu_A^N(1) - \mu_A^N(0)} \vee \frac{\mu_A^N((x' \to y')') - \mu_A^N(1)}{\mu_A^N(1) - \mu_A^N(0)} \\ =& \frac{[\mu_A^N(x) - \mu_A^N(1)] \vee [\mu_A^N((x' \to y')') - \mu_A^N(1)]}{\mu_A^N(1) - \mu_A^N(0)} \\ =& \frac{\mu_A^N(x) \vee \mu_A^N((x' \to y')') - \mu_A^N(1)}{\mu_A^N(1) - \mu_A^N(0)} \\ \geqslant& \frac{\mu_A^N(y) - \mu_A^N(1)}{\mu_A^N(1) - \mu_A^N(0)} = \mu_{\hat{A}}^N(y),\end{aligned}$$

thus $\hat{A}$ also satisfies the condition (BFI2) in Definition 2.4.

Finally, it is obvious that $\mu_{\hat{A}}^P(1) = 0 = \mu_{\hat{A}}^N(1)$, i.e., $\hat{A}(1) = (0, 0)$.

Therefore, it follows that $\hat{A} \in \mathrm{CBFI}(L)$ from Definitions 2.4, 3.1 and 3.20. □

4. Concluding remarks

As well known, ideal is an important concept for studying the structural features of non-involutive residuated lattices. In this paper, by applying the method and principle of bipolar fuzzy sets, the ideals theory in non-involutive residuated lattices is further studied. The notion of normal bipolar fuzzy ideal is introduced. Some important properties and equivalent characterizations of normal bipolar fuzzy ideals are obtained. In addition, two special types of normal bipolar fuzzy ideals are defined, which are called maxima and completely normal bipolar fuzzy ideals, respectively, and their relationships are discussed. Results obtained in this paper not only enrich the content of bipolar fuzzy ideal theory in non-involutive residuated lattices, but also show interactions of algebraic technique and bipolar fuzzy set method in the studying logic problems. We hope that more links of interval-valued fuzzy sets and logics emerge by the stipulating of this work.

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References

[1] Guojun WANG. *Nonclassical Mathematical Logic and Approximate Reasoning*. Science Press, Beijing, 2003. (in Chinese)

[2] M. WARD, R. P. DILWORTH. *Residuated lattices*. Transactions of the American Mathematical Society, 1939, **45**: 335–354.

[3] P. HÁJEK. *Metamathematics of Fuzzy Logic*. Kluwer Academic Publishers, Dordrecht, 1998.

[4] Guojun WANG, Hongjun ZHOU. *Introduction to Mathematical Logic and Resolution Principle*. 2nd Ed., Science Press, Beijing, 2009.

[5] Yiquan ZHU, Yang XU. *On filter theory of residuated lattices*. Inform. Sci., 2010, **180**(16): 3614–3632.

[6] B. DUMITRU, P. DANA. *Some types of filters on residuated lattices*. Soft Computing, 2014, **18**(5): 825–837.

[7] C. LELE, J. B. NGANOU. *MV-algebras derived from ideals in BL-algebras*. Fuzzy Sets and Systems, 2013, **218**(4): 103–113.

[8] Chunhui LIU. *LI-ideals theory in negative non-involutive residuated lattices*. Appl. Math. J. Chinese Univ. Ser. A, 2015, **30**(4): 445–456.

[9] Yi LIU, Yao QIN, Xiaoyan QIN, et al. *Ideals and fuzzy ideals on residuated lattices*. International Journal of Machine Learning and Cybernetics, 2017, **8**(1): 239–253.

[10] L. A. ZADEH. *Fuzzy Sets*. Information Control, 1965, **8**: 338–353.

[11] W. R. ZHANG. *Bipolar fuzzy sets and relations: A computational framework for cognitive modeling and multiagent decision analysis*. Proceedings of the First International Joint Conference of the North American Fuzzy Information Processing Society Biannual Conference, Berln: Springer, 1994.

[12] W. R. ZHANG. *Yin Yang bipolar relativity: a unifying theory of nature, agents and causality wihth applications in quantum computing, cogintive informatics and lift sciences*. US: IGI Global, 2011.

[13] K. M. LEE. *Bipolar-Valued Fuzzy Sets and Their Operations*. Proceedings of International Conference on Intelligent Technologies, Bangkok Thailand: Springer, 2000.

[14] M. AKRAM. *Bipolar fuzzy graphs*. Inform. Sci., 2011, **181**(24): 5548–5564.

[15] A. B. SAEID, M. K. RAFSANJANI. *Some results in bipolar-valued fuzzy BCK/BCI-algebras*. Proceedings of International Conference on Networked Digital Technologies, Prague, Czech Republic: Springer, 2010.

[16] S. K. MAJUNDER. *Bipolar-valued fuzzy in* Γ*-semigroups*. Mathematica Aeterna, 2012, **2**(3): 203–213.

[17] K. MAHMOOD K, K. HAYAT. *Characterizations of Hemi-rings by their bipolar-valued fuzzy h-ideals*. Information Sciences Letters, 2015, **4**(2): 51–59.

[18] C. JANA, M. PAL, A. B. SAEID. $(\in, \in \vee q)$*-bipolar-valued fuzzy BCK/BCI-algebras*. Missouri J. Math. Sci., 2017, **29**(5): 101–112.

[19] Chunhui LIU, Yumao LI, Haiyan ZHANG. *Bipolar fuzzy ideals in negative non-involutive residuated lattices*. Journal of Shandong University (Natural Science), 2019, **54**(5): 88–98.

[20] Chunhui LIU, Zhiting JIANG, Xuecheng QIN. *BF-ideals in negative non-involutive residuated lattices*. Journal of Mathematics in Practice and Theory, 2019, **49**(13): 245–251.