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Estimations for the Spectral Radius of Nonnegative Tensors

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Abstract In this paper, a lower bound and an upper bound for the spectral radius of nonnegative tensors are obtained. Our new bounds are tighter than the corresponding bounds obtained by Li et al. (J. Inequal. Appl. 2015). A numerical example is given to show the effectiveness of theoretical results.

Keywords bounds; spectral radius; nonnegative tensor; irreducible

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1. Introduction

A tensor $\mathcal{A}=(a_{i_1,\ldots,i_m})$, which is a multidimensional array with the entries

$$a_{i_1,\ldots,i_m}\in\mathbb{R},\ i_j\in N=\{1,\ldots,n\}\text{ for } j\in M=\{1,\ldots,m\},$$

is called an m-order n-dimensional real tensor, denoted by $\mathcal{A}\in\mathbb{R}^{[m,n]}$, where $\mathbb{R}$ is mean real number. Moreover, a real tensor $\mathcal{A}$ is called nonnegative (positive) if $a_{i_1\cdots i_m}\geq(>)\ 0$, denoted by $\mathcal{A}\geq(>)\ 0$. Given a tensor $\mathcal{A}=(a_{i_1,\ldots,i_m})$, if there exists a nonempty proper index subset $I\subset\{1,\ldots,n\}$ such that

$$a_{i_1,\ldots,i_m}=0,\quad\forall\, i_1\in I,\quad\forall\, i_2,\ldots,i_m\notin I,$$

then, we say $\mathcal{A}$ is reducible. Otherwise, we call $\mathcal{A}$ irreducible [1].

Definition 1.1 ([2,3]) *Let $\mathcal{A}=(a_{i_1,\ldots,i_m})\in\mathbb{R}^{[m,n]}$ and a number $\lambda\in\mathbb{C}$ ($\mathbb{C}$ denotes complex number). If there is a non-zero vector $x=(x_1,x_2,\ldots,x_n)^T\in\mathbb{C}^n$, such that*

$$\mathcal{A}x^{m-1}=\lambda x^{[m-1]},$$

then λ is called an eigenvalue and x an eigenvector associated with λ of $\mathcal{A}$, where $\mathcal{A}x^{m-1}$ and $x^{[m-1]}$ are n-dimensional vectors, whose i-th entries are

$$(\mathcal{A}x^{m-1})_i=\sum_{i_2,\ldots,i_m\in N}a_{ii_2,\ldots,i_m}x_{i_2},\ldots,x_{i_m}\quad\text{and}\quad(x^{[m-1]})_i=x_i^{m-1}.$$

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Eigenvalues of tensors have a wide range of applications [1–6], so methods how to calculate or estimate the eigenvalues has become an important topic in numerical multilinear algebra. In addition, the spectral radius $\rho(\mathcal{A})$ of a tensor $\mathcal{A}$ is defined as [7]

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\}.$$

Chang et al. extended the Perron-Frobenius theorem for irreducible nonnegative matrices to irreducible nonnegative tensors.

Theorem 1.2 ([1]) *If $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is irreducible nonnegative, then $\rho(\mathcal{A})$ is a positive eigenvalue of $\mathcal{A}$ with an entrywise positive eigenvector x, i.e., $x > 0$, corresponding to it.*

Yang et al. generalized this theorem to nonnegative tensors and provided a lower bound and an upper bound for the spectral radius of nonnegative tensors [8].

Theorem 1.3 ([8]) *If $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is nonnegative, then $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with an entrywise nonnegative eigenvector x, i.e., $x \geq 0$, $x \neq 0$, corresponding to it.*

Theorem 1.4 ([8]) *Let $\mathcal{A} = (a_{i_1,\ldots,i_m}) \in \mathbb{R}^{[m,n]}$ be nonnegative. Then*

$$R_{\min} \leq \rho(\mathcal{A}) \leq R_{\max},$$

where $R_{\min} = \min_{i\in N} R_i(\mathcal{A})$, $R_{\max} = \max_{i\in N} R_i(\mathcal{A})$, and $R_i(\mathcal{A}) = \sum_{i_2,\ldots,i_m\in N} a_{ii_2,\ldots,i_m}$.

Li et al. [9, 10] had given several bounds which are tighter than those in Theorem 1.4.

Theorem 1.5 ([9]) *Let $\mathcal{A} = (a_{i_1,\ldots,i_m}) \in \mathbb{R}^{[m,n]}$ be nonnegative with $n \geq 2$. Then*

$$\rho(\mathcal{A}) \leq \Omega_{\max},$$

where

$$\Omega_{\max} = \max_{\substack{i,j\in N\\ j\neq i}} \frac{1}{2}\Big(a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \sqrt{(a_{i\ldots i} - a_{j\ldots j} + r_i^j(\mathcal{A}))^2 + 4a_{ij\ldots j}r_j(\mathcal{A})}\ \Big).$$

$$r_i(\mathcal{A}) = \sum_{\substack{i_2,\ldots,i_m\in N\\ \delta_{ii_2\ldots i_m}=0}} a_{ii_2\ldots i_m}, \qquad r_i^j(\mathcal{A}) = \sum_{\substack{\delta_{ii_2\ldots i_m}=0\\ \delta_{ji_2\ldots i_m}=0}} a_{ii_2\ldots i_m} = r_i(\mathcal{A}) - a_{ij\ldots j}.$$

Furthermore, $\Omega_{\max} \leq R_{\max}$.

The method used in Theorem 1.5 also provides a lower bound on $\Omega_{\min}$ of $\mathcal{A}$.

Theorem 1.6 *Let $\mathcal{A} = (a_{i_1,\ldots,i_m}) \in \mathbb{R}^{[m,n]}$ be nonnegative with $n \geq 2$. Then*

$$\rho(\mathcal{A}) \geq \Omega_{\min},$$

where

$$\Omega_{min} = \min_{\substack{i,j\in N\\ j\neq i}} \frac{1}{2}\Big(a_{i\ldots i} + a_{j\ldots j} + r_i^j(\mathcal{A}) + \sqrt{(a_{i\ldots i} - a_{j\ldots j} + r_i^j(\mathcal{A}))^2 + 4a_{ij\ldots j}r_j(\mathcal{A})}\ \Big).$$

Furthermore, $\Omega_{\min} \leq \rho(\mathcal{A}) \leq \Omega_{\max}$.

In order to obtain tighter bounds, let $\mathcal{A} = (a_{i_1,\ldots,i_m}) \in \mathbb{R}^{[m,n]}$ be nonnegative with $n \geq 2$.

We denote

$$\Theta_i = \{(i_2, i_3, \ldots, i_m) : i_j = i \text{ for some } j \in \{2, \ldots, m\}, \text{ where } i, i_2, \ldots, i_m \in N\},$$

$$\overline{\Theta}_i = \{(i_2, i_3, \ldots, i_m) : i_j \neq i \text{ for any } j \in \{2, \ldots, m\}, \text{ where } i, i_2, \ldots, i_m \in N\},$$

$$r_i^{\Theta_i}(\mathcal{A}) = \sum_{\substack{(i_2,\ldots,i_m)\in\Theta_i\\ \delta_{ii_2,\ldots,i_m}=0}} a_{ii_2,\ldots,i_m}, \quad r_i^{\overline{\Theta}_i}(\mathcal{A}) = \sum_{(i_2,\ldots,i_m)\in\overline{\Theta}_i} a_{ii_2,\ldots,i_m},$$

where

$$\delta_{i_1\ldots i_m} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $r_i(\mathcal{A}) = r_i^{\Theta_i}(\mathcal{A}) + r_i^{\overline{\Theta}_i}(\mathcal{A})$ and $r_i^j(\mathcal{A}) = r_i^{\Theta_i}(\mathcal{A}) + r_i^{\overline{\Theta}_i}(\mathcal{A}) - a_{ij\ldots j}$.

Theorem 1.7 ([10]) *Let $\mathcal{A} = (a_{i_1,\ldots,i_m}) \in \mathbb{R}^{[m,n]}$ be nonnegative with $n \geq 2$. Then*

$$\Delta_{\min} \leq \rho(\mathcal{A}) \leq \Delta_{\max},$$

where

$$\Delta_{\min} = \min_{\substack{i,j\in N,\\ j\neq i}} \Delta_{i,j}(\mathcal{A}), \quad \Delta_{\max} = \max_{\substack{i,j\in N,\\ j\neq i}} \Delta_{i,j}(\mathcal{A}),$$

$$\Delta_{i,j}(\mathcal{A}) = \frac{1}{2}(a_{i,\ldots,i} + a_{j,\ldots,j} + r_i^{\Theta_i}(\mathcal{A}) + \Lambda^{\frac{1}{2}}), \quad \Lambda = (a_{i,\ldots,i} - a_{j,\ldots,j} + r_i^{\Theta_i}(A))^2 + 4r_i^{\overline{\Theta}_i}(\mathcal{A})r_j(\mathcal{A}).$$

Li et al. [10] had compared these bounds in Theorems 1.4 and 1.7.

Theorem 1.8 ([10]) *Let $\mathcal{A} = (a_{i_1,\ldots,i_m}) \in \mathbb{R}^{[m,n]}$ be nonnegative with $n \geq 2$. Then*

$$R_{\min} \leq \Delta_{\min} \leq \Delta_{\max} \leq R_{\max}.$$

In this paper, we continue to focus on the bounds for the spectral radius of nonnegative tensors and propose one new lower bound and one new upper bound. It is proved that these new bounds are tighter than the corresponding bounds in [8–10]. At last a numerical example is given to verify the effectiveness of theory results.

2. New bounds for the spectral radius of nonnegative tensors

Now we establish the new bounds for spectral radius of nonnegative tensors.

Theorem 2.1 *Let $\mathcal{A} = (a_{i_1,\ldots,i_m}) \in \mathbb{R}^{[m,n]}$ be a nonnegative tensor with $n \geq 2$. Then*

$$\Psi_{\min} \leq \rho(\mathcal{A}) \leq \Psi_{\max},$$

where

$$\Psi_{\min} = \min_{\substack{i,j\in N,\\ j\neq i}} \Psi_{i,j}(\mathcal{A}), \quad \Psi_{\max} = \max_{\substack{i,j\in N,\\ j\neq i}} \Psi_{i,j}(\mathcal{A}),$$

and

$$\Psi_{i,j}(\mathcal{A}) = \frac{1}{2}(a_{i,\ldots,i} + a_{j,\ldots,j} + r_i^{\Theta_i}(\mathcal{A}) + r_j^{\overline{\Theta}_i}(\mathcal{A}) + \nabla^{\frac{1}{2}}),$$

$$\nabla = (a_{i,\ldots,i} - a_{j,\ldots,j} + r_i^{\Theta_i}(A) - r_j^{\overline{\Theta}_i}(\mathcal{A}))^2 + 4r_i^{\overline{\Theta}_i}(\mathcal{A})r_j^{\Theta_i}(\mathcal{A}).$$

Proof Let $x=(x_1,x_2,\ldots,x_n)^T$ be an eigenvector to $\rho(\mathcal{A})$ of $\mathcal{A}$ and satisfy

$$\mathcal{A}x^{m-1}=\rho(\mathcal{A})x^{[m-1]}. \tag{2.1}$$

Without loss of generality, we suppose that

$$x_{t_p}\ge x_{t_{p-1}}\ge\cdots\ge x_{t_2}\ge x_{t_1}>x_{n-p}=x_{n-p-1}=\cdots=x_1=0,\quad p>0,\ p\in N.$$

(i) Firstly, we prove

$$\Psi_{\min}=\min_{\substack{i,j\in N,\\ j\neq i}}\Psi_{i,j}(\mathcal{A})\le\rho(\mathcal{A}).$$

By Eq. (2.1), we have

$$(\rho(\mathcal{A})-a_{t_1,\ldots,t_1})x_{t_1}^{m-1}=\sum_{\substack{(i_2,\ldots,i_m)\in\Theta_{t_1}\\ \delta_{t_1i_2\cdots i_m}=0}}a_{t_1i_2,\ldots,i_m}x_{i_2},\ldots,x_{i_m}+\sum_{(i_2,\ldots,i_m)\in\overline{\Theta}_{t_1}}a_{t_1i_2,\ldots,i_m}x_{i_2},\ldots,x_{i_m}.$$

Hence,

$$\begin{aligned}(\rho(\mathcal{A})-a_{t_1\ldots t_1})x_{t_1}^{m-1}&\ge\sum_{\substack{(i_2,\ldots,i_m)\in\Theta_{t_1}\\ \delta_{t_2i_2,\ldots,i_m}=0}}a_{t_1i_2,\ldots,i_m}x_{t_1}^{m-1}+\sum_{(i_2,\ldots,i_m)\in\overline{\Theta}_{t_1}}a_{t_1i_2,\ldots,i_m}x_{t_2}^{m-1}\\ &=r_{t_1}^{\Theta_{t_1}}(\mathcal{A})x_{t_1}^{m-1}+r_{t_1}^{\overline{\Theta}_{t_1}}(\mathcal{A})x_{t_2}^{m-1},\end{aligned}$$

i.e.,

$$(\rho(\mathcal{A})-a_{t_1,\ldots,t_1}-r_{t_1}^{\Theta_{t_1}}(\mathcal{A}))x_{t_1}^{m-1}\ge r_{t_1}^{\overline{\Theta}_{t_1}}(\mathcal{A})x_{t_2}^{m-1}\ge 0. \tag{2.2}$$

Similarly, from Eq. (2.1), we obtain,

$$(\rho(\mathcal{A})-a_{t_2,\ldots,t_2}-r_{t_2}^{\overline{\Theta}_{t_1}}(\mathcal{A}))x_{t_2}^{m-1}\ge r_{t_2}^{\Theta_{t_1}}(\mathcal{A})x_{t_1}^{m-1}\ge 0. \tag{2.3}$$

Multiplying Inequality (2.2) with Inequality (2.3) yields

$$(\rho(\mathcal{A})-a_{t_1,\ldots,t_1}-r_{t_1}^{\Theta_{t_1}}(\mathcal{A}))(\rho(\mathcal{A})-a_{t_2,\ldots,t_2}-r_{t_2}^{\overline{\Theta}_{t_1}}(\mathcal{A}))\ge r_{t_1}^{\overline{\Theta}_{t_1}}(\mathcal{A})r_{t_2}^{\Theta_{t_1}}(\mathcal{A}).$$

Then solving for $\rho(\mathcal{A})$ gives

$$\rho(\mathcal{A})\ge\Psi_{t_1,t_2}(\mathcal{A})\ge\min_{\substack{i,j\in N,\\ i\neq j}}\Psi_{i,j}(\mathcal{A})=\Psi_{\min}.$$

(ii) Similar to the argument in (i), we easily get

$$\rho(\mathcal{A})\le\Psi_{t_p,t_{p-1}}(\mathcal{A})\le\max_{\substack{i,j\in N,\\ i\neq j}}\Psi_{i,j}(\mathcal{A})=\Psi_{\max}.$$

The conclusion follows by combining (i) and (ii). □

We next compare the bounds in Theorems 1.4, 1.6, 1.7 and 2.1.

Theorem 2.2 *Let $\mathcal{A}\in\mathbb{R}^{[m,n]}$ be a nonnegative tensor with $n\ge 2$. Then*

$$R_{\min}\le\Omega_{\min}\le\Delta_{\min}\le\Psi_{\min}\le\rho(\mathcal{A})\le\Psi_{\max}\le\Delta_{\max}\le\Omega_{\max}\le R_{\max}. \tag{2.4}$$

Proof Without loss of generality, we suppose that $R_i(\mathcal{A})\le R_j(\mathcal{A})$, $\forall\, i,j\in N$, $i\neq j$, then

$$a_{ii,\ldots,i}-a_{jj,\ldots,j}+r_i^j(\mathcal{A})+a_{ij,\ldots,j}\le r_j(\mathcal{A}).$$

Hence,

$$\begin{aligned}
&(a_{ii,\dots,i} - a_{jj,\dots,j} + r_i^j(\mathcal{A}))^2 + 4a_{ij,\dots,j}r_j(\mathcal{A}) \\
&\geq (a_{ii,\dots,i} - a_{jj,\dots,j} + r_i^j(\mathcal{A}))^2 + 4a_{ij,\dots,j}(a_{ii,\dots,i} - a_{jj,\dots,j} + r_i^j(\mathcal{A}) + a_{ij,\dots,j}) \\
&= (a_{ii,\dots,i} - a_{jj,\dots,j} + r_i^j(\mathcal{A}))^2 + 4a_{ij,\dots,j}(a_{ii,\dots,i} - a_{jj,\dots,j} + r_i^j(\mathcal{A})) + 4a_{ij,\dots,j}^2 \\
&= (a_{ii,\dots,i} - a_{jj,\dots,j} + r_i^j(\mathcal{A}) + 2a_{ij,\dots,j})^2.
\end{aligned}$$

When $a_{ii,\dots,i} - a_{jj,\dots,j} + r_i^j(\mathcal{A}) + 2a_{ij,\dots,j} > 0$, we obtain

$$\begin{aligned}
&a_{ii,\dots,i} + a_{jj,\dots,j} + r_i^j(\mathcal{A}) + \sqrt{(a_{ii,\dots,i} - a_{jj,\dots,j} + r_i^j(\mathcal{A}))^2 + 4a_{ij,\dots,j}r_j(\mathcal{A})} \\
&\geq a_{ii,\dots,i} + a_{jj,\dots,j} + r_i^j(\mathcal{A}) + a_{ii,\dots,i} - a_{jj,\dots,j} + r_i^j(\mathcal{A}) + 2a_{ij,\dots,j} \\
&= 2(a_{ii,\dots,i} + r_i^j(\mathcal{A}) + a_{ij,\dots,j}) = 2R_i(A).
\end{aligned}$$

When $a_{ii,\dots,i} - a_{jj,\dots,j} + r_i^j(\mathcal{A}) + 2a_{ij,\dots,j} \leq 0$, that is, $a_{ii,\dots,i} + r_i^j(\mathcal{A}) + 2a_{ij,\dots,j} \leq a_{jj,\dots,j}$, we obatin

$$\begin{aligned}
&a_{ii,\dots,i} + a_{jj,\dots,j} + r_i^j(\mathcal{A}) + \sqrt{(a_{ii,\dots,i} - a_{jj,\dots,j} + r_i^j(\mathcal{A}))^2 + 4a_{ij,\dots,j}r_j(\mathcal{A})} \\
&\geq a_{ii,\dots,i} + a_{jj,\dots,j} + r_i^j(\mathcal{A}) + \sqrt{(a_{ii,\dots,i} - a_{jj,\dots,j} + r_i^j(\mathcal{A}))^2} \\
&= a_{ii,\dots,i} + a_{jj,\dots,j} + r_i^j(\mathcal{A}) - a_{ii,\dots,i} + a_{jj,\dots,j} - r_i^j(\mathcal{A}) \\
&= 2a_{jj,\dots,j} \geq 2(a_{ii,\dots,i} + r_i^j(\mathcal{A}) + 2a_{ij,\dots,j}) \\
&\geq 2(a_{ii,\dots,i} + r_i^j(\mathcal{A}) + a_{ij,\dots,j}) = 2R_i(A).
\end{aligned}$$

Therefore,

$$\frac{1}{2}\Big(a_{ii,\dots,i} + a_{jj,\dots,j} + r_i^j(\mathcal{A}) + \sqrt{(a_{ii,\dots,i} - a_{jj,\dots,j} + r_i^j(\mathcal{A}))^2 + 4a_{ij,\dots,j}r_j(\mathcal{A})}\ \Big) \geq R_i(\mathcal{A}),$$

which implies

$$\min_{\substack{i,j\in N\\ i\neq j}} \frac{1}{2}\Big(a_{ii,\dots,i} + a_{jj,\dots,j} + r_i^j(\mathcal{A}) + \sqrt{(a_{ii,\dots,i} - a_{jj,\dots,j} + r_i^j(\mathcal{A}))^2 + 4a_{ij,\dots,j}r_j(\mathcal{A})}\ \Big) \geq \min_{i\in N} R_i(\mathcal{A}),$$

i.e., $R_{\min} \leq \Omega_{\min}$. Similarly, we can prove that the other inequalities in (2.4) also hold. The proof is completed. □

Example 2.3 Consider the nonnegative tensor

$$\mathcal{A} = [A(:,:,1), A(:,:,2), A(:,:,3)] \in \mathbb{R}^{[3,3]},$$

where

$$A(:,:,1) = \begin{pmatrix} 0.2192 & 0.4411 & 0.5232 \\ 0.7637 & 0.5239 & 0.8330 \\ 0.7993 & 0.3710 & 0.5328 \end{pmatrix}, \quad A(:,:,2) = \begin{pmatrix} 0.4380 & 0.0482 & 0.1325 \\ 0.1803 & 0.6729 & 0.1809 \\ 0.3773 & 0.1079 & 0.8965 \end{pmatrix},$$

$$A(:,:,3) = \begin{pmatrix} 0.0779 & 0.1982 & 0.4691 \\ 0.5135 & 0.8284 & 0.7352 \\ 0.1135 & 0.1163 & 0.8645 \end{pmatrix}.$$

We now compute the bounds for $\rho(\mathcal{A})$.

By Theorem 1.4, we have $2.5474 \leq \rho(\mathcal{A}) \leq 5.2318$.

By Theorem 1.6, we have $2.6125 \leq \rho(\mathcal{A}) \leq 5.0753$.

By Theorem 1.7, we have $3.0097 \leq \rho(\mathcal{A}) \leq 4.7894$.

By Theorem 2.1, we have $3.2137 \leq \rho(\mathcal{A}) \leq 4.6547$.

It is easy to see that the bounds in Theorem 2.1 are tighter than all the others. In fact [11,12], the spectral radius $\rho(\mathcal{A}) = 3.7883$.

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